Toeplitz operators with unbounded symbols of several complex variables

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Abstract

In this note we construct a function \( \varphi \) in \( L^2(\mathbb{B}_n, dA) \) which is unbounded on any neighborhood of each boundary point of \( \mathbb{B}_n \) such that \( T_\varphi \) is a trace class operator on Bergman space \( L^2_a(\mathbb{B}_n) \) for several complex variables. In addition, we also discuss the compactness of Toeplitz operators with \( L^1 \) symbols.

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1. Introduction

Let \( \mathbb{B}_n \) denote the open unit ball of the complex space \( \mathbb{C}^n \) and \( \partial \mathbb{B}_n \) its boundary. The Bergman space \( L^2_a(\mathbb{B}_n) \), consists of holomorphic functions \( f \) on \( \mathbb{B}_n \) for which the norm

\[
\|f\|_2 = \left( \int_{\mathbb{B}_n} |f(z)|^2 \, dA(z) \right)^{\frac{1}{2}}
\]

is finite. Where \( dA \) denotes the normalized Lebesgue volume measure on \( \mathbb{B}_n \). For \( \varphi \in L^2(\mathbb{B}_n, dA) \), where \( L^2(\mathbb{B}_n, dA) \) is the space of square-integrable functions with respect to \( dA \), we let \( T_\varphi \) denote the Toeplitz operator with symbol \( f \) on \( L^2_a \) defined by

\[
T_\varphi f = P(\varphi f),
\]

where \( P \) is the orthogonal projection from \( L^2(\mathbb{B}_n, dA) \) onto \( L^2_a \). In general, if \( \varphi \notin L^\infty(\mathbb{B}_n) \), the space of essentially bounded functions on \( \mathbb{B}_n \), then \( T_\varphi \) is densely defined only. In the case of Hardy space, it is well known that \( T_\varphi \) is bounded if and only if \( \varphi \) is essentially bounded, and \( T_\varphi \) is compact if and only if \( \varphi = 0 \) (cf. see Douglas [1], Jewell [2]). However, there are indeed bounded and compact Toeplitz operators with unbounded symbols on Bergman...
space of one complex variable, in fact, Miao and Zheng [3] have introduced a class of functions, called BT, which contains \( L^\infty \), for \( \varphi \in BT \). \( T_\varphi \) is compact on \( L^2_a \) if and only if the Berezin transform of \( \varphi \) vanishes on the unit circle. Zorboska [4] has proved that if \( \varphi \) belongs to the hyperbolic BMO space, the \( T_\varphi \) is compact if and only if the Berezin transform of \( \varphi \) vanishes on the unit circle. Cima and Cuckovic [5] construct a class of unbounded functions build over Cantor set, the Toeplitz operators with these functions are compact. Essentially, if the values of the function \( \varphi \) vanishes rapidly near the unit circle in the sense of measure \( dA \), then \( T_\varphi \) will be compact.

The analogous results are true on the Bergman space of several complex variables. In this paper, we construct a class of unbounded functions on \( \mathbb{B}_n \). The Toeplitz operators with these symbols are compact, our construction does not rely on properties of Cantor set. For each countable dense subset in the boundary of \( \mathbb{B}_n \) we can construct functions \( \varphi \) which are unbounded on any neighborhood of each boundary point of \( \mathbb{B}_n \), such that \( T_\varphi \) is compact. We also construct a function \( \varphi \) which is unbounded on any neighborhood of each boundary point of \( \mathbb{B}_n \), such that \( T_\varphi \) is a trace class operator.

2. Trace class Toeplitz operators with unbounded symbols

If \( z = (z_1, \ldots, z_n) \), \( w = (w_1, \ldots, w_n) \in \mathbb{C}^n \), where \( \mathbb{C}^n \) denotes the \( n \)-dimensional complex space, then the inner product of \( z \) and \( w \) is defined as

\[
\langle z, w \rangle = z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n.
\]

It is clearly that \( \langle z, z \rangle = |z|^2 = \sum_{j=1}^{n} |z_j|^2 \). For \( \delta > 0 \), \( \xi \in \partial \mathbb{B}_n \), write

\[
\Omega(\xi, \delta) = \left\{ z \in \mathbb{B}_n \left| \frac{(1 - |\xi|)^2}{2} \cdot |z - \xi| < |\text{Re}(\xi, (z - \xi))|, \right. \text{Re}(z, \xi) > 0 \right\},
\]

then \( \Omega(\xi, \delta) \) is an open subset of \( \mathbb{B}_n \), the domain is said to be a circular-like cone with vertex \( \xi \). For any \( 0 < r < 1 \), let \( B_n(r) = \{ z \in \mathbb{C}^n \left| |z| < r \right. \} \) be the ball with radius \( r \), \( \partial B_n(r) \) its boundary. We denote the area measure on the sphere by \( d\sigma_r \). It is clear that \( \sigma_r(\partial B_n(r)) = O(r^{2n-1}) \). For \( r = 1 \), write \( d\sigma = d\sigma_1 \), the normalized area measure on the unit sphere \( \partial B_n(1) = \partial \mathbb{B}_n \). Assume \( b \) is an arbitrary positive number, it is obvious that we may choose a suitable \( \delta = \delta(b) > 0 \) such that for arbitrary \( 0 < r < 1 \), and \( \xi \in \partial \mathbb{B}_n \),

\[
\sigma_r(\Omega(\xi, \delta) \cap \partial B_n(r)) < d(1 - r^2)^b,
\]

where \( d \) is a constant which is independent of \( \xi \) and \( r \). Write \( \Omega_b(\xi) = \Omega(\xi, \delta(b)) \) for simplicity.

**Proposition 1.** Suppose \( c > 0 \), \( U_c(z) = (1 - |z|^2)^{-c} \), \( z \in \mathbb{B}_n \), for arbitrary \( \xi \in \mathbb{B}_n \), let \( b \geq n + 2c + 2 \), write the characteristic function of the set \( \Omega_b(\xi) \) as characteristic function \( \chi_{\Omega_b(\xi)}(z) \). Then \( \varphi(z) = \chi_{\Omega_b(\xi)}(z) \cdot U_c(z) \) introduces a compact Toeplitz operator on \( L^2_a(\mathbb{B}_n, dA) \).

**Proof.** Suppose \( \{ f_k \} \subset L^2_a \) with \( \| f_k \| \leq 1 \) is a sequence which converges to zero, it is enough to prove that

\[
\| T_\varphi f_k \| \to 0.
\]

Note

\[
T_\varphi f_k(z) = \int_{\mathbb{B}_n} \frac{\varphi(w) f_k(w)}{(1 - \langle z, w \rangle)^{n+1}} dA(w),
\]

we see that

\[
\| T_\varphi f_k \|^2 = \int_{\mathbb{B}_n} \left| \int_{\mathbb{B}_n} \frac{\varphi(w) f_k(w)}{(1 - \langle z, w \rangle)^{n+1}} dA(w) \right|^2 dA(z)
\]

\[
= \int_{\mathbb{B}_n} \left| \int_{\Omega_b(\xi)} \frac{f_k(w)(1 - |w|^2)^{-c}}{(1 - \langle z, w \rangle)^{n+1}} dA(w) \right|^2 dA(z).
\]

For \( m \in (0, 1) \), set \( \Omega_b(\xi, m) = \{ z \in \Omega_b(\xi) \left| |z| > m \right. \} \), then
\[
\int \left| \int_{\Omega_b(\xi)} f_k(w)(1 - |w|^2)^{-c} dA(w) \right|^2 dA(z)
= \int \left( \int_{\Omega_b(\xi)} \frac{f_k(w)(1 - |w|^2)^{-c}}{(1 - \langle z, w \rangle)^{n+1}} dA(w) + \int_{\Omega_b(\xi) - \Omega_b(\xi,m)} \frac{f_k(w)(1 - |w|^2)^{-c}}{(1 - \langle z, w \rangle)^{n+1}} dA(w) \right) \left| \int_{\Omega_b(\xi,m)} \frac{f_k(w)(1 - |w|^2)^{-c}}{(1 - \langle z, w \rangle)^{n+1}} dA(w) \right|^2 dA(z)
\leq 2 \left( \int_{\Omega_b(\xi,m)} \frac{f_k(w)(1 - |w|^2)^{-c}}{(1 - \langle z, w \rangle)^{n+1}} dA(w) \right)^2 + \left| \int_{\Omega_b(\xi) - \Omega_b(\xi,m)} \frac{f_k(w)(1 - |w|^2)^{-c}}{(1 - \langle z, w \rangle)^{2n+1}} dA(w) \right|^2 dA(z) \right].
\]

Note
\[
\left| \int_{\Omega_b(\xi,m)} \frac{f_k(w)(1 - |w|^2)^{-c}}{(1 - \langle z, w \rangle)^{n+1}} dA(w) \right|^2 \leq \int_{\Omega_b(\xi,m)} |f_k|^2 dA(w) \cdot \int_{\Omega_b(\xi,m)} \frac{(1 - |w|^2)^{-2c}}{|1 - \langle z, w \rangle|^{2n+2}} dA(w)
\leq \int_{\Omega_b(\xi,m)} \frac{(1 - |w|^2)^{-2c}}{|1 - \langle z, w \rangle|^{2n+2}} dA(w).
\]

Thus
\[
\int \left( \int_{\Omega_b(\xi,m)} \frac{f_k(w)(1 - |w|^2)^{-c}}{(1 - \langle z, w \rangle)^{n+1}} dA(w) \right)^2 dA(z) \leq \int \int_{\Omega_b(\xi,m)} (1 - |w|^2)^{-2c} dA(w) dA(z)
= \int_{\Omega_b(\xi,m)} (1 - |w|^2)^{-2c} \int \frac{1}{|1 - \langle z, w \rangle|^{2n+2}} dA(z) dA(w)
= \int_{\Omega_b(\xi,m)} \frac{1}{|1 - |w|^2|^{n+2c+1}} dA(w)
= 2n \int_0^{2n-1} r^{2n-1} \int_{\partial \Omega_b(\xi,m)} \chi_{\Omega_b(\xi,m)}(r\eta) \frac{1}{(1 - r^2)^{n+2c+1}} d\sigma(\eta)
= c_0 \int_m^1 dr \int_{\Omega_b(\xi,m) \cap \partial \Omega_n(r)} \frac{1}{(1 - r^2)^{n+2c+1}} d\sigma_r
\leq c_1 \int_m^1 \frac{1}{(1 - r^2)^{n+2c+1}} \cdot (1 - r^2)^b dr^2
= c_1 \frac{(1 - m^2)^{b-n-2c}}{m^{b-n-2c}} \leq \frac{c_1}{m} (1 - m^2)^2,
\]
where \(c_0, c_1\) are constants. It is clear that for arbitrary \(\varepsilon > 0\), there is \(m_0 \in (0, 1)\) such that
\[
\int \left( \int_{\Omega_b(\xi,m)} \frac{f_k(w)(1 - |w|^2)^{-c}}{(1 - \langle z, w \rangle)^{n+1}} dA(w) \right)^2 dA(z) \leq \frac{c_1}{m} (1 - m^2)^2 < \varepsilon
\]
for \(m \in [m_0, 1)\).
On the other hand, since $\Omega_b(\xi) - \Omega_b(\xi, m_0) \subset B_n(m_0)$, we see that $f_k(w) \to 0$ uniformly on $\Omega_b(\xi) - \Omega_b(\xi, m_0)$. Hence for any $\varepsilon > 0$, there is $K_0$, such that for $k > K_0$, we have $|f_k(w)| < \varepsilon$ for arbitrary $w \in \Omega_b(\xi) - \Omega_b(\xi, m_0)$.

Consequently,

$$\left|\int_{\Omega_b(\xi) - \Omega_b(\xi, m)} f_k(w)(1 - |w|^2)^{-c} \frac{1}{(1 - \langle z, w \rangle)^{n+1}} dA(w)\right| \leq \varepsilon \int_{\Omega_b(\xi) - \Omega_b(\xi, m)} \frac{1}{(1 - |w|^2)^{n+c+1}} dA(w) \leq \frac{\varepsilon}{(1 - m_0^n)^{n+c+1}}.$$

This follows that

$$\|T\phi f_k\| \to 0.$$  \hfill $\square$

**Proposition 2.** There is a function $\phi$ in $L^2(B_n, dA)$ which is unbounded on any neighborhood of each boundary point of $B_n$ (i.e. for arbitrary $\xi \in \partial B_n$, and $r > 0$,

$$\text{ess sup}_{z \in B_n \cap B(\xi, r)} |\phi(z)| = \infty,$$

where $B(\xi, r) = \{z \mid |z - \xi| < r\}$), such that $T\phi$ is a compact operator on $L^2_a$.

**Proof.** Assume $c > 0, b \geq 2n + 2c + 4$, and $U_c(z)$ is the function in Proposition 1, choose a countable dense subset $\{\xi_j\}$ of $\partial B_n$. For each $\xi_j$, write $\phi_j = \chi_{\Omega_b(\xi_j)} U_c$, then $T\phi_j$ is a compact operator by Proposition 1 since $b > n + 2c + 2$. For any $f \in L^2_a$,

$$\|T\phi_j f\|^2 = \int_{B_n} \left|\int_{B_n} \frac{\phi_j(w) f(w)}{(1 - \langle z, w \rangle)^{n+1}} dA(w)\right|^2 dA(z)
\leq \|f\|^2 \int_{B_n} \left|\int_{\Omega_b(\xi_j)} \frac{f(w)(1 - |w|^2)^{-c}}{(1 - \langle z, w \rangle)^{n+1}} dA(w)\right|^2 dA(z)
\leq \|f\|^2 \int_{\Omega_b(\xi_j)} \left(\frac{1 - |w|^2)^{-2c}}{(1 - \langle z, w \rangle)^{2n+2}} dA(w) dA(z)
\leq c \|f\|^2 \int_{\Omega_b(\xi_j)} \frac{1}{(1 - |w|^2)^{2n+2c+2}} dA(w)
\leq c d \|f\|^2 \int_0^1 \frac{1}{(1 - r^2)^{2n+2c+2}} dr \leq c d \|f\|^2,$$

where $c$ is a constant. Hence $\|T\phi_j\| \leq c d$. Set $T_N = \sum_{j=1}^N \frac{1}{2^j} T\phi_j$, then $T_N$ is compact, and for any $M, N$ and $f \in L^2_a$,

$$\left\|\sum_{j=N}^{M} \frac{1}{2^j} T\phi_j f\right\| \leq c d \|f\| \sum_{j=N}^{M} \frac{1}{2^j}.$$

This shows that

$$\left\|\sum_{j=N}^{M} \frac{1}{2^j} T\phi_j\right\| \leq c d \sum_{j=N}^{M} \frac{1}{2^j},$$
hence $T = \sum_{j=1}^{\infty} \frac{1}{2^j} T_{\varphi_j}$ converges in norm. Further, $T$ is a compact operator. It is easy to check that $\varphi_j \in L^2(\mathbb{B}_n, dA)$ and $\|\varphi_j\|_2 \leq 1$. Thus $\sum_{j=1}^{\infty} \frac{1}{2^j} \varphi_j$ converges to an $L^2$-function $\varphi$. It is not difficult to see that for each polynomial $p$,

$$\| (T_{\varphi} - T_N) p \| = \| T_N \sum_{j=N+1}^{\infty} \frac{1}{2^j} \varphi_j p \| \lesssim cd \| p \| \sum_{j=N+1}^{\infty} \frac{1}{2^j} \to 0.$$ 

Thus $T = T_{\varphi}$, that is, $T$ is a Toeplitz operator with symbol $\varphi = \sum_{j=1}^{\infty} \frac{1}{2^j} \varphi_j$. Since $\{\xi_j\}$ is dense in $\partial \mathbb{B}_n$, we have $\text{ess sup}_{\xi \in \partial \mathbb{B}_n} \varphi(\xi) = \infty$. □

The proposition follows from this.

**Theorem 3.** There is a function $\varphi \in L^2(\mathbb{B}_n, dA)$ which is unbounded on any neighborhood of each boundary point of $\mathbb{B}_n$ (i.e. for arbitrary $\xi \in \partial \mathbb{B}_n$, and $r > 0$,

$$\text{ess sup}_{z \in \mathbb{B}_n \cap B(\xi, r)} |\varphi(z)| = \infty,$$

such that $T_{\varphi}$ is a trace class operator.

**Proof.** For every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, write $|\alpha| = \sum_{j=1}^{n} \alpha_j$, $\alpha! = \alpha_1! \cdots \alpha_n!$, then for $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, we have $\|z^\alpha\|_2^2 = \frac{|\alpha|!}{(n |\alpha| + n)!}$ (see W. Rudin [6]). Thus $\{e_{\alpha}\} = \{z^\alpha / \|z^\alpha\|_2\} = \left\{\sqrt{\frac{1}{(n |\alpha| + n)!}} z^\alpha\right\}$ forms the orthogonal base in $L^2_{\alpha}(\mathbb{B}_n)$. Let $\{\xi_j\}$ be a countable dense subset of $\partial \mathbb{B}_n$, and $U_c = (1 - |z|^2)^{-c}$ ($c > 0$), for any $m \in (0, 1)$, set

$$\varphi_j(z) = \chi_{\Omega_b(\xi_j, m)} U_c(z),$$

where $b \geq 2n + 2c + 3$. Then

$$\left| \left( T_{\varphi} e_\alpha, e_\alpha \right) \right| = \frac{(|\alpha| + n)!}{n! \alpha!} \left| \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \varphi_j(w) w^\alpha \overline{z}^\alpha (1 - \langle z, w \rangle)^{n+1} dA(z) dA(w) \right|$$

$$= \frac{(|\alpha| + n)!}{n! \alpha!} \left| \int_{\mathbb{B}_n} \varphi_j(w) w^\alpha \overline{w}^\alpha dA(w) \right|$$

$$= \frac{(|\alpha| + n)!}{n! \alpha!} \left| \int_{\Omega_b(\xi_j, m)} \left(1 - |w|^2\right)^{-c} w^\alpha \overline{w}^\alpha dA(w) \right|$$

$$= c \frac{(|\alpha| + n)!}{n! \alpha!} \left| \int_{m \cap \Omega_b(\xi_j, m)} \left(1 - |w|^2\right)^{-c} r^{2|\alpha|} d\sigma_r dr \right|$$

$$\leq c \frac{(|\alpha| + n)!}{n! \alpha!} \left| \int_{m \cap \Omega_b(\xi_j, m)} \left(1 - r^2\right)^{-c} r^{2|\alpha|} d\sigma_r dr \right|$$

$$\leq cd \frac{(|\alpha| + n)!}{n! \alpha!} \left| \int_{m \cap \Omega_b(\xi_j, m)} \left(1 - r^2\right)^{-c} (1 - r^2)^b r^{2|\alpha|} dr \right|$$

$$\leq cd \frac{(|\alpha| + n)!}{n! \alpha!} \left| \int_{m \cap \Omega_b(\xi_j, m)} \left(1 - r^2\right)^{2n+3} r^{2|\alpha|} dr \right|.$$
where \( c \) is a constant independent on \( \alpha \). Using the integration by parts, we may prove that

\[
\int_{m}^{1} (1 - r^2)^{2n+3} r^{2|\alpha|} \, dr \leq \frac{d(n)}{(2|\alpha| + 1)^{2n+4}},
\]

where \( d(n) \) is a constant which is relative to \( n \). Let \( T = \sum_{j=1}^{\infty} \frac{1}{2j} T_{\psi_j} \), then \( T \) is compact by Proposition 2. Note \( T \) is clearly positive, thus

\[
\sum_{\alpha} |\langle T e_{\alpha}, e_{\alpha} \rangle| = \sum_{\alpha} \langle T e_{\alpha}, e_{\alpha} \rangle
\]

\[
= \sum_{\alpha} \sum_{j=1}^{\infty} \frac{1}{2j} \frac{(|\alpha| + n)!}{n! \alpha!} \int_{B_n} \frac{\psi_j(w) w^{\alpha} z^{\alpha}}{(1 - \langle z, w \rangle)^{n+1}} \, dA(z) \, dA(w)
\]

\[
\leq \sum_{\alpha} \sum_{j=1}^{\infty} \frac{1}{2j} \frac{(|\alpha| + n)!}{n! \alpha!} \frac{d(n) cd}{(2|\alpha| + 1)^{2n+4}}
\]

\[
= \sum_{j=1}^{\infty} \frac{1}{2j} \sum_{\alpha} \frac{(|\alpha| + n)!}{n! \alpha!} \frac{d(n) cd}{(2|\alpha| + 1)^{2n+4}}
\]

\[
= \sum_{k=0}^{\infty} \sum_{|\alpha| = k} \frac{(k + n)!}{n! \alpha!} \frac{d(n) cd}{(2k + 1)^{2n+4}}
\]

\[
= \sum_{k=0}^{\infty} \left( \sum_{|\alpha| = k} \frac{(k + n)!}{n! \alpha!} \right) \frac{d(n) cd}{(2k + 1)^{2n+4}}.
\]

Note for any continuous function \( f \) on \( B_n \), we have

\[
\int_{B_n} f(z) \, dA(z) = \int_{0}^{2\pi} \int_{0}^{\pi} \left( \int_{B_{n-1}(\sqrt{1-r^2})} f(z', re^{i\theta}) \, dA(z') \right) r \, dr \, d\theta,
\]

where \( z' = (z_1, \ldots, z_{n-1}) \), \( z_n = re^{i\theta} \). Thus, by the inductive method, we see easily that

\[
\sum_{|\alpha| = k} \frac{(k + n)!}{n! \alpha!} = O((k + n)^n).
\]

Further, there are constants \( M_1, M_2 > 0 \) such that

\[
\sum_{\alpha} |\langle T e_{\alpha}, e_{\alpha} \rangle| \leq \sum_{k=0}^{\infty} \left( \sum_{|\alpha| = k} \frac{(k + n)!}{n! \alpha!} \right) \frac{d(n) cd}{(2k + 1)^{2n+4}}
\]

\[
\leq M_1 \sum_{k=0}^{\infty} \frac{(k + n)^n}{(2k + 1)^{2n+4}} \leq M_2 \sum_{k=0}^{\infty} \frac{1}{(2k + 1)^{n+4}} < +\infty.
\]

That is, \( T \) is a trace class operator. \( \square \)

3. **Toeplitz operators with \( L^1 \) symbols**

For \( S \) a bounded operator on \( L^p_\alpha \) \((1 < p < \infty)\), the Berezin transform of \( S \) is the function on \( B_n \) defined by
\[ \tilde{S}(z) = \langle SK_z, K_z \rangle, \]

where \( \langle u, v \rangle = \int_{\mathbb{B}_n} u \bar{v} \, dA(w) \) whenever \( u \bar{v} \in L^1(\mathbb{B}_n, dA) \). Let \( \tilde{T}_\varphi \) denote and let

\[ BT = \left\{ \varphi \in L^1(\mathbb{B}_n, dA) \mid \| \varphi \|_{BT} = \sup_{z \in B_n} |\tilde{\varphi}(z)| < \infty \right\}. \]

The BT function is introduced by J. Miao and D. Zheng [3] in the case of unit disk of complex plane, they proved that if \( S \) is a finite sum of operators \( T_{\varphi_1} \cdots T_{\varphi_n} \), where each \( \varphi_j \in BT \), then \( S \) is compact on \( L^p_\alpha(1 < p < \infty) \) if and only if \( \tilde{S}(z) \to 0 \) as \( z \to \partial D \). In particular, if \( \varphi \in BT \), then \( T_\varphi \) is compact if and only if \( \tilde{\varphi}(z) \to 0 \) as \( z \to \partial D \). It is not difficult to see that the examples constructed by J. Cima and Z. Cuckvic [5] are BT functions. The key of Miao and Zheng’s proofs is the following lemma.

**Lemma 4.** Suppose \( d\mu \) is a positive Borel measure on \( D \) and \( 1 \leq p < \infty \). Then the following four quantities are equivalent:

(a) \( \sup \{ \int_D |f|^p \, d\mu / \int_D |f|^p : f \in L^p_\alpha \} \);
(b) \( \sup \{ \mu(D(z))/A(D(z)) : z \in D \} \), where \( D(z) = \{ w \in D : \beta(z, w) < 1/2 \} \);
(c) \( \sup \{ \mu(S(\zeta, r))/A(S(\zeta, r)) : \zeta \in \partial D, r \in [0, 1] \} \), where

\[ S(\zeta, r) = \left\{ z \in D : r < |z| < 1, \arg \frac{1 - r}{2} < \arg z < \arg \zeta + \frac{1 - r}{2} \right\} \]

is the Carleson square;
(d) \( \sup \{ \tilde{\mu}(z) : z \in D \} \).

Furthermore, the constants of equivalence depend only on \( p \).

Let \( \mu \) be a finite positive Borel measure on \( \mathbb{B}_n \) and \( p \geq 1 \). If there exists a finite constant \( C > 0 \) such that

\[ \int_{\mathbb{B}_n} |f|^p \, d\mu \leq C \int_{\mathbb{B}_n} |f|^p \, dA \]

for all \( f \in L^p_\alpha \), then \( \mu \) is said to be a Carleson measure on Bergman space \( L^p_\alpha \). For \( n \geq 1 \), we have following

**Theorem 5.** Suppose \( \varphi \in L^1(\mathbb{B}_n, dA) \) is positive, then \( T_\varphi \) is bounded on \( L^2_\alpha(\mathbb{B}_n, dA) \) if and only if \( d\mu = \varphi \, dA \) is a Carleson measure.

**Proof.** If \( T_\varphi \) is bounded, then for any \( f \in L^2_\alpha \),

\[ \| T_\varphi f \| \leq \| T_\varphi \| \left[ \int_{\mathbb{B}_n} |f|^2 \, dA \right]^{1/2}. \]

Thus

\[ \langle T_\varphi f, f \rangle \leq \| T_\varphi f \| \| f \| \leq \| T_\varphi \| \| f \|^2. \]

That is

\[ \int_{\mathbb{B}_n} |f|^2 \varphi \, dA \leq \| T_\varphi \| \int_{\mathbb{B}_n} |f|^2 \, dA. \]

This follows that \( \varphi \, dA \) is a Carleson measure.

Conversely, if \( \varphi \, dA \) is a Carleson measure, then there is \( M > 0 \), such that

\[ \int_{\mathbb{B}_n} |f|^2 \varphi \, dA \leq M \int_{\mathbb{B}_n} |f|^2 \, dA \quad \text{for arbitrary } f \in L^2_\alpha. \]
Thus
\[ \left| \langle T_\phi f, f \rangle \right| \leq M \int_{\mathbb{B}_n} |f|^2 dA. \]

Note \( T_\phi \) is positive. We see that
\[ \| T_\phi \| = \omega(T_\phi) = \sup_{\|f\|=1} \left| \langle T_\phi f, f \rangle \right| \leq M. \]

Hence \( T_\phi \) is bounded.

It is not difficult to see that if \( \varphi \) is positive, then \( \varphi dA \) is a Carleson measure if and only if \( \varphi \) is a BT function by K.H. Zhu [7,9].

For any \( \xi \in \partial \mathbb{B}_n \), and \( r \in (0, 1) \), write
\[ S(\xi, r) = \{ z \in \mathbb{B}_n \ | \ 1 - \langle z, \xi \rangle < r \}, \]
then we may check easily that for a positive measure \( d\mu \) on \( \mathbb{B}_n \), \( d\mu \) is a Carleson measure if and only if
\[ \sup_r \frac{\mu(S(\xi, r))}{A(S(\xi, r))} < \infty \quad \text{for each } \xi \in \partial \mathbb{B}_n \]
(cf. see C.C. Cowen and B. MacCluer [10]). We say \( d\mu \) is a vanishing Carleson measure if
\[ \lim_{r \to 0} \frac{\mu(S(\xi, r))}{A(S(\xi, r))} = 0 \quad \text{for arbitrary } \xi \in \partial \mathbb{B}_n. \]

**Lemma 6.** (See Cima and Wogen [8], Cowen and MacCluer [10].) For \( \mu \) a finite, positive Borel measure on \( \mathbb{B}_n \), the following are equivalent:

1. There is a constant \( K < \infty \) so that for all \( \xi \in \partial \mathbb{B}_n \),
   \[ \mu(S(\xi, r)) \leq KA(S(\xi, r)). \]
2. There is a constant \( c < \infty \),
   \[ \int_{\mathbb{B}_n} |f(z)|^2 d\mu \leq C \int_{\mathbb{B}_n} |f(z)|^2 dA \]
   for all \( f \) in \( L^2_\alpha(D) \). When \( K \) is small, so is \( C \).

By Lemma 6, we have immediately following

**Theorem 7.** Suppose \( \varphi \in L^1(\mathbb{B}_n, dA) \) is a positive function, then \( T_\phi \) is compact on \( L^2_\alpha(\mathbb{B}_n) \) if and only if \( \varphi dA \) is a vanishing Carleson measure.

In [11] Grudsky and Vasilevski proved that the Toeplitz operator \( T_\phi \) with radial symbol is bounded (compact) on the Bergman space \( L^2_\alpha(D) \) if and only if the sequence
\[ \gamma_k(\varphi) = \int_0^1 \varphi \left( \frac{r}{k^{1/2}} \right) dr \]
belongs to \( l_\infty(\mathbb{Z}+) \) (respectively to \( C_0(\mathbb{Z}+) \)). In the case of \( n > 1 \), if \( \varphi \) is a radial function in \( L^1(\mathbb{B}_n, dA) \), then for any multi-index \( \alpha \in \mathbb{N}^n \).
\[
\{T_{\phi}z^\alpha, z^\alpha\} = \int_{\mathbb{B}_n} \phi |z^\alpha|^2 \, dA \\
= 2n \int_0^1 \varphi(r) \int_{\partial \mathbb{B}_n} r^{|\alpha|} |\xi^\alpha|^2 \, d\sigma(\xi) r^{2n-1} \, dr \\
= 2n \frac{(n-1)!|\alpha|!}{(n-1 + |\alpha|)!} \int_0^1 \varphi(r) r^{|\alpha|+2n-1} \, dr.
\]

Thus, for \( \{e_\alpha\} = \{z^\alpha/\|z^\alpha\|_2\} \), we have
\[
\gamma_\alpha(\phi) = \langle T_{\phi}e_\alpha, e_\alpha \rangle = \int_0^1 \varphi(r) r^{|\alpha|+p} \, dr.
\]

It is not difficult to check that \( T_{\phi}e_\alpha = \gamma_\alpha(\phi)e_\alpha \), that is, \( T_{\phi} \) is a diagonal operator which has the form
\[T_{\phi} = \text{diag}(\gamma_\alpha(\phi))_{\alpha \in \mathbb{N}^n}\]
relative to the base \( \{e_\alpha\}_{\alpha \in \mathbb{N}^n} \). This shows immediately the following

**Theorem 8.** Suppose \( \varphi \in L^1(\mathbb{B}_n, dA) \) is a radial function, then:

1. \( T_{\phi} \) is bounded if and only if \( \{\gamma_\alpha(\phi)\}_{\alpha \in \mathbb{N}^n} \) is a bounded sequence.
2. \( T_{\phi} \) is compact if and only if
\[
\lim_{|\alpha| \to \infty} \gamma_\alpha(\phi) = 0.
\]
3. \( T_{\phi} \) is a \( S_p \)-class \( (p \geq 1) \) operator if and only if \( \{\gamma_\alpha(\phi)\}_{\alpha \in \mathbb{N}^n} \in l^p \), and whenever \( T_{\phi} \in S_p \), we have
\[
\|T_{\phi}\|_p = \left[ \sum_{\alpha \in \mathbb{N}^n} |\gamma_\alpha(\phi)|^p \right]^\frac{1}{p}.
\]

**References**