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Universal deformation rings and generalized quaternion defect groups

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Abstract

We determine the universal deformation rings $R(G, V)$ of certain mod 2 representations V of a finite group G which belong to a 2-modular block of G whose defect groups are isomorphic to a generalized quaternion group D. We show that for these V, a question raised by the author and Chinburg concerning the relation of $R(G, V)$ to D has an affirmative answer. We also show that $R(G, V)$ is a complete intersection even though $R(G/N, V)$ need not be for certain normal subgroups N of G which act trivially on V.

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Keywords: Universal deformation rings; Quaternion defect groups; Dihedral defect groups

1. Introduction

Let k be an algebraically closed field of positive characteristic p, let $W = W(k)$ be the ring of infinite Witt vectors over k, and let G be a finite group. It is a classical question to ask whether a given finitely generated kG-module V can be lifted to W or to a more general complete local commutative Noetherian ring R with residue field k. It is useful to formulate this question in terms of deformation rings. For example, Green's liftability theorem can be stated as saying that there is a surjection from the versal deformation ring $R(G, V)$ of V to W if there are no non-trivial 2-extensions of V by itself. A natural question is then to determine $R(G, V)$. In this paper,

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we determine $R(G, V)$ for certain mod 2 representations V of finite groups G which belong to 2-modular blocks of G with generalized quaternion defect groups.

The topological ring $R(G, V)$ is characterized by the property that every lift of V over a complete local commutative Noetherian ring R with residue field k arises from a local ring homomorphism $\alpha : R(G, V) \to R$ and that α is unique if R is the ring of dual numbers $k[\epsilon]/(\epsilon^2)$. In case α is unique for all R, $R(G, V)$ is called the universal deformation ring of V. For precise definitions, see Section 2. Note that all these rings R, including $R(G, V)$, have a natural structure as W-algebras.

One of the main motivations for studying universal deformation rings for finite groups is that they provide a good test case for various conjectures concerning the ring theoretic properties of universal deformation rings for profinite Galois groups. For example, in [6,7] it was shown that the universal deformation ring of the non-trivial irreducible mod 2 representation of the symmetric group S_4 is not a complete intersection. This led to infinitely many examples of real quadratic fields L such that the universal deformation ring of the inflation of this representation to the Galois group over L of the maximal totally unramified extension of L is not a complete intersection. The advantage of computing universal deformation rings for representations of finite groups is that one can use deep results from modular representation theory due to Brauer, Erdmann [17], Linckelmann [23,24], Carlson and Thévenaz [13,14], and others.

Suppose now that G is an arbitrary finite group and V is a finitely generated kG-module such that the stable endomorphism ring $\underline{\mathrm{End}}_{kG}(V)$ is isomorphic to k. By [5, Prop. 2.1], it follows that the versal deformation ring $R(G, V)$ of V is universal. In [5], the following question was raised which would relate $R(G, V)$ to the defect groups of the block of kG associated to V.

Question 1.1. Let B be a block of kG, let D be a defect group of B, and suppose V is a finitely generated kG-module whose stable endomorphism ring is isomorphic to k such that the unique (up to isomorphism) non-projective indecomposable summand of V belongs to B. Is the universal deformation ring $R(G, V)$ of V isomorphic to a subquotient ring of the group ring WD?

In [2–5,9], the isomorphism types of $R(G, V)$ have been determined for V belonging to cyclic blocks, respectively to various tame blocks with dihedral defect groups. It was shown that Question 1.1 has a positive answer in all these cases. Moreover, in [4, Cor. 5.1.2] an infinite series of finite groups G and mod 2 representations V was given for which $R(G, V)$ is not a complete intersection.

In this paper, we consider the case when k has characteristic 2 and B is a block of kG with generalized quaternion defect groups of order $2^{d+1} \geqslant 8$ such that the center of G/O_B has even order, where O_B is the maximal normal subgroup of G of odd order which acts trivially on all kG-modules belonging to B. Let $z \in G$ be an involution such that zO_B lies in the center of G/O_B, let $N = \langle O_B, z\rangle$ and let $\overline{B}$ be a block of $k[G/N]$ which is contained in the image of B under the natural surjection $\pi : kG \to k[G/N]$. Then $\overline{B}$ has dihedral defect groups of order $2^d \geqslant 4$, and B and $\overline{B}$ have the same number of isomorphism classes of simple modules (see Remark 3.3). In [11] (resp. [26]), Brauer (resp. Olsson) proved that a block with dihedral (resp. generalized quaternion) defect groups contains at most three simple modules up to isomorphism. In this paper, we consider the largest case when both B and $\overline{B}$ have precisely three isomorphism classes of simple modules.

Note that all principal blocks with generalized quaternion defect groups and precisely three isomorphism classes of simple modules are included in our discussion (see Remark 3.2). If B is principal and $d \geqslant 3$, then $\overline{B}$ is one of the blocks considered in [4] (see Remark 3.5).

We now summarize our main result; a more detailed statement can be found in Theorem 6.1. As above, if B is a block of kG then O_B is the maximal normal subgroup of G of odd order which acts trivially on all kG-modules belonging to B.

Theorem 1.2. *Suppose k has characteristic 2, G is a finite group, and B is a block of kG with a generalized quaternion defect group D of order $2^{d+1} \geqslant 8$ such that there are precisely three isomorphism classes of simple B-modules and such that the center of G/O_B has even order. Let $z \in G$ be an involution such that zO_B is central in G/O_B, let $N = \langle O_B, z\rangle$ and let $\overline{B}$ be a block of $k[G/N]$ which is contained in the image of B under the natural surjection $\pi : kG \to k[G/N]$. Let V be a finitely generated kG-module whose stable endomorphism ring is isomorphic to k and which is inflated from a $k[G/N]$-module belonging to $\overline{B}$. Then either*

(i) *$R(G,V)/2R(G,V) \cong k$, in which case $R(G,V)$ is isomorphic to W, or*
(ii) *$R(G,V)/2R(G,V) \cong k[[t]]/(t^{2^{d-1}-1})$, in which case $R(G,V)$ is isomorphic to $W[[t]]/(p_{d+1}(t))$ for a certain monic polynomial $p_{d+1}(t) \in W[t]$ of degree $2^{d-1} - 1$ whose non-leading coefficients are all divisible by 2.*

In all cases, $R(G,V)$ is isomorphic to a subquotient ring of WD and a complete intersection.

It is an interesting question how the universal deformation ring changes when one inflates a module V from a quotient group of G to G. Theorem 1.2 together with the results in [2,4] and Lemma 6.2 give an answer to this question for the quotient group G/N. Namely, using the notation of Theorem 1.2, the universal deformation ring $R(G/N,V)$ is as follows.

- If V is as in Theorem 1.2(i), then $R(G/N,V)$ is isomorphic to a quotient ring of W.
- If V is as in Theorem 1.2(ii), then $R(G/N,V) \cong W[[t]]/(tp_d(t), 2p_d(t))$, where we set $p_2(t) = 1$. In particular, if $d \geqslant 3$, $R(G/N,V)$ is not a complete intersection.

It is natural to ask whether Theorem 1.2 can be used to construct deformation rings arising from arithmetic in the following sense. Suppose L is a number field and S is a finite set of places of L. Let L_S be the maximal algebraic extension of L which is unramified outside S, and denote by $G_{L,S}$ the Galois group of L_S over L. Suppose k, G, G/N and V are as in Theorem 1.2, and let H be G or G/N. As in [7], one can ask whether there are L and S such that there is a surjection $\psi : G_{L,S} \to H$ which induces an isomorphism of deformation rings $R(G_{L,S},V) \to R(H,V)$ when V is viewed as a representation for $G_{L,S}$ via ψ. It was shown in [7, Lemma 3.3] that a sufficient condition for $R(G_{L,S},V) \to R(H,V)$ to be an isomorphism for all such V is that $\mathrm{Ker}(\psi)$ has no non-trivial pro-2 quotient. If this condition is satisfied, we say the group H caps L for 2 at S.

The prototypes for the groups G/N are $\mathrm{PSL}_2(\mathbb{F}_q)$ where q is an odd prime power and the alternating group A_7. Suppose G/N is one of these groups and that 8 divides the order of G/N. One can show similarly to the proof of [7, Thm. 3.7(i)] that G/N does not cap $\mathbb{Q}$ for 2 at any finite set of places S of $\mathbb{Q}$. However, it is an interesting question whether the group G does cap $\mathbb{Q}$ for 2 at certain S. For example, it was shown in [8] that although the symmetric group S_4 does not cap $\mathbb{Q}$ for 2 at any S, the double cover $\hat{S}_4$ does cap $\mathbb{Q}$ for 2 at certain S. This situation was similar to the present one, since the universal deformation ring of the simple kS_4-module E of dimension 2 is $R(S_4,E) \cong W[[t]]/(t^2, 2t) = W[[t]]/(tp_3(t), 2p_3(t))$, but the universal deformation ring of the inflation of E to $\hat{S}_4$ is $R(\hat{S}_4,E) \cong W[[t]]/(t^3 - 2t) = W[[t]]/(p_4(t))$.

The paper is organized as follows. In Section 2, we recall the definitions of deformations and deformation rings and we state some useful results from [4,5]. In Section 3, we concentrate on finite groups G and their quotient groups G/N as in Theorem 1.2 and use [17] to describe the 2-modular blocks B and $\overline{B}$. In Section 4, we determine all indecomposable kG-modules V whose stable endomorphism rings are isomorphic to k and which are inflated from $k[G/N]$-modules belonging to $\overline{B}$. In particular, we show that the endomorphism ring of each such V is isomorphic to k, leading to a finite set of possibilities for such V (see Remark 4.3). In Section 5, we describe results of Olsson [26] about the ordinary irreducible characters of G belonging to B. In Section 6, we determine the universal deformation rings of all the kG-modules V found in Section 4, by first determining the universal deformation rings modulo 2 and then using Olsson's results on ordinary characters to complete the computation (see Theorem 6.1).

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2. Preliminaries

In this section, we give a brief introduction to versal and universal deformation rings and deformations. For more background material, we refer the reader to [25] and [16].

Let k be an algebraically closed field of characteristic $p > 0$, let W be the ring of infinite Witt vectors over k, and let F be the fraction field of W. Let $\hat{\mathcal{C}}$ be the category of all complete local commutative Noetherian rings with residue field k. The morphisms in $\hat{\mathcal{C}}$ are continuous W-algebra homomorphisms which induce the identity map on k.

Suppose G is a finite group and V is a finitely generated kG-module. A lift of V over an object R in $\hat{\mathcal{C}}$ is a pair (M, ϕ) where M is a finitely generated RG-module which is free over R, and $\phi : k \otimes_R M \to V$ is an isomorphism of kG-modules. Two lifts (M, ϕ) and (M', ϕ') of V over R are isomorphic if there is an isomorphism $f : M \to M'$ with $\phi = \phi' \circ (k \otimes f)$. The isomorphism class $[M, \phi]$ of a lift (M, ϕ) of V over R is called a deformation of V over R, and the set of all such deformations is denoted by $\mathrm{Def}_G(V, R)$. The deformation functor

$$\hat{F}_V : \hat{\mathcal{C}} \to \mathrm{Sets}$$

is a covariant functor which sends an object R in $\hat{\mathcal{C}}$ to $\mathrm{Def}_G(V, R)$ and a morphism $\alpha : R \to R'$ in $\hat{\mathcal{C}}$ to the map $\mathrm{Def}_G(V, R) \to \mathrm{Def}_G(V, R')$ defined by $[M, \phi] \mapsto [R' \otimes_{R,\alpha} M, \phi_\alpha]$, where $\phi_\alpha = \phi$ after identifying $k \otimes_{R'} (R' \otimes_{R,\alpha} M)$ with $k \otimes_R M$.

Suppose there exists an object $R(G, V)$ in $\hat{\mathcal{C}}$ and a deformation $[U(G, V), \phi_U]$ of V over $R(G, V)$ with the following property: For each R in $\hat{\mathcal{C}}$ and for each lift (M, ϕ) of V over R there exists a morphism $\alpha : R(G, V) \to R$ in $\hat{\mathcal{C}}$ such that $\hat{F}_V(\alpha)([U(G, V), \phi_U]) = [M, \phi]$, and moreover α is unique if R is the ring of dual numbers $k[\epsilon]/(\epsilon^2)$. Then $R(G, V)$ is called the versal deformation ring of V and $[U(G, V), \phi_U]$ is called the versal deformation of V. If the morphism α is unique for all R and all lifts (M, ϕ) of V over R, then $R(G, V)$ is called the universal deformation ring of V and $[U(G, V), \phi_U]$ is called the universal deformation of V. In other words, $R(G, V)$ is universal if and only if $R(G, V)$ represents the functor $\hat{F}_V$ in the sense that $\hat{F}_V$ is naturally isomorphic to the Hom functor $\mathrm{Hom}_{\hat{\mathcal{C}}}(R(G, V), -)$.

By [25], every finitely generated kG-module V has a versal deformation ring. By a result of Faltings (see [16, Prop. 7.1]), V has a universal deformation ring if $\mathrm{End}_{kG}(V) = k$.

Remark 2.1. Note that the above definition of deformations differs from the definition used in [2,5] as follows. Let G and V be as above, let R be an object in $\hat{\mathcal{C}}$ and let (M,ϕ) be a lift of V over R. In [2,5], the isomorphism class of M as an RG-module was called a deformation of V over R, without taking into account the specific isomorphism $\phi : k \otimes_R M \to V$. Some authors call a deformation according to the latter definition a weak deformation of V over R (see [22, §5.2]).

In general, a weak deformation of V over R identifies more lifts than a deformation of V over R that respects the isomorphism ϕ of a representative (M,ϕ). However, if the stable endomorphism ring $\underline{\mathrm{End}}_{kG}(V)$ is isomorphic to k, these two definitions of deformations coincide. This can be seen as follows.

Suppose first that $\alpha : A \to A_0$ is a surjective morphism of Artinian rings A, A_0 in $\hat{\mathcal{C}}$ and (M,ϕ) is a lift of V over A. Let $(M_0,\phi_0) = (A_0 \otimes_{A,\alpha} M, \phi_\alpha)$ and let $u_0 \in \mathrm{End}_{A_0G}(M_0)$ be such that u_0 factors through a projective A_0G-module P_0, say $u_0 = h_0 \circ g_0$. We claim that u_0 can be lifted to an AG-module endomorphism u of M that factors through a projective AG-module P. By using induction on the length of A, it is enough to prove this when α is a small extension, i.e. when the kernel of α is a non-zero principal ideal tA where t is annihilated by the maximal ideal m_A of A. Since M_0 is a finitely generated A_0G-module, we can assume that P_0 is a finitely generated projective A_0G-module. Let P be a projective AG-module with $A_0 \otimes_{A,\alpha} P \cong P_0$. Consider the short exact sequence of AG-modules $0 \to tP \to P \xrightarrow{\alpha_P} P_0 \to 0$ which results from tensoring $0 \to tA \to A \xrightarrow{\alpha} A_0 \to 0$ with P over A. Since $k \otimes_A P$ is an injective kG-module, we have that $\mathrm{Ext}^1_{AG}(M, tP) \cong \mathrm{Ext}^1_{kG}(k \otimes_A M, k \otimes_A P) = 0$. Therefore, $\mathrm{Hom}_{AG}(M,P) \xrightarrow{(\alpha_P)_*} \mathrm{Hom}_{AG}(M,P_0)$ is surjective. This implies that $g_0 : M_0 \to P_0$ can be lifted to an AG-module homomorphism $g : M \to P$. Since P is a projective AG-module, it follows that $h_0 : P_0 \to M_0$ can be lifted to an AG-module homomorphism $h : P \to M$. Hence $u = h \circ g$ is an AG-module endomorphism of M which factors through P and which lifts u_0. Since by [5, Lemma 2.3] the ring homomorphism $\sigma_M : A \to \underline{\mathrm{End}}_{AG}(M)$ coming from the action of A on M via scalar multiplication is surjective, it follows that every endomorphism $f_0 \in \mathrm{End}_{A_0G}(M_0)$ can be lifted to an endomorphism $f \in \mathrm{End}_{AG}(M)$.

Now suppose that R is an arbitrary ring in $\hat{\mathcal{C}}$, i.e. $R = \varprojlim R/m_R^n$ where m_R is the maximal ideal of R. If (M,ϕ) and (M',ϕ') are two lifts of V over R, then $((R/m_R^n) \otimes_R M, \phi_n)$ and $((R/m_R^n) \otimes_R M', \phi'_n)$ are lifts of V over R/m_R^n for all n, where $\phi_n = \phi$ (resp. $\phi'_n = \phi'$) after identifying $k \otimes_{R/m_R^n} ((R/m_R^n) \otimes_R M)$ with $k \otimes_R M$ (resp. $k \otimes_{R/m_R^n} ((R/m_R^n) \otimes_R M')$ with $k \otimes_R M'$). Suppose there exists an RG-module isomorphism $f : M \to M'$. Then $f_n = (R/m_R^n) \otimes_R f : (R/m_R^n) \otimes_R M \to (R/m_R^n) \otimes_R M'$ is an $(R/m_R^n)G$-module isomorphism for all n. Define $g_1 = \phi^{-1} \circ \phi' \circ f_1$, which is a kG-module automorphism of $k \otimes_R M$. By what we showed above, we can inductively lift g_1 to an $(R/m_R^n)G$-module endomorphism g_n of $(R/m_R^n) \otimes_R M$ such that g_{n+1} lifts g_n for all n. Since $k \otimes g_n = g_1$ is an automorphism of $k \otimes_R M$ and $(R/m_R^n) \otimes_R M$ is free over R/m_R^n, it follows that g_n is an $(R/m_R^n)G$-module automorphism of $(R/m_R^n) \otimes_R M$. For all $n \geqslant 1$, define $h_n = f_n \circ g_n^{-1}$. Then $h_n : (R/m_R^n) \otimes_R M \to (R/m_R^n) \otimes_R M'$ is an $(R/m_R^n)G$-module isomorphism such that $\phi'_n \circ (k \otimes h_n) = \phi_n$ and such that h_{n+1} lifts h_n for all n. Hence $\{h_n\}$ defines an inverse system and its inverse limit defines an RG-module isomorphism $h : M \to M'$ with $\phi' \circ (k \otimes h) = \phi$. In particular, $[M,\phi] = [M',\phi']$.

The following three results were proved in [5] and in [4], respectively. Here Ω denotes the syzygy, or Heller, operator for kG (see for example [1, §20]).

Proposition 2.2. *(See [5, Prop. 2.1].) Suppose V is a finitely generated kG-module whose stable endomorphism ring $\underline{\mathrm{End}}_{kG}(V)$ is isomorphic to k. Then V has a universal deformation ring $R(G,V)$.*

Lemma 2.3. *(See [5, Cors. 2.5 and 2.8].) Let V be a finitely generated kG-module with $\underline{\mathrm{End}}_{kG}(V)\cong k$.*

(i) *Then $\underline{\mathrm{End}}_{kG}(\Omega(V))\cong k$, and $R(G,V)$ and $R(G,\Omega(V))$ are isomorphic.*
(ii) *There is a non-projective indecomposable kG-module V_0 (unique up to isomorphism) such that $\underline{\mathrm{End}}_{kG}(V_0)\cong k$, V is isomorphic to $V_0\oplus P$ for some projective kG-module P, and $R(G,V)$ and $R(G,V_0)$ are isomorphic.*

Lemma 2.4. *(See [4, Lemma 2.3.2].) Let V be a finitely generated kG-module such that there is a non-split short exact sequence of kG-modules*

$$0\to Y_2\to V\to Y_1\to 0$$

with $\mathrm{Ext}^1_{kG}(Y_1,Y_2)\cong k$. Suppose that for $i\in\{1,2\}$, there exists a WG-module X_i which defines a lift of Y_i over W. Suppose further that

$$\dim_F \mathrm{Hom}_{FG}(F\otimes_W X_1, F\otimes_W X_2)=\dim_k \mathrm{Hom}_{kG}(Y_1,Y_2)-1.$$

Then there exists a WG-module X which defines a lift of V over W.

We also need the following result which is a more general version of [4, Lemma 2.3.1].

Lemma 2.5. *Let Y be a finitely generated uniserial kG-module satisfying $\mathrm{End}_{kG}(Y)\cong k$ and $\mathrm{Ext}^1_{kG}(Y,Y)\cong k$. Suppose that the radical length of Y is $\ell\geqslant 1$ and that the composition factors in the descending radical series are given as $(T_1,T_2,\ldots,T_\ell)$ where $T_1,\ldots,T_\ell$ are simple kG-modules, not necessarily distinct. Assume there exists an integer $r\geqslant 2$ such that the projective cover P_{T_1} of T_1 has a quotient module $\overline{U}$ which is a uniserial kG-module of radical length ℓr such that the composition factors in the descending radical series are given as*

$$(T_1,T_2,\ldots,T_\ell,T_1,T_2,\ldots,T_\ell,\ldots,T_1,T_2,\ldots,T_\ell).$$

Suppose there are kG-module isomorphisms $\phi:\overline{U}/\mathrm{rad}^\ell(\overline{U})\to Y$ and $\psi:\overline{U}/\mathrm{rad}^{\ell(r-1)}(\overline{U})\to \mathrm{rad}^\ell(\overline{U})$, and assume that $\mathrm{Ext}^1_{kG}(\overline{U},Y)=0$. Then the universal deformation ring of Y modulo p is $\overline{R}=R(G,Y)/pR(G,Y)\cong k[t]/(t^r)$, and the universal mod *p deformation of Y over $\overline{R}$ is $[\overline{U},\phi]$ where the action of t on $\overline{U}$ is induced by ψ.*

Proof. By assumption, $\mathrm{Ext}^1_{kG}(Y,Y)\cong k$, which implies that $\overline{R}\cong k[[t]]$ or $\overline{R}\cong k[t]/(t^m)$ for some $m\geqslant 2$. The action of t on $\overline{U}$ is given by the composition

$$\overline{U}\xrightarrow{\pi_U}\overline{U}/\mathrm{rad}^{\ell(r-1)}(\overline{U})\xrightarrow{\psi}\mathrm{rad}^\ell(\overline{U})\xrightarrow{\iota_U}\overline{U}$$

where π_U (resp. ι_U) is the natural surjection (resp. inclusion). It follows that $\mathrm{rad}^\ell(\overline{U}) = t\overline{U}$ and that $\overline{U}$ is a free $k[t]/(t^r)$-module under this action. In particular, ϕ defines a kG-module isomorphism $k \otimes_{k[t]/(t^r)} \overline{U} = \overline{U}/\mathrm{rad}^\ell(\overline{U}) \to Y$, and hence $(\overline{U}, \phi)$ defines a lift of Y over $k[t]/(t^r)$. This means that there is a unique k-algebra homomorphism

$$\alpha : \overline{R} \to k[t]/\bigl(t^r\bigr)$$

corresponding to the deformation $[\overline{U}, \phi]$. Since $\overline{U}$ is indecomposable as a kG-module, it follows that α is surjective. We now show that α is a k-algebra isomorphism. Suppose this is false. Then there exists a surjective k-algebra homomorphism $\alpha_1 : \overline{R} \to k[t]/(t^{r+1})$ such that $\tau \circ \alpha_1 = \alpha$ where $\tau : k[t]/(t^{r+1}) \to k[t]/(t^r)$ is the natural projection. Let $(\overline{U}_1, \phi_1)$ be a lift of Y over $k[t]/(t^{r+1})$ corresponding to α_1. Then $k[t]/(t^r) \otimes_{k[t]/(t^{r+1}),\tau} \overline{U}_1 \cong \overline{U}$ and $t^r\overline{U}_1 \cong Y$. We have a short exact sequence of $k[t]/(t^{r+1})G$-modules

$$0 \to t^r\overline{U}_1 \to \overline{U}_1 \to \overline{U} \to 0. \tag{2.1}$$

We now show that this sequence cannot split as a sequence of kG-modules. Suppose it splits. Then $\overline{U}_1 \cong Y \oplus \overline{U}$ as kG-modules. Let $z = \binom{y}{u} \in Y \oplus \overline{U} \cong \overline{U}_1$. Then t acts on z as multiplication by a matrix of the form

$$M_t = \begin{pmatrix} 0 & \gamma \\ 0 & \mu_t \end{pmatrix}$$

where $\gamma : \overline{U} \to Y$ is a surjective kG-module homomorphism, and μ_t is multiplication by t on $\overline{U}$. Since $t^r\overline{U}_1 \cong Y$, $(M_t)^r$ cannot be the zero matrix. Because $\mathrm{End}_{kG}(Y) \cong k$, γ corresponds, up to a non-zero scalar multiple, to the isomorphism $\phi : \overline{U}/t\overline{U} \to Y$, which means that the kernel of γ is $t\overline{U}$. This implies that $(M_t)^r$ is the zero matrix, which is a contradiction. Hence the short exact sequence (2.1) does not split as a sequence of kG-modules. Since $\mathrm{Ext}^1_{kG}(\overline{U}, Y) = 0$ by assumption and $t^r\overline{U}_1 \cong Y$, this is impossible. Therefore, $\overline{U}_1$ does not exist, which means that α is a k-algebra isomorphism. Thus $\overline{R} \cong k[t]/(t^r)$, and the universal mod p deformation of Y over $\overline{R}$ is $[\overline{U}, \phi]$. □

3. Blocks with generalized quaternion defect groups

For the remainder of this paper, we make the following assumptions.

Hypothesis 3.1. Let k be an algebraically closed field of characteristic 2. Suppose G is a finite group and B is a block of kG with a generalized quaternion defect group D of order $2^{d+1} \geqslant 8$ such that there are precisely three isomorphism classes of simple B-modules. Assume further that the center of G/O_B has even order, where O_B is the maximal normal subgroup of G of odd order which acts trivially on all kG-modules belonging to B. Let $z \in G$ be an involution such that zO_B lies in the center of G/O_B, and let $N = \langle O_B, z\rangle$. Let $\overline{B}$ be a block of $k[G/N]$ which is contained in the image of B under the natural surjection $\pi : kG \to k[G/N]$.

Note that Olsson proved in [26] that a block with generalized quaternion defect groups contains at most three simple modules up to isomorphism. Hence we consider the largest case.

Remark 3.2. Suppose k is an algebraically closed field of characteristic 2 and G is a finite group with generalized quaternion Sylow 2-subgroups of order $2^{d+1} \geqslant 8$. If B is the principal block of kG, then $O_B = O_{2'}(G)$, where $O_{2'}(G)$ is the maximal normal subgroup of G of odd order. By a result of Brauer and Suzuki [12,10], the center of $G/O_{2'}(G)$ has order 2. Hence all assumptions in Hypothesis 3.1 are satisfied provided there are precisely three isomorphism classes of simple B-modules.

The following remark discusses the basic properties of B and $\overline{B}$ as in Hypothesis 3.1.

Remark 3.3. Assume Hypothesis 3.1. Since O_B has odd order and the characteristic of k is 2, the blocks of $k[G/O_B]$ correspond to the blocks of kG whose primitive central idempotents occur in the decomposition of the central idempotent $\frac{1}{\#O_B}\sum_{g\in O_B} g$. Because O_B acts trivially on all kG-modules belonging to B, it follows that B can be identified with a block B_{O_B} of $k[G/O_B]$ and all kG-modules V, V' belonging to B can be identified with $k[G/O_B]$-modules belonging to B_{O_B}. We obtain that $\mathrm{Hom}_{kG}(V, V') = \mathrm{Hom}_{k[G/O_B]}(V, V')$, $\underline{\mathrm{Hom}}_{kG}(V, V') = \underline{\mathrm{Hom}}_{k[G/O_B]}(V, V')$ and $\mathrm{Ext}^i_{kG}(V, V') = \mathrm{Ext}^i_{k[G/O_B]}(V, V')$ for all $i \geqslant 1$.

Let S and S' be simple kG-modules belonging to B. Since their restriction to N is trivial, they can also be viewed as $k[G/N]$-modules. The Lyndon–Hochschild–Serre spectral sequence gives an exact sequence

$$0 \to \mathrm{Ext}^1_{k[G/N]}\big(S, S'\big) \to \mathrm{Ext}^1_{k[G/O_B]}\big(S, S'\big) \to \mathrm{Hom}_{k[G/N]}\big(S, S'\big).$$

This implies immediately that $\mathrm{Ext}^1_{kG}(S, S')$ and $\mathrm{Ext}^1_{k[G/N]}(S, S')$ have the same k-dimension for each choice of non-isomorphic S and S'. Since $\overline{B}$ is contained in the image of B under the natural surjection $\pi : kG \to k[G/N]$, it follows by [1, Prop. 13.3] that $\overline{B}$ has the same number of isomorphism classes of simple modules as B. Hence we can identify the simple B-modules with the simple $\overline{B}$-modules. It follows that the restriction of π to B gives a surjective k-algebra homomorphism $\pi_B : B \to \overline{B}$. In particular, this implies that if V is a $k[G/N]$-module belonging to $\overline{B}$, then its inflation to kG via π belongs to B.

Define $D_{O_B} = DO_B/O_B$ and $\overline{D} = DN/N$. Then D_{O_B} is a defect group of B_{O_B}. By [1, Thm. 13.6], zO_B lies in D_{O_B}, which implies $\overline{D} \cong D_{O_B}/\langle zO_B\rangle$. Since the simple B-modules can be identified with the simple $\overline{B}$-modules, it follows by [15, Prop. (56.32)] that the order of the defect groups of $\overline{B}$ is $2^d = \#\overline{D}$. Because all indecomposable B-modules are (G, D)-projective, it follows that all indecomposable $\overline{B}$-modules are $(G/N, \overline{D})$-projective. Since there exists an indecomposable $\overline{B}$-module which has a defect group of $\overline{B}$ as a vertex (see [15, Thm. (59.10)]), it follows that $\overline{D}$ is a defect group of $\overline{B}$. In particular, this implies that the defect groups of $\overline{B}$ are dihedral groups of order 2^d.

Lemma 3.4. *Assume Hypothesis* 3.1*, and let $\pi_B : B \to \overline{B}$ be the surjective k-algebra homomorphism from Remark* 3.3*. Suppose $\overline{P}$ is a projective indecomposable $\overline{B}$-module, and denote its inflation to B via π_B also by $\overline{P}$. Let P be a projective indecomposable B-module which is a projective cover of $\overline{P}$. Then there is a short exact sequence of B-modules*

$$0 \to \overline{P} \to P \to \overline{P} \to 0.$$

Proof. By Remark 3.3, we can assume without loss of generality that O_B is trivial, which means that the involution z lies in the center of G and $N = \langle z\rangle$. Hence the kernel of the natural projection

$\pi : kG \to k[G/N]$ is $\mathrm{Ker}(\pi) = (1+z)kG$, which implies that the kernel of π_B is $\mathrm{Ker}(\pi_B) = (1+z)B$. Because P is a projective B-module, we obtain a short exact sequence of B-modules

$$0 \to (1+z)P \to P \to \overline{P} \to 0.$$

Since the map $f : (1+z)P \to \overline{B} \otimes_B P$, defined by $f((1+z)x) = 1 \otimes x$ for all $x \in P$, is a B-module isomorphism, Lemma 3.4 follows. □

Assume Hypothesis 3.1. From Erdmann's classification of all blocks of tame representation type in [17], and by using Lemma 3.4, it follows that the quivers of the basic algebras of B and $\overline{B}$ can be identified and that, up to Morita equivalence, there are precisely three families (I), (II) and (III) of blocks $\overline{B}$ and B.

Using [17], we now give a description of these families as follows. For each of the three families, we give a quiver Q and ideals I and $\bar{I}$ of kQ such that B is Morita equivalent to $\Lambda = kQ/I$ and $\overline{B}$ is Morita equivalent to $\overline{\Lambda} = kQ/\bar{I}$. We denote the simple Λ-modules by S_0, S_1, S_2, or, using short-hand, by $0, 1, 2$. The corresponding projective indecomposable Λ-modules are denoted by P_0, P_1 and P_2. We describe the radical series of P_0, P_1 and P_2 and also provide the decomposition matrix of B.

3.1. Family (I)

For family (I), $d \geqslant 2$. The quiver Q has the form

$$Q = 1 \bullet \underset{\gamma}{\overset{\beta}{\rightleftarrows}} \overset{0}{\bullet} \underset{\eta}{\overset{\delta}{\rightleftarrows}} \bullet\, 2$$

and the ideals I and $\bar{I}$ are given as

$$\begin{aligned} I = \big\langle &\beta\gamma\beta - \eta\delta\beta(\gamma\eta\delta\beta)^{2^{d-1}-1}, \gamma\beta\gamma - \gamma\eta\delta(\beta\gamma\eta\delta)^{2^{d-1}-1}, \\ &\eta\delta\eta - \beta\gamma\eta(\delta\beta\gamma\eta)^{2^{d-1}-1}, \delta\eta\delta - \delta\beta\gamma(\eta\delta\beta\gamma)^{2^{d-1}-1}, \\ &\delta\beta\gamma\beta, \gamma\eta\delta\eta\big\rangle, \\ \bar{I} = \big\langle &\gamma\beta, \delta\eta, (\eta\delta\beta\gamma)^{2^{d-2}} - (\beta\gamma\eta\delta)^{2^{d-2}}\big\rangle. \end{aligned}$$

The radical series of the projective indecomposable Λ-modules P_0, P_1 and P_2 are described in Fig. 1 where the radical series length of each of these modules is $2^{d+1} + 1$. The decomposition matrix of B is given in Fig. 2.

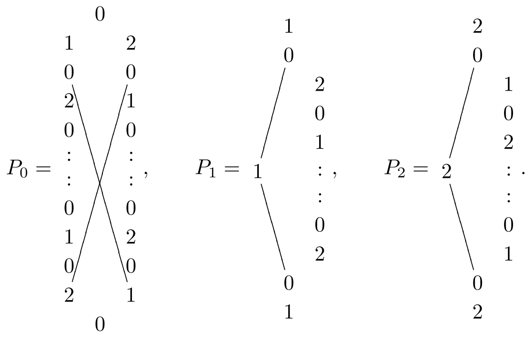


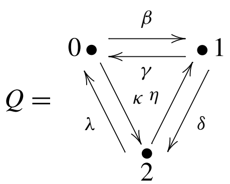
Fig. 1. The projective indecomposable modules for blocks B in family (I).

	φ_0	φ_1	φ_2	
χ_1	1	0	0	
χ_2	1	1	0	
χ_3	1	0	1	
χ_4	1	1	1	
χ_5	0	1	0	
χ_6	0	0	1	
$\chi_{7,i}$	2	1	1	$1 \le i \le 2^{d-1}-1.$

Fig. 2. The decomposition matrix for blocks B in family (I).

3.2. Family (II)

For family (II), $d \geqslant 2$. The quiver Q has the form



and the ideals I and $\bar{I}$ are given as

$$\begin{aligned} I = \big\langle & \delta\beta - \kappa\lambda\kappa, \gamma\eta - \lambda\kappa\lambda, \lambda\delta - \gamma\beta\gamma, \eta\kappa - \beta\gamma\beta, \\ & \beta\lambda - \eta(\delta\eta)^{2^{d-1}-1}, \kappa\gamma - \delta(\eta\delta)^{2^{d-1}-1}, \\ & \delta\beta\gamma, \gamma\eta\delta, \eta\kappa\lambda\big\rangle, \\ \bar{I} = \big\langle & \delta\beta, \lambda\delta, \beta\lambda, \kappa\gamma, \eta\kappa, \gamma\eta, \gamma\beta - \lambda\kappa, \kappa\lambda - (\delta\eta)^{2^{d-2}}, (\eta\delta)^{2^{d-2}} - \beta\gamma\big\rangle. \end{aligned}$$

The radical series of the projective indecomposable Λ-modules P_0, P_1 and P_2 are described in Fig. 3 where the radical series length of P_0 is 5 and the radical series length of P_1 and P_2 is $2^d + 1$. The decomposition matrix of B is given in Fig. 4.

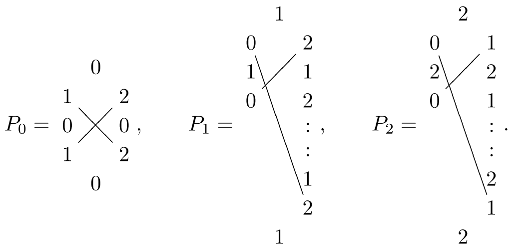


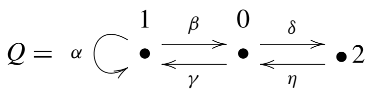
Fig. 3. The projective indecomposable modules for blocks B in family (II).

	φ_0	φ_1	φ_2	
χ_1	1	0	0	
χ_2	0	1	0	
χ_3	0	0	1	
χ_4	1	1	1	
χ_5	1	1	0	
χ_6	1	0	1	
$\chi_{7,i}$	0	1	1	$1 \le i \le 2^{d-1} - 1.$

Fig. 4. The decomposition matrix for blocks B in family (II).

3.3. Family (III)

For family (III), $d \geqslant 3$. The quiver Q has the form



and the ideals I and $\bar{I}$ are given as

$$\begin{aligned} I = \big\langle & \beta\alpha - \eta\delta\beta(\gamma\eta\delta\beta), \alpha\gamma - \gamma\eta\delta(\beta\gamma\eta\delta), \\ & \delta\eta\delta - \delta\beta\gamma(\eta\delta\beta\gamma), \eta\delta\eta - \beta\gamma\eta(\delta\beta\gamma\eta), \\ & \gamma\beta - \alpha^{2^{d-1}-1}, \beta\alpha^2, \delta\eta\delta\beta\big\rangle, \\ \bar{I} = \big\langle & \beta\alpha, \alpha\gamma, \gamma\beta, \delta\eta, \eta\delta\beta\gamma - \beta\gamma\eta\delta, \alpha^{2^{d-2}} - \gamma\eta\delta\beta\big\rangle. \end{aligned}$$

The radical series of the projective indecomposable Λ-modules P_0, P_1 and P_2 are described in Fig. 5 where the radical series length of P_0 and P_2 is 9 and the radical series length of P_1 is 9 if $d = 3$ and $2^{d-1} + 1$ if $d \geqslant 4$. The decomposition matrix of B is given in Fig. 6.

Remark 3.5. It follows from [17] that we have the following Morita equivalences for the blocks $\overline{B}$ and B:

(i) In family (I), $\overline{B}$ is Morita equivalent to the principal 2-modular block of $\mathrm{PSL}_2(\mathbb{F}_q)$ and B is Morita equivalent to the principal 2-modular block of $\mathrm{SL}_2(\mathbb{F}_q)$ when q is a prime power with $q \equiv 1 \bmod 4$ such that 2^{d+1} is the maximal 2-power dividing $(q^2 - 1)$.

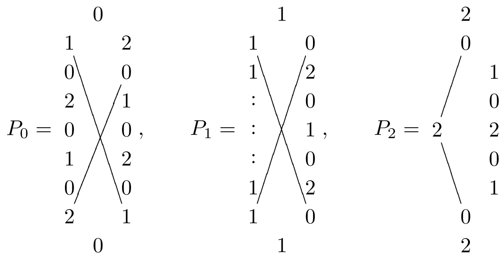


Fig. 5. The projective indecomposable modules for blocks B in family (III).

$$\begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \\ \chi_6 \\ \chi_{7,i} \end{array} \begin{array}{c} \varphi_0 \ \varphi_1 \ \varphi_2 \\ \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \end{array} \quad 1 \le i \le 2^{d-1} - 1.$$

Fig. 6. The decomposition matrix for blocks B in family (III).

(ii) In family (II), $\overline{B}$ is Morita equivalent to the principal 2-modular block of $\mathrm{PSL}_2(\mathbb{F}_q)$ and B is Morita equivalent to the principal 2-modular block of $\mathrm{SL}_2(\mathbb{F}_q)$ when q is a prime power with $q \equiv 3 \bmod 4$ such that 2^{d+1} is the maximal 2-power dividing $(q^2 - 1)$.

(iii) In family (III), if $d = 3$ then $\overline{B}$ is Morita equivalent to the principal 2-modular block of the alternating group A_7 and B is Morita equivalent to the principal 2-modular block of the non-trivial double cover $\tilde{A}_7$ of A_7. If $d \geqslant 4$, by [17, §X.4], it remains open whether there are blocks $\overline{B}$ and B that are Morita equivalent to the algebras $\overline{\Lambda}$ and Λ, respectively.

If $\overline{B}$ is the principal block of $k[G/N]$, then we can exclude the blocks in family (III) with $d \geqslant 4$ as follows. Since N contains $O_{2'}(G)$ by Remark 3.2, G/N has no non-trivial normal subgroup of odd order. By [19], it then follows that G/N is isomorphic to either a subgroup of $\mathrm{P\Gamma L}_2(\mathbb{F}_q)$ containing $\mathrm{PSL}_2(\mathbb{F}_q)$ for some odd prime power q, or to the alternating group A_7. Using a theorem by Clifford [21, Hauptsatz V.17.3], we see that the only possibility for $\overline{B}$ to be Morita equivalent to a block in family (III) occurs when $d = 3$ and $\overline{B}$ is Morita equivalent to the principal 2-modular block of the alternating group A_7. Therefore, if $d \geqslant 3$ then the blocks $\overline{B}$ in the families (I), (II) and (III) that are Morita equivalent to principal blocks are precisely the blocks considered in [4].

In the following remark we introduce the notation that will enable us to go back and forth between B, $\overline{B}$ and Λ, $\overline{\Lambda}$.

Remark 3.6. Assume Hypothesis 3.1. By Remark 3.3, the natural projection $\pi : kG \to k[G/N]$ induces by restriction to B a surjective k-algebra homomorphism $\pi_B : B \to \overline{B}$. Let e be a sum of orthogonal primitive idempotents in B such that eBe is basic and Morita equivalent to B, and let $\bar{e} = \pi_B(e)$. Then $\bar{e}\overline{B}\bar{e}$ is basic and Morita equivalent to $\overline{B}$, and the restriction of π_B to eBe gives a surjective k-algebra homomorphism $\pi_e : eBe \to \bar{e}\overline{B}\bar{e}$.

Suppose $\Lambda = kQ/I$ and $\overline{\Lambda} = kQ/\bar{I}$ are as described in Section 3.1, 3.2 or 3.3 such that B is Morita equivalent to Λ and $\overline{B}$ is Morita equivalent to $\overline{\Lambda}$. Then there are k-algebra isomorphisms $f_\Lambda : eBe \to \Lambda$ and $f_{\overline{\Lambda}} : \bar{e}\overline{B}\bar{e} \to \overline{\Lambda}$. Moreover, π_e induces a surjective k-algebra homomorphism $\pi_\Lambda : \Lambda \to \overline{\Lambda}$ such that the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{\pi_B} & \overline{B} \\ {\scriptstyle \text{incl.}}\uparrow & & \uparrow{\scriptstyle \text{incl.}} \\ eBe & \xrightarrow{\pi_e} & \bar{e}\overline{B}\bar{e} \\ {\scriptstyle f_\Lambda}\uparrow & & \uparrow{\scriptstyle f_{\overline{\Lambda}}} \\ \Lambda & \xrightarrow{\pi_\Lambda} & \overline{\Lambda} \end{array} \tag{3.1}$$

By Remark 3.3, we can identify the simple $\overline{B}$-modules with the simple B-modules via inflation using π_B. This implies that the simple $\bar{e}\overline{B}\bar{e}$-modules can also be identified with the simple eBe-modules via inflation using π_e, and hence the simple $\overline{\Lambda}$-modules can be identified with the simple Λ-modules via inflation using π_Λ. For $i \in \{0, 1, 2\}$, let $e_i \in \Lambda$ (resp. $\bar{e}_i \in \overline{\Lambda}$) be the primitive idempotent corresponding to the vertex i in Q. Using Lemma 3.4 and the structure of the projective indecomposable Λ-modules, we see that we can replace $f_{\overline{\Lambda}}$ by $f_{\overline{\Lambda}} \circ \Xi$ for a suitable k-algebra automorphism Ξ of $\overline{\Lambda}$ induced by a quiver automorphism of Q so as to be able to assume that $\overline{\Lambda} \otimes_{\Lambda,\pi_\Lambda} \Lambda e_i \cong \overline{\Lambda}\pi_\Lambda(e_i) \cong \overline{\Lambda}\bar{e}_i$ for $i \in \{0, 1, 2\}$. We make this assumption from now on. Hence we obtain for $i \in \{0, 1, 2\}$ that the simple Λ-module $S_i = \Lambda e_i/\mathrm{rad}(\Lambda)e_i$ is isomorphic to the inflation via π_Λ of the simple $\overline{\Lambda}$-module $\overline{\Lambda}\bar{e}_i/\mathrm{rad}(\overline{\Lambda})\bar{e}_i$.

4. Stable endomorphism rings

Assume Hypothesis 3.1. Let V be a finitely generated $k[G/N]$-module and denote its inflation to kG via π also by V. By Higman's criterion (see [20, Thm. 1]), the kG-module endomorphisms of V that factor through projective kG-modules are precisely those in the image of the trace map $\mathrm{Tr}_1^G : \mathrm{End}_k(V) \to \mathrm{End}_{kG}(V)$, where $\mathrm{Tr}_1^G(\psi)(v) = \sum_{g\in G} g\psi(g^{-1}v)$ for all $\psi \in \mathrm{End}_k(V)$ and all $v \in V$. Because N acts trivially on V, Tr_1^N is multiplication by $\#N = 2 \cdot (\#O_B)$. Hence Tr_1^N is zero, which implies that $\mathrm{Tr}_1^G = \mathrm{Tr}_N^G \circ \mathrm{Tr}_1^N$ is also zero. We obtain the following result.

Proposition 4.1. *Assume Hypothesis* 3.1. *Suppose V is an indecomposable $k[G/N]$-module belonging to $\overline{B}$, and denote its inflation to kG via π also by V. Then V belongs to B, and* $\underline{\mathrm{End}}_{kG}(V) \cong \mathrm{End}_{k[G/N]}(V)$. *In particular,* $\underline{\mathrm{End}}_{kG}(V) \cong k$ *if and only if* $\mathrm{End}_{k[G/N]}(V) \cong k$.

We now describe all $k[G/N]$-modules belonging to $\overline{B}$ whose endomorphism rings are isomorphic to k by describing the corresponding $\overline{\Lambda}$-modules and their inflations via π_Λ to Λ-modules, using the notation introduced in Remark 3.6. We first need to define some indecomposable Λ-modules.

Definition 4.2. Let $\Lambda = kQ/I$ where Q and I are either as in Section 3.1, 3.2 or 3.3.

(i) Let $w = \zeta_n \cdots \zeta_2\zeta_1$ be a path of length $n \geqslant 1$ in kQ whose image modulo I does not lie in $\mathrm{soc}_2(\Lambda)$. For $1 \leqslant j \leqslant n$, let v_j be the end vertex of ζ_j, and let v_0 be the starting vertex of ζ_1. Define a kQ-module M_w of k-dimension $n+1$ with respect to a given k-basis $\{b_0, \dots, b_n\}$ as follows. Let $0 \leqslant j \leqslant n$. If v is a vertex in Q, define $vb_j = b_j$ if $v = v_j$, and $vb_j = 0$ otherwise. If ζ is an arrow in Q, define $\zeta b_j = b_{j+1}$ if $\zeta = \zeta_{j+1}$ and $j \leqslant n-1$, otherwise define $\zeta b_j = 0$. By our assumption on w, the ideal I of kQ acts as zero on M_w. Hence M_w defines a Λ-module, which we also denote by M_w. Moreover, M_w is a uniserial Λ-module whose descending composition factors are $(S_{v_0}, S_{v_1}, \dots, S_{v_n})$.

(ii) Let ζ_1, ζ_2 be two arrows in Q which start (resp. end) at the same vertex v_1. Let v_0, v_2 be the end vertices (resp. starting vertices) of ζ_1, ζ_2. Let $w = \zeta_2\zeta_1^{-1}$ (resp. $w = \zeta_2^{-1}\zeta_1$). Define a kQ-module M_w of k-dimension 3 with respect to a given k-basis $\{b_0, b_1, b_2\}$ as follows. If v is a vertex in Q and $j \in \{0,1,2\}$, define $vb_j = b_j$ if $v = v_j$, and $vb_j = 0$ otherwise. Define $\zeta_1 b_1 = b_0$ and $\zeta_2 b_1 = b_2$ (resp. $\zeta_1 b_0 = b_1$ and $\zeta_2 b_2 = b_1$). If ζ is an arrow in Q and $j \in \{0,1,2\}$ such that $(\zeta, j) \notin \{(\zeta_1, 1), (\zeta_2, 1)\}$ (resp. $(\zeta, j) \notin \{(\zeta_1, 0), (\zeta_2, 2)\}$), define $\zeta b_j = 0$. Since the ideal I of kQ acts as zero on M_w, this defines a Λ-module, which we also denote by M_w. Moreover, M_w is a Λ-module with $M_w/\mathrm{rad}(M_w) \cong S_{v_1}$ (resp. $\mathrm{soc}(M_w) \cong S_{v_1}$) and $\mathrm{rad}(M_w) \cong S_{v_0} \oplus S_{v_2}$ (resp. $M/\mathrm{soc}(M_w) \cong S_{v_0} \oplus S_{v_2}$). In particular, $\mathrm{soc}(M_w) = \mathrm{rad}(M_w)$.

Remark 4.3. Assume Hypothesis 3.1. Suppose $\Lambda = kQ/I$ and $\overline{\Lambda} = kQ/\overline{I}$ are as described in Section 3.1, 3.2 or 3.3 such that B is Morita equivalent to Λ and $\overline{B}$ is Morita equivalent to $\overline{\Lambda}$. Let $\pi_\Lambda : \Lambda \to \overline{\Lambda}$ be the surjective k-algebra homomorphism from Remark 3.6. In particular, for $i \in \{0,1,2\}$, the simple Λ-module S_i is isomorphic to the inflation via π_Λ of the simple $\overline{\Lambda}$-module corresponding to the vertex i in Q, and we denote the latter also by S_i.

We now describe the $\overline{\Lambda}$-modules whose endomorphism rings are isomorphic to k and their inflations via π_Λ to Λ-modules, up to isomorphism, using the following notation:

A module of the form $\begin{smallmatrix} v_0 \\ \vdots \\ v_n \end{smallmatrix}$ denotes a uniserial $\overline{\Lambda}$-module whose factors in the descending radical series are $(S_{v_0}, \dots, S_{v_n})$ and which is the unique such $\overline{\Lambda}$-module up to isomorphism. A module of the form $\begin{smallmatrix} & v_1 & \\ v_2 & & v_0 \end{smallmatrix}$ (resp. $\begin{smallmatrix} v_2 & & v_0 \\ & v_1 & \end{smallmatrix}$) denotes an indecomposable $\overline{\Lambda}$-module whose top (resp. socle) is simple and whose factors in the descending radical series are $(S_{v_1}, S_{v_0} \oplus S_{v_2})$ (resp. $(S_{v_0} \oplus S_{v_2}, S_{v_1})$) and which is the unique such $\overline{\Lambda}$-module up to isomorphism.

(i) If Λ and $\overline{\Lambda}$ are as in Section 3.1, then a complete list of the $\overline{\Lambda}$-modules whose endomorphism ring is isomorphic to k, up to isomorphism, is given by

$$S_0, S_1, S_2, \begin{matrix}1\\0\end{matrix}, \begin{matrix}0\\1\end{matrix}, \begin{matrix}0\\2\end{matrix}, \begin{matrix}2\\0\end{matrix}, \begin{matrix}1\\0\\2\end{matrix}, \begin{matrix}2\\0\\1\end{matrix},$$

$$\begin{matrix}0\\1\\0\\2\end{matrix}, \begin{matrix}0\\2\\0\\1\end{matrix}, \begin{matrix}1\\0\\2\\0\end{matrix}, \begin{matrix}2\\0\\1\\0\end{matrix}, \begin{matrix} & 0 & \\ 1 & & 2 \end{matrix}, \begin{matrix} 1 & & 2 \\ & 0 & \end{matrix}.$$

Since $\mathrm{Ext}^1_\Lambda(S_0, S_i) \cong k \cong \mathrm{Ext}^1_\Lambda(S_i, S_0)$ for $i \in \{1, 2\}$, the inflations via π_Λ of the two-dimensional $\overline{\Lambda}$-modules in this list are isomorphic to M_β, M_γ, M_δ, M_η (as defined in Definition 4.2). Because $\mathrm{Ext}^1_\Lambda(M_\beta, S_2) \cong k \cong \mathrm{Ext}^1_\Lambda(M_\eta, S_1)$ and $\mathrm{Ext}^1_\Lambda(M_\gamma, S_2) \cong k \cong \mathrm{Ext}^1_\Lambda(S_1, M_\eta)$, the inflations via π_Λ of the three-dimensional $\overline{\Lambda}$-modules in this list are isomorphic to $M_{\delta\beta}$, $M_{\gamma\eta}$, $M_{\gamma\delta^{-1}}$, $M_{\beta^{-1}\eta}$. Since $\mathrm{Ext}^1_\Lambda(S_0, M_{\delta\beta}) \cong k \cong \mathrm{Ext}^1_\Lambda(S_0, M_{\gamma\eta})$ and $\mathrm{Ext}^1_\Lambda(M_{\delta\beta}, S_0) \cong k \cong \mathrm{Ext}^1_\Lambda(M_{\gamma\eta}, S_0)$, the inflations via π_Λ of the four-dimensional $\overline{\Lambda}$-modules in this list are isomorphic to $M_{\delta\beta\gamma}$, $M_{\gamma\eta\delta}$, $M_{\eta\delta\beta}$, $M_{\beta\gamma\eta}$.
Therefore, a complete list of the Λ-modules which are inflated via π_Λ from the above $\overline{\Lambda}$-modules, up to isomorphism, is given by

$$S_0, S_1, S_2, M_\beta, M_\gamma, M_\delta, M_\eta, M_{\delta\beta}, M_{\gamma\eta},$$

$$M_{\delta\beta\gamma}, M_{\gamma\eta\delta}, M_{\eta\delta\beta}, M_{\beta\gamma\eta}, M_{\gamma\delta^{-1}}, M_{\beta^{-1}\eta}.$$

(ii) If Λ and $\overline{\Lambda}$ are as in Section 3.2, then a complete list of the $\overline{\Lambda}$-modules whose endomorphism ring is isomorphic to k, up to isomorphism, is given by

$$S_0, S_1, S_2, \begin{smallmatrix}0\\1\end{smallmatrix}, \begin{smallmatrix}1\\0\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}0\\2\end{smallmatrix}, \begin{smallmatrix}2\\0\end{smallmatrix},$$

$$\begin{smallmatrix}&0&\\1&&2\end{smallmatrix}, \begin{smallmatrix}&1&\\0&&2\end{smallmatrix}, \begin{smallmatrix}&2&\\0&&1\end{smallmatrix}, \begin{smallmatrix}1&&2\\&0&\end{smallmatrix}, \begin{smallmatrix}2&&0\\&1&\end{smallmatrix}, \begin{smallmatrix}0&&1\\&2&\end{smallmatrix}.$$

Using similar Ext calculations as in (i), we see that a complete list of the Λ-modules which are inflated via π_Λ from these $\overline{\Lambda}$-modules, up to isomorphism, is given by

$$S_0, S_1, S_2, M_\beta, M_\gamma, M_\delta, M_\eta, M_\kappa, M_\lambda,$$

$$M_{\beta\kappa^{-1}}, M_{\gamma\delta^{-1}}, M_{\lambda\eta^{-1}}, M_{\gamma^{-1}\lambda}, M_{\beta^{-1}\eta}, M_{\kappa^{-1}\delta}.$$

(iii) If Λ and $\overline{\Lambda}$ are as in Section 3.3, a complete list of the $\overline{\Lambda}$-modules whose endomorphism ring is isomorphic to k, up to isomorphism, is given by

$$S_0, S_1, S_2, \begin{smallmatrix}1\\0\end{smallmatrix}, \begin{smallmatrix}0\\1\end{smallmatrix}, \begin{smallmatrix}0\\2\end{smallmatrix}, \begin{smallmatrix}2\\0\end{smallmatrix}, \begin{smallmatrix}1\\0\\2\end{smallmatrix}, \begin{smallmatrix}2\\0\\1\end{smallmatrix},$$

$$\begin{smallmatrix}0\\1\\0\\2\end{smallmatrix}, \begin{smallmatrix}0\\2\\0\\1\end{smallmatrix}, \begin{smallmatrix}1\\0\\2\\0\end{smallmatrix}, \begin{smallmatrix}2\\0\\1\\0\end{smallmatrix}, \begin{smallmatrix}&0&\\1&&2\end{smallmatrix}, \begin{smallmatrix}1&&2\\&0&\end{smallmatrix}.$$

Using similar Ext calculations as in (i), we see that a complete list of the Λ-modules which are inflated via π_Λ from these $\overline{\Lambda}$-modules, up to isomorphism, is given by

$$S_0, S_1, S_2, M_\beta, M_\gamma, M_\delta, M_\eta, M_{\delta\beta}, M_{\gamma\eta},$$

$$M_{\delta\beta\gamma}, M_{\gamma\eta\delta}, M_{\eta\delta\beta}, M_{\beta\gamma\eta}, M_{\gamma\delta^{-1}}, M_{\beta^{-1}\eta}.$$

5. Ordinary characters for blocks with generalized quaternion defect groups

Assume Hypothesis 3.1. In the notation of [26, §2], this means that we are in Case (aa) (see [26, Thm. 3.17]). Let W be the ring of infinite Witt vectors over k, and let F be the fraction field of W. For $2 \leqslant \ell \leqslant d$, let ζ_{2^ℓ} be a fixed primitive 2^ℓ-th root of unity in an algebraic closure of F. Let

$$\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \qquad \chi_{7,i}, \quad 1 \leqslant i \leqslant 2^{d-1} - 1,$$

be the ordinary irreducible characters of G belonging to B. Let σ be an element of order 2^d in D. By [26], there is a block b_σ of $kC_G(\sigma)$ with $b_\sigma^G = B$ which contains a unique 2-modular character $\varphi^{(\sigma)}$ such that the following is true. There is an ordering of $(1, 2, \ldots, 2^{d-1} - 1)$ such that for $1 \leqslant i \leqslant 2^{d-1} - 1$ and r odd,

$$\chi_{7,i}(\sigma^r) = \left(\zeta_{2^d}^{ri} + \zeta_{2^d}^{-ri}\right) \cdot \varphi^{(\sigma)}(1). \tag{5.1}$$

Note that W contains all roots of unity of order not divisible by 2. Hence by [26] and by [18], the characters $\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6$ correspond to simple FG-modules. On the other hand, the characters $\chi_{7,i}$, $i = 1, \ldots, 2^{d-1} - 1$, fall into $d - 1$ Galois orbits $\mathcal{O}_2, \ldots, \mathcal{O}_d$ under the action of $\mathrm{Gal}(F(\zeta_{2^d} + \zeta_{2^d}^{-1})/F)$. Namely for $2 \leqslant \ell \leqslant d$, $\mathcal{O}_\ell = \{\chi_{7,2^{d-\ell}(2u-1)} \mid 1 \leqslant u \leqslant 2^{\ell-2}\}$. The field generated by the character values of each $\xi_\ell \in \mathcal{O}_\ell$ over F is $F(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1})$. Hence by [18], each ξ_ℓ corresponds to an absolutely irreducible $F(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1})G$-module X_ℓ. By [21, Satz V.14.9], this implies that for $2 \leqslant \ell \leqslant d$, the Schur index of each $\xi_\ell \in \mathcal{O}_\ell$ over F is 1. Hence we obtain $d - 1$ non-isomorphic simple FG-modules $V_2, \ldots, V_d$ with characters $\rho_2, \ldots, \rho_d$ satisfying

$$\rho_\ell = \sum_{\xi_\ell \in \mathcal{O}_\ell} \xi_\ell = \sum_{u=1}^{2^{\ell-2}} \chi_{7,2^{d-\ell}(2u-1)} \quad \text{for } 2 \leqslant \ell \leqslant d. \tag{5.2}$$

By [21, Hilfssatz V.14.7], $\mathrm{End}_{FG}(V_\ell)$ is a commutative F-algebra isomorphic to the field generated over F by the character values of any $\xi_\ell \in \mathcal{O}_\ell$. This means

$$\mathrm{End}_{FG}(V_\ell) \cong F\left(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1}\right) \quad \text{for } 2 \leqslant \ell \leqslant d. \tag{5.3}$$

By [26], the characters $\chi_{7,i}$ have the same degree x for $1 \leqslant i \leqslant 2^{d-1} - 1$. The characters $\chi_1, \chi_2, \chi_3, \chi_4$ have height 0, χ_5, χ_6 have height $d - 1$, and $\chi_{7,i}$, $1 \leqslant i \leqslant 2^{d-1} - 1$, have height 1. Hence $x = 2^{a-(d+1)+1} x^*$ where $\#G = 2^a \cdot g^*$ and x^* and g^* are odd. Since the centralizer $C_G(\sigma)$ contains $\langle\sigma\rangle$, we have $\#C_G(\sigma) = 2^d \cdot 2^b \cdot m^*$ where $b \geqslant 0$ and m^* is odd. Suppose $\varphi^{(\sigma)}(1) = 2^c \cdot n^*$ where $c \geqslant 0$ and n^* is odd. Note that if ψ is an ordinary irreducible character of $C_G(\sigma)$ belonging to the block b_σ, then by [27, p. 61], $\psi(1)$ divides $(\#C_G(\sigma))/(\#\langle\sigma\rangle) = 2^b \cdot m^*$. Because $\psi(1) = s_\psi \cdot \varphi^{(\sigma)}(1)$ for some positive integer s_ψ, we have $c \leqslant b$.

Let C be the conjugacy class in G of σ, and let $t(C) \in WG$ be the class sum of C. We want to determine the action of $t(C)$ on V_ℓ for $2 \leqslant \ell \leqslant d$. For this, we identify $\mathrm{End}_{FG}(V_\ell) \cong F(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1})$ with $\mathrm{End}_{F(\zeta_{2^\ell}+\zeta_{2^\ell}^{-1})G}(X_\ell)$ for one particular absolutely irreducible $F(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1})G$-

constituent X_ℓ of V_ℓ with character ξ_ℓ. By (5.2), we can choose $\xi_\ell = \chi_{7,2^{d-\ell}}$. Then, under this identification, for $2 \leqslant \ell \leqslant d$, the action of $t(C)$ on V_ℓ is given as multiplication by

$$\frac{\#C}{\xi_\ell(1)} \cdot \xi_\ell(\sigma) = 2^{c-b} \frac{g^* \cdot n^*}{m^* \cdot x^*} \cdot \left(\zeta_{2^d}^{2^{d-\ell}} + \zeta_{2^d}^{-2^{d-\ell}}\right) \tag{5.4}$$

where, as shown above, $c \leqslant b$. Note that $\frac{g^* \cdot n^*}{m^* \cdot x^*}$ is a unit in W, since $g^* \cdot n^*$ and $m^* \cdot x^*$ are odd. Since $t(C) \in WG$, we must have $c \geqslant b$, i.e. $c = b$. Therefore, (5.4) implies that there exists a unit ω in W such that for $2 \leqslant \ell \leqslant d$, the action of $t(C)$ on V_ℓ is given as multiplication by

$$\omega \cdot \left(\zeta_{2^d}^{2^{d-\ell}} + \zeta_{2^d}^{-2^{d-\ell}}\right) \tag{5.5}$$

when we identify $\mathrm{End}_{FG}(V_\ell)$ with $\mathrm{End}_{F(\zeta_{2^\ell}+\zeta_{2^\ell}^{-1})G}(X_\ell)$ for an absolutely irreducible $F(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1})G$-constituent X_ℓ of V_ℓ whose character is $\chi_{7,2^{d-\ell}}$.

Definition 5.1. Use the notation introduced above.

(i) Define

$$p_{d+1}(t) = \prod_{\ell=2}^{d} \mathrm{min.pol.}_F\left(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1}\right),$$

and let $R' = W[[t]]/(p_{d+1}(t))$.

(ii) Let $Z = \langle\sigma\rangle$ be a cyclic group of order 2^d, and let $\tau : Z \to Z$ be the group automorphism sending σ to σ^{-1}. Then τ can be extended to a W-algebra automorphism of the group ring WZ which will again be denoted by τ. Let $T(\sigma^2) = 1 + \sigma^2 + \sigma^4 + \cdots + \sigma^{2^d-2}$, and define

$$S' = (WZ)^{\langle\tau\rangle}/\left(T\left(\sigma^2\right), \sigma T\left(\sigma^2\right)\right).$$

Remark 5.2. The minimal polynomial $\mathrm{min.pol.}_F(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1})$ for $\ell \geqslant 2$ is as follows:

$$\mathrm{min.pol.}_F\left(\zeta_{2^2} + \zeta_{2^2}^{-1}\right)(t) = t,$$
$$\mathrm{min.pol.}_F\left(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1}\right)(t) = \left(\mathrm{min.pol.}_F\left(\zeta_{2^{\ell-1}} + \zeta_{2^{\ell-1}}^{-1}\right)(t)\right)^2 - 2 \quad \text{for } \ell \geqslant 3.$$

The W-algebra R' from Definition 5.1 is a complete local commutative Noetherian ring with residue field k. Moreover,

$$F \otimes_W R' \cong \prod_{\ell=2}^{d} F\left(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1}\right) \quad \text{as } F\text{-algebras},$$
$$k \otimes_W R' \cong k[t]/\left(t^{2^{d-1}-1}\right) \quad \text{as } k\text{-algebras}.$$

Additionally, for any sequence $(r_\ell)_{\ell=2}^d$ of odd integers, R' is isomorphic to the W-subalgebra of

$$\prod_{\ell=2}^{d} W\left[\zeta_{2^\ell} + \zeta_{2^\ell}^{-1}\right]$$

generated by the element $(\zeta_{2^\ell}^{r_\ell} + \zeta_{2^\ell}^{-r_\ell})_{\ell=2}^d$.

Lemma 5.3. *Using the notation of Definition* 5.1*, there is a continuous W-algebra isomorphism $\rho : R' \to S'$ with $\rho(t) = \sigma + \sigma^{-1}$. In particular, R' is isomorphic to a subquotient algebra of the group ring over W of a generalized quaternion group of order 2^{d+1}.*

Proof. It follows from [4, Lemma 2.3.6] that $\rho : R' \to S'$ is a continuous W-algebra isomorphism. Hence the description of S' in Definition 5.1 shows that S' is isomorphic to a subquotient algebra of the group ring over W of a generalized quaternion group of order 2^{d+1}. □

Lemma 5.4. *Assume Hypothesis* 3.1 *and use the notation introduced above. Let U' be a WG-module which is free over W and whose F-character is*

$$\sum_{\ell=2}^{d} \rho_\ell = \sum_{i=1}^{2^{d-1}-1} \chi_{7,i}.$$

Then U' is an $R'G$-module which is free as an R'-module and $\mathrm{End}_{WG}(U') \cong R'$.

Proof. We first prove that R' is isomorphic to a W-subalgebra of $\mathrm{End}_{WG}(U')$. Let σ be the same element of order 2^d in D as in (5.1), and let $t(C)$ be the class sum of the conjugacy class C of σ in G. Since $t(C)$ lies in the center of WG, multiplication by $t(C)$ defines a WG-module endomorphism of U'. Since $\mathrm{End}_{WG}(U')$ can naturally be identified with a subring of $\mathrm{End}_{FG}(F \otimes_W U')$, $t(C)$ acts on U' as multiplication by a scalar λ_C in $F \otimes_W R' \cong \prod_{\ell=2}^d F(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1})$. Moreover, λ_C can be read off from the action of $t(C)$ on $F \otimes_W U' \cong \bigoplus_{\ell=2}^d V_\ell$. By (5.5), λ_C equals $\omega \cdot (\zeta_{2^\ell}^{r_\ell} + \zeta_{2^\ell}^{-r_\ell})_{\ell=2}^d$ for some unit $\omega \in W$ and some sequence $(r_\ell)_{\ell=2}^d$ of odd integers. Since R' is isomorphic to the W-subalgebra of $\prod_{\ell=2}^d W[\zeta_{2^\ell} + \zeta_{2^\ell}^{-1}]$ generated by $(\zeta_{2^\ell}^{r_\ell} + \zeta_{2^\ell}^{-r_\ell})_{\ell=2}^d$, this implies that R' is isomorphic to a W-subalgebra of $\mathrm{End}_{WG}(U')$. Hence U' is an $R'G$-module.

We next prove that U' is free as an R'-module. Since U' is finitely generated as a W-module, it is also finitely generated as an R'-module. Since R' is a local ring with maximal ideal $m_{R'}$, it follows by Nakayama's Lemma that any k-basis $\{\bar{b}_1, \ldots, \bar{b}_s\}$ of $U'/m_{R'}U'$ can be lifted to a set $\{b_1, \ldots, b_s\}$ of generators of U' over R'. Since $F \otimes_W U'$ is a free $(F \otimes_W R')$-module of rank s, it follows that $b_1, \ldots, b_s$ are linearly independent over R'. Because $\mathrm{End}_{FG}(F \otimes_W U') \cong F \otimes_W R'$, this then implies that $\mathrm{End}_{WG}(U') \cong R'$. □

6. Universal deformation rings

As in Section 5, let W be the ring of infinite Witt vectors over k and let F be the fraction field of W.

Theorem 6.1. *Assume Hypothesis* 3.1. *Suppose B is Morita equivalent to $\Lambda = kQ/I$ where Q and I are as in Section* 3.1 (*resp. Section* 3.2, *resp. Section* 3.3)*. Denote the three simple B-modules by T_0, T_1 and T_2, where T_i corresponds to S_i, for $i \in \{0,1,2\}$, under the Morita equivalence between B and Λ. Suppose M is an indecomposable B-module whose stable endomorphism ring is isomorphic to k and which is inflated from a $\overline{B}$-module. For $d \geqslant 2$, let $p_{d+1}(t) \in W[t]$ be as in Definition* 5.1.

- (i) *Suppose Λ is as in Section* 3.1 *and $d \geqslant 2$. If M is a uniserial module of length* 4, *then $R(G,M) \cong W[[t]]/(p_{d+1}(t))$. Otherwise $R(G,M) \cong W$.*
- (ii) *Suppose Λ is as in Section* 3.2 *and $d \geqslant 2$. If M is a uniserial module of length* 2 *with composition factors T_1, T_2, then $R(G,M) \cong W[[t]]/(p_{d+1}(t))$. Otherwise $R(G,M) \cong W$.*
- (iii) *Suppose Λ is as in Section* 3.3 *and $d \geqslant 3$. If M is isomorphic to T_1, then $R(G,M) \cong W[[t]]/(p_{d+1}(t))$. Otherwise $R(G,M) \cong W$.*

In all cases (i)–(iii)*, $R(G,M)$ is isomorphic to a subquotient ring of WD and a complete intersection.*

Proof. By Proposition 4.1, M is inflated from a $\overline{B}$-module whose endomorphism ring is isomorphic to k. Hence we can use the lists given in Remark 4.3 to describe the possibilities for M. Recall that we have shown in Remark 4.3 that all these M are uniquely determined by the factors in their descending radical series.

We prove part (i) of Theorem 6.1, the proofs of parts (ii) and (iii) being similar. Using the list in Remark 4.3(i) and the description of the projective indecomposable B-modules in Fig. 1, it follows for all $d \geqslant 2$ that $\mathrm{Ext}^1_{kG}(M,M) = 0$ if M is not a uniserial module of length 4. If M is uniserial of length 4, then $\mathrm{Ext}^1_{kG}(M,M) = 0$ if $d = 2$ and $\mathrm{Ext}^1_{kG}(M,M) \cong k$ if $d \geqslant 3$.

We first show that each M corresponding to a Λ-module in the list in Remark 4.3(i) has a lift over W. We see directly from the decomposition matrix in Fig. 2 that if M is simple then M has a lift over W. We divide the remaining M into two subsets $\mathcal{M}_1$ and $\mathcal{M}_2$ where

$$\mathcal{M}_1 = \left\{ \begin{matrix} T_1 \\ T_0 \end{matrix}, \begin{matrix} T_0 \\ T_1 \end{matrix}, \begin{matrix} T_0 \\ T_2 \end{matrix}, \begin{matrix} T_2 \\ T_0 \end{matrix}, \begin{matrix} T_1 \\ T_0 \\ T_2 \end{matrix}, \begin{matrix} T_2 \\ T_0 \\ T_1 \end{matrix}, \begin{matrix} T_1 \\ T_0 \\ T_2 \\ T_0 \end{matrix}, \begin{matrix} T_2 \\ T_0 \\ T_1 \\ T_0 \end{matrix}, \begin{matrix} & T_0 & \\ T_1 & & T_2 \end{matrix} \right\}$$

$$\text{and} \quad \mathcal{M}_2 = \left\{ \begin{matrix} T_0 \\ T_1 \\ T_0 \\ T_2 \end{matrix}, \begin{matrix} T_0 \\ T_2 \\ T_0 \\ T_1 \end{matrix}, \begin{matrix} T_1 & & T_2 \\ & T_0 & \end{matrix} \right\}.$$

For $M \in \mathcal{M}_1$, let P_M^W be a projective indecomposable WG-module such that $k \otimes_W P_M^W$ is a projective kG-module cover of M. Using the decomposition matrix in Fig. 2 and [15, Prop. (23.7)], it follows for $M \in \mathcal{M}_1$ that there is a W-pure WG-sublattice X_M of P_M^W such that $U_M = P_M^W / X_M$ defines a lift over W of a B-module N which has the same top and the same composition factors as M. Moreover, it follows from the description of the projective indecomposable B-modules in Fig. 1 that the factors in the descending radical series of N and M are the same. Hence it follows from Remark 4.3(i) that N has to be isomorphic to M.

For $M \in \mathcal{M}_2$, let Q_M^W be the projective indecomposable WG-module such that $k \otimes_W Q_M^W$ is a projective kG-module cover of $\Omega^{-1}(M)$. Using the decomposition matrix in Fig. 2, [15, Prop. (23.7)] and the description of the projective indecomposable B-modules in Fig. 1, it follows for $M \in \mathcal{M}_2$ that there is a W-pure WG-sublattice Y_M of Q_M^W such that $V_M = Q_M^W / Y_M$ defines a lift of $\Omega^{-1}(M)$ over W. Therefore, Y_M defines a lift of M over W.

Hence, every B-module M which is inflated from a $\overline{B}$-module whose endomorphism ring is isomorphic to k has a lift over W. This implies that $R(G, M) \cong W$ for all such M in case $d = 2$, and for all such M that are not uniserial of length 4 in case $d \geqslant 3$. In particular, this proves part (i) of Theorem 6.1 in case $d = 2$ since $p_3(t) = t$.

To finish the proof of part (i) in general, assume now that $d \geqslant 3$ and M is a uniserial module of length 4. Let $i \neq j$ in $\{1, 2\}$ and define

$$M_{i0j0} = \begin{matrix} T_i \\ T_0 \\ T_j \\ T_0 \end{matrix} \quad \text{and} \quad M_{0i0j} = \begin{matrix} T_0 \\ T_i \\ T_0 \\ T_j \end{matrix}.$$

Then $M \cong M_{i0j0}$ (resp. $M \cong M_{0i0j}$) for some choice of $i \neq j$ in $\{1, 2\}$, and $\mathrm{Ext}^1_{kG}(M, M) \cong k$. Considering the projective B-module cover P_{T_i} of T_i, it follows that P_{T_i} has a uniserial quotient module $\overline{U}_{i0j0}$ of length $4 \cdot (2^{d-1} - 1)$ whose composition factors in the descending radical series are given as

$$(T_i, T_0, T_j, T_0, T_i, T_0, T_j, T_0, \ldots, T_i, T_0, T_j, T_0)$$

such that there are kG-module isomorphisms

$$\phi_{i0j0} : \overline{U}_{i0j0}/\mathrm{rad}^4(\overline{U}_{i0j0}) \to M_{i0j0},$$
$$\psi_{i0j0} : \overline{U}_{i0j0}/\mathrm{rad}^{4(2^{d-1}-2)}(\overline{U}_{i0j0}) \to \mathrm{rad}^4(\overline{U}_{i0j0}),$$

and $\mathrm{Ext}^1_{kG}(\overline{U}_{i0j0}, M_{i0j0}) = 0$. Namely, the B-module $\overline{U}_{1020}$ (resp. $\overline{U}_{2010}$) corresponds to the Λ-module $M_{\eta\delta\beta(\gamma\eta\delta\beta)^{2^{d-1}-2}}$ (resp. $M_{\beta\gamma\eta(\delta\beta\gamma\eta)^{2^{d-1}-2}}$), as defined in Definition 4.2, under the Morita equivalence between B and Λ. Similarly, the projective B-module cover P_{T_0} of T_0 has a uniserial quotient module $\overline{U}_{0i0j}$ of length $4 \cdot (2^{d-1} - 1)$ whose composition factors in the descending radical series are given as

$$(T_0, T_i, T_0, T_j, T_0, T_i, T_0, T_j, \ldots, T_0, T_i, T_0, T_j)$$

such that there are kG-module isomorphisms

$$\phi_{0i0j} : \overline{U}_{0i0j}/\mathrm{rad}^4(\overline{U}_{0i0j}) \to M_{0i0j},$$
$$\psi_{0i0j} : \overline{U}_{0i0j}/\mathrm{rad}^{4(2^{d-1}-2)}(\overline{U}_{0i0j}) \to \mathrm{rad}^4(\overline{U}_{0i0j}),$$

and $\mathrm{Ext}^1_{kG}(\overline{U}_{0i0j}, M_{0i0j}) = 0$. Namely, the B-module $\overline{U}_{0102}$ (resp. $\overline{U}_{0201}$) corresponds to the Λ-module $M_{\delta\beta\gamma(\eta\delta\beta\gamma)^{2^{d-1}-2}}$ (resp. $M_{\gamma\eta\delta(\beta\gamma\eta\delta)^{2^{d-1}-2}}$) under the Morita equivalence between B and Λ. Hence it follows by Lemma 2.5 in case M is isomorphic to M_{i0j0} (resp. to M_{0i0j}) that

$\overline{R} = R(G,M)/2R(G,M) \cong k[t]/(t^{2^{d-1}-1})$. Moreover, the universal mod 2 deformation of M over $\overline{R}$ is $[\overline{U}_{i0j0}, \phi_{i0j0}]$ (resp. $[\overline{U}_{0i0j}, \phi_{0i0j}]$) where the action of t on $\overline{U}_{i0j0}$ (resp. $\overline{U}_{0i0j}$) is given by the kG-module endomorphism $\mu_{t,i0j0}$ (resp. $\mu_{t,0i0j}$) of $\overline{U}_{i0j0}$ (resp. $\overline{U}_{0i0j}$) which is induced by ψ_{i0j0} (resp. ψ_{0i0j}).

We now use Lemma 2.4 to show that $\overline{U}_{i0j0}$ (resp. $\overline{U}_{0i0j}$) has a lift over W. Consider the following submodule Z_{i0j0} (resp. quotient module Z_{0j0i}) of the projective indecomposable B-module P_{T_i}:

$$Z_{i0j0} = \Omega(\overline{U}_{i0j0}) = \begin{array}{cccc} & & & T_i \\ & & T_0 & \\ T_i & T_j & & \\ T_0 & & & \\ T_i & & & \end{array}, \qquad Z_{0j0i} = \Omega^{-1}(\overline{U}_{0j0i}) = \begin{array}{cccc} T_i & & & \\ T_0 & & & \\ T_i & T_j & & \\ & & T_0 & \\ & & & T_i \end{array}.$$

If $Y_{s0t} = \begin{array}{c} T_s \\ T_0 \\ T_t \end{array}$ for $s,t \in \{1,2\}$ then we have two non-split short exact sequences of B-modules

$$0 \to Y_{i0i} \to Z_{i0j0} \to Y_{i0j} \to 0$$

and

$$0 \to Y_{j0i} \to Z_{0j0i} \to Y_{i0i} \to 0$$

where $\mathrm{Ext}^1_{kG}(Y_{i0j}, Y_{i0i}) \cong k$ and $\mathrm{Ext}^1_{kG}(Y_{i0i}, Y_{j0i}) \cong k$. Moreover, it follows from the decomposition matrix in Fig. 2 and from the description of the projective indecomposable B-modules in Fig. 1 that Y_{s0t} has a lift (X_{s0t}, ξ_{s0t}) over W for all $s,t \in \{1,2\}$ such that the following holds. If $s \neq t$ then the F-character of X_{s0t} is χ_4, the F-character of X_{101} is $\chi_2 + \chi_5$, and the F-character of X_{202} is $\chi_3 + \chi_6$. Therefore, we have

$$\mathrm{Hom}_{FG}(F \otimes_W X_{i0j}, F \otimes_W X_{i0i}) = 0 = \mathrm{Hom}_{FG}(F \otimes_W X_{i0i}, F \otimes_W X_{j0i}).$$

Since

$$\mathrm{Hom}_{kG}(Y_{i0j}, Y_{i0i}) \cong k \cong \mathrm{Hom}_{kG}(Y_{i0i}, Y_{j0i}),$$

it follows from Lemma 2.4 that both Z_{i0j0} and Z_{0j0i} have a lift over W. Moreover, if $(i,j) = (1,2)$ then the F-character of these lifts is $\chi_2 + \chi_4 + \chi_5$, and if $(i,j) = (2,1)$ then the F-character of these lifts is $\chi_3 + \chi_4 + \chi_6$. Since $\overline{U}_{i0j0} \cong \Omega^{-1}(Z_{i0j0})$ (resp. $\overline{U}_{0i0j} \cong \Omega(Z_{0i0j})$), we obtain that $\overline{U}_{i0j0}$ (resp. $\overline{U}_{0i0j}$) also has a lift (U'_{i0j0}, ν'_{i0j0}) (resp. (U'_{0i0j}, ν'_{0i0j})) over W. Because of the F-characters of Z_{i0j0} and Z_{0i0j}, it follows that the F-character of U'_{i0j0} (resp. U'_{0i0j}) is

$$\sum_{\ell=2}^{d} \rho_\ell = \sum_{u=1}^{2^{d-1}-1} \chi_{7,u}.$$

If M is isomorphic to M_{i0j0} (resp. M_{0i0j}), let $\overline{U}$ be $\overline{U}_{i0j0}$ (resp. $\overline{U}_{0i0j}$), let ϕ be ϕ_{i0j0} (resp. ϕ_{0i0j}), let μ_t be $\mu_{t,i0j0}$ (resp. $\mu_{t,0i0j}$), let U' be U'_{i0j0} (resp. U'_{0i0j}) and let ν' be ν'_{i0j0} (resp.

ν'_{0i0j}). By Lemma 5.4, U' is an $R'G$-module which is free as an R'-module and $\mathrm{End}_{WG}(U') \cong R'$ where $R' = W[[t]]/(p_{d+1}(t))$. Let $\Psi'_t : U' \to U'$ be the WG-module endomorphism of U' which defines the action of $t \in R'$ on U' and let $\overline{\Psi'_t} : U'/2U' \to U'/2U'$ be the induced kG-module endomorphism of $U'/2U'$. Since $\nu' : U'/2U' \to \overline{U}$ is a kG-module isomorphism and $\mathrm{End}_{kG}(U'/2U') \cong R'/2R' \cong k[t]/(t^{2^{d-1}-1})$, it follows that the kG-module endomorphism $\psi'_t = \nu' \circ \overline{\Psi'_t} \circ (\nu')^{-1}$ of $\overline{U}$ satisfies $\psi'_t(\overline{U}) = \mu_t(\overline{U}) = t\overline{U}$. Thus we obtain a composition of kG-modules $k \otimes_{R'} U' = U'/(2,t)U' \xrightarrow{\overline{\nu'}} \overline{U}/t\overline{U} \xrightarrow{\phi} M$, where $(2,t)$ denotes the ideal of R' generated by 2 and t and $\overline{\nu'}$ is the kG-module isomorphism induced by ν'. If $\phi' = \phi \circ \overline{\nu'}$, then (U', ϕ') is a lift of M over $R' = W[[t]]/(p_{d+1}(t))$. We therefore have a continuous W-algebra homomorphism $\tau : R(G,M) \to R'$ relative to (U', ϕ'). Since $U'/2U' \cong \overline{U}$ is indecomposable as a kG-module, τ must be surjective. Hence τ induces a surjective k-algebra homomorphism $\overline{\tau} : R(G,M)/2R(G,M) \to R'/2R'$. Since $R(G,M)/2R(G,M) \cong R'/2R'$ are both finite dimensional over k, this implies that $\overline{\tau}$ is an isomorphism. Because R' is a free W-module of finite rank, it follows that τ is an isomorphism. By Lemma 5.3, R' is isomorphic to a subquotient ring of WD. This completes the proof of part (i), and hence of Theorem 6.1. □

Since the case $d \geqslant 4$ in family (III) was excluded in [4], we now determine the universal deformation rings $R(G/N, M)$ when $\mathrm{End}_{k[G/N]}(M) \cong k$ and M belongs to a block $\overline{B}$ as in Section 3.3 for $d \geqslant 4$. We prove a slightly more general result.

Lemma 6.2. *Let k be an algebraically closed field of characteristic 2, let H be a finite group and let $d \geqslant 4$ be an integer. Suppose B_H is a block of kH with a dihedral defect group D_H of order 2^d such that B_H is Morita equivalent to $\overline{\Lambda} = kQ/\overline{I}$ where Q and $\overline{I}$ are as in Section 3.3. Denote the three simple B_H-modules by T_0, T_1 and T_2, where T_i corresponds to the simple $\overline{\Lambda}$-module S_i, for $i \in \{0,1,2\}$, under the Morita equivalence. Suppose M is a kH-module belonging to B_H with $\mathrm{End}_{kH}(M) \cong k$.*

(i) *If M is not isomorphic to T_i for $i \in \{1,2\}$ and M is not uniserial of length 4, then $R(H,M) \cong W$.*
(ii) *If M is isomorphic to T_2 or M is uniserial of length 4, then $R(H,M) \cong k$.*
(iii) *If M is isomorphic to T_1, then $R(H,M) \cong W[[t]]/(tp_d(t), 2p_d(t))$, where $p_d(t) \in W[t]$ is as in Definition 5.1 (when $d+1$ is replaced by d).*

In all cases (i)–(iii), $R(H,M)$ is isomorphic to a subquotient ring of WD_H. If M is as in part (iii), then $R(H,M)$ is not a complete intersection.

Proof. Since B_H is Morita equivalent to $\overline{\Lambda}$ as in Section 3.3 and since the endomorphism ring of M is isomorphic to k, we can use the list in Remark 4.3(iii) to describe all possibilities for M. It follows that $\mathrm{Ext}^1_{kH}(M,M) = 0$ if M is not isomorphic to T_1 and $\mathrm{Ext}^1_{kH}(T_1,T_1) \cong k$. By [17, p. 295], the decomposition matrix of B_H has the form

$$\begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_{5,i} \end{array} \begin{array}{c} \begin{array}{ccc} \varphi_0 & \varphi_1 & \varphi_2 \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{array} \right] \end{array} \; 1 \leqslant i \leqslant 2^{d-2}-1. \tag{6.1}$$

Moreover, the description in [4, §3.4] of the ordinary characters belonging to blocks with dihedral defect groups and precisely three isomorphism classes of simple modules is in particular valid for B_H. It follows from this description that $\chi_1, \chi_2, \chi_3, \chi_4$ are the characters of simple FH-modules and that $\sum_{i=1}^{2^{d-2}-1} \chi_{5,i}$ is the character of a semisimple FH-module.

Using the decomposition matrix in (6.1) together with [15, Prop. (23.7)], it follows that the modules M in parts (i) and (iii) of Lemma 6.2 have a lift over W but not the modules M in part (ii). In particular, this implies that $R(H, M) \cong W$ for all M as in part (i).

If M is as in part (ii), then M belongs to the boundary of a 3-tube of the stable Auslander–Reiten quiver of B_H. Hence we can use the same arguments as in the proof of [4, Cor. 5.2.5] to see that $R(H, M) \cong k$ for all M as in part (ii).

Finally, let M be as in part (iii), i.e. $M \cong T_1$. Let $R = R(H, T_1)$ and $\overline{R} = R/2R$. Considering the projective B_H-module cover P_{T_1} of T_1, it follows that P_{T_1} has a uniserial quotient module $\overline{U}$ of length 2^{d-2} whose composition factors are all isomorphic to T_1 such that $\mathrm{Ext}^1_{kH}(\overline{U}, T_1) = 0$ and such that there are kH-module isomorphisms $\phi : \overline{U}/\mathrm{rad}(\overline{U}) \to T_1$, $\phi' : \mathrm{rad}(\overline{U})/\mathrm{rad}^2(\overline{U}) \to T_1$ and $\psi : \overline{U}/\mathrm{rad}^{2^{d-2}-1}(\overline{U}) \to \mathrm{rad}(\overline{U})$. Hence it follows by Lemma 2.5 that $\overline{R} \cong k[t]/(t^{2^{d-2}})$ and that the universal mod 2 deformation of T_1 over $\overline{R}$ is $[\overline{U}, \phi]$, where the action of t on $\overline{U}$ is given by the kH-module endomorphism μ_t of $\overline{U}$ which is induced by ψ. Let $\overline{U'} = \mathrm{rad}(\overline{U})$. Then $(\overline{U'}, \phi')$ is a lift of T_1 over $k[t]/(t^{2^{d-2}-1})$, where the action of t on $\overline{U'}$ is given by the kH-module endomorphism μ'_t of $\overline{U'}$ which is the restriction of μ_t to $\overline{U'}$. Using the decomposition matrix in (6.1) together with [15, Prop. (23.7)], we see that $\overline{U'}$ has a lift (U', ξ') over W such that the F-character of U' is equal to $\sum_{i=1}^{2^{d-2}-1} \chi_{5,i}$. Using similar arguments as in the proof of [4, Thm. 5.1] and employing the description of the ordinary characters belonging to B_H in [4, §3.4], it follows that U' defines a lift (U', ν') of T_1 over $W[[t]]/(p_d(t))$, where $p_d(t)$ is as in Definition 5.1 (when $d+1$ is replaced by d). We therefore have a continuous W-algebra homomorphism $\tau' : R = R(H, T_1) \to W[[t]]/(p_d(t))$ relative to (U', ν'). Since $k \otimes_W U' \cong \overline{U'}$ is indecomposable as a kH-module, τ' must be surjective. By [4, Lemma 2.3.3], it follows that $R \cong W[[t]]/(p_d(t)(t-2c), a2^m p_d(t))$ for certain $c \in W$, $a \in \{0, 1\}$ and $0 < m \in \mathbb{Z}$. Let $[U, \nu]$ be the universal deformation of T_1 over R. Since U is free over R and since we can identify $k \otimes_W R = R/2R = \overline{R}$, it follows that we can also identify $k \otimes_W U = U/2U = \overline{R} \otimes_R U$ as $\overline{R}H$-modules. Hence $[U/2U, \nu]$ is equal to the universal mod 2 deformation $[\overline{U}, \phi]$ of T_1 over $\overline{R}$. In particular, there is a kH-module isomorphism $\xi : U/2U \to \overline{U}$. If $a = 0$, then $R \cong W[[t]]/(p_d(t)(t-2c))$ is free over W, which implies that (U, ξ) is a lift of $\overline{U}$ over W. If $a = 1$, then $(W/2^m W) \otimes_W R \cong (W/2^m W)[[t]]/(p_d(t)(t-2c))$ is free over $W/2^m W$, which implies that $((W/2^m W) \otimes_W U, \xi)$ is a lift of $\overline{U}$ over $W/2^m W$. Since $\overline{U}$ lies in the Ω-orbit of a uniserial B_H-module of length 4, it follows that $R(H, \overline{U}) \cong k$ by what we have proved above for the modules in part (ii). Hence $a = 1$ and $m = 1$, which implies $R = R(H, T_1) \cong W[[t]]/(t p_d(t), 2 p_d(t))$. □

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