On maps preserving square-zero matrices

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1. Introduction

The problem of characterizing linear operators on matrix algebras that leave invariant certain functions, subsets or relations has attracted the attention of many mathematicians (see survey papers [20–22,27] for details). For instance, linear operators preserving zero-product of matrices are studied in [15,16,29,31]; linear operators preserving idempotent matrices are studied in [1,11,13,14]; linear operators preserving matrices annihilated by a fixed polynomial are studied in [12,13,18]; linear operators preserving nilpotent matrices are studied in [9]; and linear operators preserving square-zero matrices are studied in [29].

Most linear preserver problems were investigated in the case of matrix algebras over fields. In contrast, not too much is known about matrix algebras over commutative rings. To the best of our knowledge, beside the papers by McDonald [25] and Waterhouse [30], only Brešar and Šemrl [11] used elementary calculations to describe linear maps preserving idempotent matrices over commutative rings. The reason why people seldom study matrix algebras over commutative rings is probably that one might encounter some difficulties that are not easy to overcome. For example, unlike matrix algebras over fields, the automorphisms on matrix algebras over commutative rings can fail to be inner (see [19,28]).
The breakthrough in the case of matrix algebras over commutative rings is connected with the papers by McDonald [25] and Waterhouse [30]. Since the paper [24] by Marcus and Moyls was published in 1959, it was known that many of the questions regarding linear preservers can be reduced to the problem of determining the set of linear maps which carry the matrices of rank one into themselves (see survey [23]). But to describe linear maps which preserve rank one matrices over commutative rings was really a very hard job! In the paper by McDonald [25] this problem was solved by using module theory and localization techniques from commutative algebras, and in the paper by Waterhouse [30] the group scheme approach was applied. For more details about linear algebra over commutative rings the reader is referred to the book by McDonald [26].

The central place of this paper occupies the description of maps preserving square-zero matrices over unital commutative rings. For matrices over complex numbers an analogous result was obtained by Šemrl [29] by means of linear algebra. Like in papers [25,30], in order to get a similar result for matrix algebras over commutative rings we need a new tool—the theory of functional identities. We shall provide in Section 3 the essential information on functional identities which is needed in this paper. Interested readers are referred to the surveys [8,10] for details. Using functional identities, we shall prove a key result (Theorem 3.3) which describes an additive map preserving “equal Jordan products” on a Lie ideal. As an application, we shall prove in Section 4 the main theorem (Theorem 4.1) that a surjective linear map preserving square-zero elements in a Lie ideal of a matrix ring must be a scalar multiple of the sum of a homomorphism and an antihomomorphism.

Note that a map $\phi$ preserving “equal Jordan products” certainly preserves “zero Jordan products,” that is, $\phi(x) \circ \phi(y) = 0$ whenever $x \circ y = 0$ and such a map preserves square-zero elements (provided the range of $\phi$ is 2-torsion free). In a forthcoming paper [17], we show that a surjective additive map preserving zero Jordan product of matrices over any unital ring must be a scalar multiple of a Jordan homomorphism.

2. Algebras generated by matrix units

Let $F$ be a unital commutative ring and $A$ an $F$-algebra which is generated by the matrix units $\{e_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq n\}$. We set $a_{ij} = ae_{ij}$ for any $a \in F$, and put

$$M = \{a_{ij} \mid a \in F, i \neq j\} \cup \{a_{ii} - a_{jj} \mid a \in F, i \neq j\}.$$ 

As usual, we denote by $[x, y] = xy - yx$ and $x \circ y = xy + yx$ for any $x, y \in A$. Let $L = [A, A]$ be the additive subgroup of $A$ generated by elements of the form $[x, y]$ with $x, y \in A$. It is clear that $L$ is generated additively by $M$.

Let $A'$ be an $F$-algebra and $\theta : L \to A'$ an additive map with the property that $\theta(x)^2 = 0$ whenever $x^2 = 0$ for $x \in L$. Our goal is to show that if $x, y, u, v \in L$ are such that $x \circ y = u \circ v$, then $\theta(x) \circ \theta(y) = \theta(u) \circ \theta(v)$. First of all, we shall examine the elements $x \circ y$ and $\theta(x) \circ \theta(y)$ for $x, y \in M$. Note that, for $x, y \in M$, the element $x \circ y$ is of one of the following forms:

$$0, \quad c_{ij}, \quad c_{ii}, \quad c_{ii} + c_{jj},$$

where $i \neq j$. More precisely, it is easy to see the following facts via direct calculations.
**Remark 2.1.**

1. The following four cases for \( x \circ y \) vanish identically:
   (1a) if \( i \neq j \), \( k \neq l \), \( i \neq l \) and \( j \neq k \), then \( a_{ij} \circ b_{kl} = 0 \);
   (1b) if \( i \neq j \), \( k \neq l \) and \( i, j \notin \{k, l\} \), then \( a_{ij} \circ (b_{kk} - b_{ll}) = 0 \);
   (1c) if \( i \neq j \), \( k \neq l \) and \( i, j \notin \{k, l\} \), then \( (a_{ii} - a_{jj}) \circ (b_{kk} - b_{ll}) = 0 \);
   (1d) if \( i \neq j \), then \( a_{ij} \circ (b_{ii} - b_{jj}) = 0 \).

2. The following two cases for \( x \circ y \) are of the form \( c_{ij} \) with \( i \neq j \):
   (2a) if \( i \neq j \) and \( k \notin \{i, j\} \), then \( a_{ik} \circ b_{kj} = c_{ij} \) where \( c = ab \);
   (2b) if \( i \neq j \) and \( k \notin \{i, j\} \), then \( a_{ij} \circ (b_{ii} - b_{kk}) = a_{ij} \circ (b_{jj} - b_{kk}) = c_{ij} \) where \( c = ab \).

3. The following one case for \( x \circ y \) is of the form \( c_{ii} \):
   (3a) if \( i \neq j \) and \( k \notin \{i, j\} \), then \( (a_{ii} - a_{jj}) \circ (b_{ii} - b_{kk}) = c_{ii} \) where \( c = 2ab \).

4. The following two cases for \( x \circ y \) are of the form \( c_{ii} + c_{jj} \):
   (4a) if \( i \neq j \), then \( (a_{ii} - a_{jj}) \circ (b_{ii} - b_{jj}) = c_{ii} + c_{jj} \) where \( c = 2ab \);
   (4b) if \( i \neq j \), then \( a_{ij} \circ b_{jj} = c_{ii} + c_{jj} \) where \( c = ab \).

To simplify the notation, we shall make the following conventions. For \( i \neq j \), we put \( E_{ij} = \theta(e_{ij}) \) and \( A_{ij} = \theta(a_{ij}) \) for \( a \in F \), \( B_{ij} = \theta(b_{ij}) \) for \( b \in F \) and so on. Also, we put \( E_{ii} - E_{jj} = \theta(e_{ii} - e_{jj}) \), \( A_{ii} - A_{jj} = \theta(a_{ii} - a_{jj}) \), \( B_{ii} - B_{jj} = \theta(b_{ii} - b_{jj}) \) and so on. Moreover, we shall also use, for example, \( [A + B]_{ij} = \theta(c_{ij}) \) when \( c = a + b \) and \( [AB]_{ij} \) for \( \theta(d_{ij}) \) when \( d = ab \). These notation will be used particularly in linearizing an equation because we have that \( [A + B]_{ij} = A_{ij} + B_{ij} \), \( [mA]_{ij} = mA_{ij} \) for any integer \( m \), and \( [(-A)]_{ij} = -[AB]_{ij} \).

The following lemma shows that the elements of the set
\[
\theta(M) = \{ A_{ij} \mid a \in F, \ i \neq j \} \cup \{ A_{ii} - A_{jj} \mid a \in F, \ i \neq j \}
\]

satisfy some properties similar to those listed in Remark 2.1.

**Lemma 2.2.** Let \( F \) be a commutative ring with \( \frac{1}{F} \). For \( i, j, k, l, p, q \in \{1, \ldots, n\} \) with \( i \neq j \), \( k \neq l \), \( p \neq q \) and \( a, b, c \in F \) with \( c = ab \), we have:

(a) if \( i \neq l \) and \( j \neq k \), then \( A_{ij} \circ B_{kl} = 0 \);
(b) if \( i, j \notin \{k, l\} \), then \( A_{ij} \circ (B_{kk} - B_{ll}) = 0 \);
(c) if \( i, j \notin \{k, l\} \), then \( (A_{ii} - A_{jj}) \circ (B_{kk} - B_{ll}) = 0 \);
(d) \( 2A_{ij} \circ (B_{ii} - B_{jj}) = 0 \);
(e) \( (A_{ii} - A_{jj}) \circ (B_{ii} - B_{jj}) = (C_{ii} - C_{jj}) \circ (E_{ii} - E_{jj}) \);
(f) \( 2A_{ij} \circ (B_{ii} - B_{jj}) = (C_{ii} - C_{jj}) \circ (E_{ii} - E_{jj}) \);
(g) if \( i, j \notin \{k, p\} \), then \( A_{ik} \circ B_{kj} = C_{ij} \circ (E_{ii} - E_{pp}) = C_{ij} \circ (E_{jj} - E_{pp}) \);
(h) if \( i, j \notin \{k, p\} \), then \( A_{ij} \circ (B_{ii} - B_{kk}) = A_{ij} \circ (B_{jj} - B_{kk}) = C_{ij} \circ (E_{ii} - E_{pp}) = C_{ij} \circ (E_{jj} - E_{pp}) \);
(i) if \( i \notin \{k, l, p, q\} \), then \( (A_{ii} - A_{kk}) \circ (B_{ii} - B_{ll}) = (C_{ii} - C_{pp}) \circ (E_{ii} - E_{qq}) \).
Proof. For \(x, y \in L\) with \(x^2 = y^2 = x \circ y = 0\), we have \((x + y)^2 = 0\) and so \(\theta(x)^2 = 0\), \(\theta(y)^2 = 0\) and \((\theta(x) + \theta(y))^2 = 0\). Expanding the last equation, we get \(\theta(x) \circ \theta(y) = 0\).

In particular, when \(x = a_{ij}, y = b_{kl}\) with \(i \neq l\) and \(j \neq k\), we have \(\theta(a_{ij}) \circ \theta(b_{kl}) = 0\), that is, \(A_{ij} \circ B_{kl} = 0\). This proves (a).

Assume that \(i, j \notin [k, l]\). Then it follows from

\[
\begin{align*}
(a_{ii} + a_{ij} - a_{ji} - a_{jj})^2 &= 0, \\
b_{kk} + b_{kl} - b_{lk} - b_{ll} &\neq 0
\end{align*}
\]

that

\[
(A_{ii} + A_{ij} - A_{ji} - A_{jj}) \circ (B_{kk} + B_{kl} - B_{lk} - B_{ll}) = 0.
\]

Expanding the last equation and applying (a) and (b), we have \((A_{ii} - A_{jj}) \circ (B_{kk} + B_{kl} - B_{lk} - B_{ll}) = 0\) which is (c).

Now, for any \(x, y \in F\) and \(m \in \{1, 2, 3\}\), we have

\[
([xy]_{ii} + m[y^2]_{ji} - [xy]_{jj})^2 = 0
\]

and so

\[
([XY]_{ii} + m[X_{ij} - X_{ji}] - [XY]_{jj})^2 = 0. \tag{2.1}
\]

Expanding the last equation and using \(X_{ij}^2 = [XY]^2_{ji} = 0\), we have \(U + mV + m^{-1}W = 0\) for \(m = 1, 2, 3\), where

\[
U = ([XY]_{ii} - [XY]_{jj})^2 - X_{ij} \circ [XY^2]_{ji}, \quad V = ([XY]_{ii} - [XY]_{jj}) \circ X_{ij}, \quad \text{and} \quad W = ([XY]_{ij} - [XY]_{ii}) \circ [XY^2]_{ji}.
\]

Then a van der Monde argument concludes that \(U = 0\) and \(V = 0\). That is,

\[
([XY]_{ii} - [XY]_{jj})^2 - X_{ij} \circ [XY^2]_{ji} = 0 \tag{2.2}
\]
and

\[
([XY]_{ii} - [XY]_{jj}) \circ X_{ij} = 0.
\] (2.3)

Thus, setting \( x = 1 \) and \( y = ab \) in (2.3), we get

\[
(C_{ii} - C_{jj}) \circ E_{ij} = 0.
\] (2.4)

On the other hand, linearizing (2.3) by replacing \( x \) with \( x + z \), we obtain

\[
([XY]_{ii} - [XY]_{jj}) \circ Z_{ij} + ([ZY]_{ii} - [ZY]_{jj}) \circ X_{ij} = 0.
\] (2.5)

Setting \( z = 1, x = a \) and \( y = b \) in (2.5) and using (2.4), we have

\[
(A_{ii} \circ (B_{ii} - B_{jj})) = 0
\]

which is (d).

Linearizing (2.2) by replacing \( y \) with \( y + v \), we obtain

\[
([XY]_{ii} - [XY]_{jj}) \circ ([XV]_{ii} - [XV]_{jj}) - 2X_{ij} \circ [XYV]_{ji} = 0.
\] (2.6)

Setting \( x = 1, y = a \) and \( v = b \) in (2.6), we get

\[
(E_{ii} \circ E_{jj}) \circ (C_{ii} - C_{jj}) - 2E_{ij} \circ C_{ji} = 0.
\] (2.8)

Thus (e) is proved by (2.7) and (2.8).

Linearizing (2.6) by replacing \( x \) with \( x + u \), we have

\[
([XY]_{ii} - [XY]_{jj}) \circ ([UV]_{ii} - [UV]_{jj}) + ([UY]_{ii} - [UY]_{jj}) \circ ([XV]_{ii} - [XV]_{jj}) - 2X_{ij} \circ [XYV]_{ji} = 0.
\] (2.9)

Setting \( u = v = 1, x = a \) and \( y = b \) in (2.9) and using (2.7), we have

\[2A_{ij} \circ B_{ji} = (E_{ii} - E_{jj}) \circ (C_{ii} - C_{jj}).\]

This is (f).

Assume now that \( k \notin \{i, j\} \). Then for any \( x, y, z \in F \), we have

\[
([xy]_{li} + x_{ik} - [xy]_{ki} - [xy]_{kk})^2 = 0, \quad (z_{ij} - [yz]_{kj})^2 = 0
\]

and

\[
([xy]_{li} + x_{ik} - [xy]_{ki} - [xy]_{kk}) \circ (z_{ij} - [yz]_{kj}) = 0,
\]

and so

\[
([XY]_{li} + X_{ik} - [XY]_{ki} - [XY]_{kk}) \circ (Z_{ij} - [YZ]_{kj}) = 0.
\] (2.10)
Replacing $y$ with $-y$ in (2.10), we get
\[
\left( -[XY]_{ii} + X_{ik} - [XY^2]_{ki} + [XY]_{kk} \right) \circ (Z_{ij} + [YZ]_{kj}) = 0. \tag{2.11}
\]
Subtracting (2.11) from (2.10) and noting that $[XY^2]_{ki} \circ [YZ]_{kj} = 0$ by (a), we arrive at
\[
\left( [XY]_{ii} - [XY]_{kk} \right) \circ Z_{ij} = X_{ik} \circ [YZ]_{kj}. \tag{2.12}
\]
Setting $x = a$, $y = 1$ and $z = b$ in (2.12), we have
\[
(A_{ii} - A_{kk}) \circ B_{ij} = A_{ik} \circ B_{kj}; \tag{2.13}
\]
setting $x = 1$, $y = a$ and $z = b$ in (2.12), we have
\[
(A_{ii} - A_{kk}) \circ B_{ij} = E_{ik} \circ C_{kj}; \tag{2.14}
\]
and setting $x = y = 1$ and $z = ab$ in (2.12), we have
\[
(E_{ii} - E_{kk}) \circ C_{ij} = E_{ik} \circ C_{kj}. \tag{2.15}
\]
Therefore, Eqs. (2.13)–(2.15) yield
\[
A_{ik} \circ B_{kj} = C_{ij} \circ (E_{ii} - E_{kk}). \tag{2.16}
\]
Assume that $p \notin \{i, j\}$. As we have seen in (b), $C_{ij} \circ (E_{kk} - E_{pp}) = 0$, so
\[
A_{ik} \circ B_{kj} = C_{ij} \circ (E_{ii} - E_{pp})
\]
for all $p \notin \{i, j\}$. Since $C_{ij} \circ (E_{jj} - E_{ii}) = 0$ by (d), we obtain
\[
A_{ik} \circ B_{kj} = C_{ij} \circ (E_{jj} - E_{pp}).
\]
This proves (g). Exchanging the elements $a$ and $b$ in (2.14), we have
\[
A_{ij} \circ (B_{ii} - B_{kk}) = E_{ik} \circ C_{kj}. \tag{2.17}
\]
Since $A_{ij} \circ (B_{jj} - B_{ii}) = 0$ by (d), we have
\[
A_{ij} \circ (B_{jj} - B_{kk}) = E_{ik} \circ C_{kj}. \tag{2.18}
\]
Also, for $p \notin \{i, j\}$ we have from (g)
\[
E_{ik} \circ C_{kj} = C_{ij} \circ (E_{ii} - E_{pp}) = C_{ij} \circ (E_{jj} - E_{pp}). \tag{2.19}
\]
Combining (2.17)–(2.19), we obtain
\[ A_{ij} \circ (B_{ii} - B_{kk}) = A_{ij} \circ (B_{jj} - B_{kk}) = C_{ij} \circ (E_{ii} - E_{pp}) = C_{ij} \circ (E_{jj} - E_{pp}). \]

This proves (h).

Now we use (a)–(h) to derive (i). So assume that \( i \notin \{k, l\} \). First of all, it follows from
\[
(a_{ii} + a_{ik} + a_{il} - a_{ki} - a_{kk} - a_{kl})^2 = 0, \\
(b_{ii} + b_{ik} + b_{il} - b_{li} - b_{lk} - b_{ll})^2 = 0,
\]
and
\[
(a_{ii} + a_{ik} + a_{il} - a_{ki} - a_{kk} - a_{kl}) \circ (b_{ii} + b_{ik} + b_{il} - b_{li} - b_{lk} - b_{ll}) = 0
\]
that
\[
(A_{ii} + A_{ik} + A_{il} - A_{ki} - A_{kk} - A_{kl}) \circ (B_{ii} + B_{ik} + B_{il} - B_{li} - B_{lk} - B_{ll}) = 0. \tag{2.20}
\]

Let us set
\[
U = A_{ii} + A_{ik} - A_{ki} - A_{kk}, \quad V = B_{ii} + B_{il} - B_{li} - B_{ll}, \\
W = A_{il} - A_{kl}, \quad Z = B_{ik} - B_{lk}.
\]

Then we can write (2.20) as
\[
U \circ V + U \circ Z + W \circ V + W \circ Z = 0.
\]

Applying (h), (a), (g), and (d), we get
\[
U \circ V = (A_{ii} - A_{kk}) \circ (B_{ii} - B_{ll}) + (A_{ii} - A_{kk}) \circ B_{il} - (A_{ii} - A_{kk}) \circ B_{li} \\
+ A_{ik} \circ (B_{ii} - B_{ll}) + A_{ik} \circ B_{il} - A_{ik} \circ B_{li} \\
- A_{ki} \circ (B_{ii} - B_{ll}) - A_{ki} \circ B_{il} + A_{ki} \circ B_{li} \\
= (A_{ii} - A_{kk}) \circ (B_{ii} - B_{ll}) + C_{ii} \circ (E_{ii} - E_{kk}) - C_{il} \circ (E_{ii} - E_{kk}) \\
+ C_{ik} \circ (E_{ii} - E_{ll}) + 0 - C_{ki} \circ (E_{kk} - E_{ii}) \\
- C_{kl} \circ (E_{ll} - E_{ll}) - C_{li} \circ (E_{kk} - E_{ll}) + 0,
\]
\[
U \circ Z = (A_{ii} - A_{kk}) \circ B_{ik} - (A_{ii} - A_{kk}) \circ B_{lk} + A_{ik} \circ B_{ik} - A_{ik} \circ B_{lk} \\
- A_{ki} \circ B_{ik} + A_{kl} \circ B_{lk} \\
= 0 - C_{ik} \circ (E_{ii} - E_{kk}) + 0 - 0 - A_{ki} \circ B_{ik} + C_{kl} \circ (E_{ii} - E_{kk}),
\]
\[
W \circ V = A_{il} \circ (B_{ii} - B_{ll}) + A_{il} \circ B_{il} - A_{il} \circ B_{li} - A_{kli} \circ (B_{ii} - B_{ll}) \\
- A_{kli} \circ B_{il} + A_{kl} \circ B_{li} \\
= 0 + 0 - A_{il} \circ B_{li} - C_{kl} \circ (E_{ii} - E_{ll}) - 0 + C_{ki} \circ (E_{ii} - E_{ll}),
\]
\[
W \circ Z = A_{il} \circ B_{ik} - A_{il} \circ B_{lk} + A_{kl} \circ B_{ik} - A_{kl} \circ B_{lk} \\
- A_{ki} \circ B_{ik} + A_{kl} \circ B_{lk} \\
= 0 - C_{ik} \circ (E_{ii} - E_{kk}) + 0 - 0 - A_{ki} \circ B_{ik} + C_{kl} \circ (E_{ii} - E_{kk}).
\]
Putting these together, we obtain
\[(A_{ii} - A_{kk}) \circ (B_{ii} - B_{ll}) = A_{ki} \circ B_{ik} - A_{il} \circ B_{li} + A_{kl} \circ B_{lk}.\]

Thus, from (e), we have
\[(A_{ii} - A_{kk}) \circ (B_{ii} - B_{ll}) = \frac{1}{2}(C_{ii} - C_{kk}) \circ (E_{ii} - E_{ll}) + \frac{1}{2}(C_{ii} - C_{ll}) \circ (E_{ii} - E_{ll}) - \frac{1}{2}(C_{kk} - C_{ll}) \circ (E_{kk} - E_{ll}).\]

(2.21)

Replacing \(a\) by \(ab\) and \(b\) by 1 in (2.21), we obtain
\[(A_{ii} - A_{kk}) \circ (B_{ii} - B_{ll}) = (C_{ii} - C_{kk}) \circ (E_{ii} - E_{ll}).\]

On the other hand, if \(i, l\) and \(p\) are all distinct, we can apply (c) to get
\[(C_{ii} - C_{kk}) \circ (E_{ii} - E_{ll}) = (C_{ii} - C_{pp} + C_{pp} - C_{kk}) \circ (E_{ii} - E_{ll}) = (C_{ii} - C_{pp}) \circ (E_{ii} - E_{ll}).\]

And, if \(i, k\) and \(q\) are all distinct, we have
\[(C_{ii} - C_{kk}) \circ (E_{ii} - E_{ll}) = (C_{ii} - C_{kk}) \circ (E_{ii} - E_{qq} + E_{qq} - E_{ll}) = (C_{ii} - C_{kk}) \circ (E_{ii} - E_{qq}).\]

Thus we can conclude that
\[(C_{ii} - C_{kk}) \circ (E_{ii} - E_{ll}) = (C_{ii} - C_{pp}) \circ (E_{ii} - E_{qq}),\]
if \(i \notin \{k, l, p, q\}\). Therefore,
\[(A_{ii} - A_{kk}) \circ (B_{ii} - B_{ll}) = (C_{ii} - C_{pp}) \circ (E_{ii} - E_{qq}),\]
which is (i). Thus the proof of the lemma is now complete. □

**Theorem 2.3.** Let \(F\) be a commutative ring with \(1\), and \(A\) an \(F\)-algebra generated by the matrix units \(\{e_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq n\}\) where \(n \geq 4\). Let \(\mathcal{L} = [A, A]\), \(A'\) an \(F\)-algebra and \(\theta : \mathcal{L} \to A'\) an additive map such that \(\theta(w)^2 = 0\) for all \(w \in \mathcal{L}\) with \(w^2 = 0\). Then, for \(x_i, y_i \in \mathcal{L}\) with \(\sum_{i=1}^{m} x_i \circ y_i = 0\), we have \(\sum_{i=1}^{m} \theta(x_i) \circ \theta(y_i) = 0\). In particular, for \(x, y, u, v \in \mathcal{L}\) with \(x \circ y = u \circ v\), we have \(\theta(x) \circ \theta(y) = \theta(u) \circ \theta(v)\).
Proof. Since \( L \) is additively generated by \( M = \{ a_{ij} \mid a \in F, i \neq j \} \cup \{ a_{ii} - a_{jj} \mid a \in F, i \neq j \} \), \( \theta(L) \) is additively generated by \( \theta(M) = \{ A_{ij} \mid a \in F, i \neq j \} \cup \{ A_{ii} - A_{jj} \mid a \in F, i \neq j \} \). Thus \( x_1 \circ y_1 + \cdots + x_m \circ y_m \) is a sum of elements \( x \circ y \) with \( x, y \in M \) and \( \theta(x_1) \circ \theta(y_1) + \cdots + \theta(x_m) \circ \theta(y_m) \) is a sum of corresponding elements of the form \( \theta(x) \circ \theta(y) \) with \( x, y \in M \). Moreover, since

\[
(a_{ii} - a_{jj}) \circ (b_{ii} - b_{jj}) = (a_{ii} - a_{jj}) \circ (b_{ii} - b_{kk}) + (a_{jj} - a_{ii}) \circ (b_{jj} - b_{kk})
\]

for \( k \neq \{i, j\} \) and

\[
a_{ij} \circ b_{ji} = \frac{1}{2} (a_{ii} - a_{jj}) \circ (b_{ii} - b_{jj})
\]

by (4a) and (4b) of Remark 2.1, we can express both \( (a_{ii} - a_{jj}) \circ (b_{ii} - b_{jj}) \) and \( a_{ij} \circ b_{ji} \) in terms of \( (a_{ii} - a_{jj}) \circ (b_{ii} - b_{kk}) \) for \( k \neq \{i, j\} \). Thus we may assume further that \( x_1 \circ y_1 + \cdots + x_m \circ y_m \) is a sum of elements \( x \circ y \) of the forms (1a)–(3a) in Remark 2.1, and hence the result of \( x \circ y \) is 0, \( c_{ii} \) or \( c_{jj} \), where \( i \neq j \).

For the terms \( x \circ y \) with \( x \circ y = 0 \), that is, terms of the forms (1a)–(1d) or of the forms (2a)–(4b) with \( ab = 0 \), the corresponding terms \( \theta(x) \circ \theta(y) \) also vanish by Lemma 2.2.

Since \( x_1 \circ y_1 + \cdots + x_m \circ y_m = 0 \), for any \( i, j \) with \( i \neq j \), the sum of terms with results of the form \( c_{ij} \) is 0. This sum is resulted from terms of the form \( a_{ik} \circ b_{kj} \), \( a_{ij} \circ (b_{ii} - b_{kk}) \) or \( a_{ij} \circ (b_{jj} - b_{kk}) \) with \( k \neq \{i, j\} \) and \( c = ab \). Choose an integer \( p \neq \{i, j\} \), then the corresponding terms \( A_{ik} \circ B_{kj} \), \( A_{ij} \circ (B_{ii} - B_{kk}) \) or \( A_{ij} \circ (B_{jj} - B_{kk}) \) can be written as \( C_{ij} \circ (E_{ii} - E_{pp}) \) with \( c = ab \) by (g) and (h) of Lemma 2.2 and hence the sum of them is 0.

Finally, for any \( i \in \{1, \ldots, n\} \), the sum of terms with results of the form \( c_{ii} \) is 0. This sum is resulted from terms of the form \( (a_{ii} - a_{jj}) \circ (b_{ii} - b_{kk}) \) with \( i, j, k \) all distinct and \( c = 2ab \). Choose two integers \( p, q \) such that \( i, p, q \) are all distinct, then the corresponding terms \( (A_{ii} - A_{jj}) \circ (B_{ii} - B_{kk}) \) can be written as \( (C_{ii} - C_{pp}) \circ (E_{ii} - E_{qq}) \) with \( c = ab \) by (i) of Lemma 2.2 and hence the sum of them is 0. Therefore we have \( \theta(x_1) \circ \theta(y_1) + \cdots + \theta(x_m) \circ \theta(y_m) = 0 \). In particular, if \( x, y, u, v \in L \) are such that \( x \circ y = u \circ v \), then \( x \circ y + (-u) \circ v = 0 \), and so we get \( \theta(x) \circ \theta(y) - \theta(u) \circ \theta(v) = 0 \). \( \square \)

Note that in case the map \( \theta \) in Theorem 2.3 is not only additive but also \( F \)-linear, we need only (a)–(i) of Lemma 2.2 for the special cases when \( a = b = 1 \). As a matter of fact, from the proof of Lemma 2.2 we see that it suffices to assume that \( \theta(w)^2 = 0 \) for all \( w \in L \) of some particular forms as remarked below.

Remark 2.4. Let \( F \) be a commutative ring with \( \frac{1}{2} \), and \( A \) an \( F \)-algebra generated by the matrix units \( \{ e_{ij} | 1 \leq i \leq n, 1 \leq j \leq n \} \) where \( n \geq 4 \). Let \( L = [A, A] \), and \( w \in L \) be of one of the following forms:

1. \( e_{ij} \) where \( i \neq j \);
2. \( e_{ij} + e_{kl} \) where \( i \neq j, k \neq l, i \neq l \) and \( j \neq k \);
3. \( e_{ii} + e_{ij} - e_{ji} - e_{jj} \) where \( i \neq j \);
Then, for any such \( w \), there exists an idempotent \( e \) such that \( w \in e\mathcal{A}(1-e) \). Moreover, suppose that \( \mathcal{A} \) is an \( F \)-algebra and \( \theta : \mathcal{L} \rightarrow \mathcal{A} \) is an \( F \)-linear map such that \( \theta(w)^2 = 0 \) for all \( w \in \mathcal{L} \) of one of the forms (1)–(9). Then, for \( x_1, y_1, \ldots, x_m, y_m \in \mathcal{L} \) with \( x_1 \circ y_1 + \cdots + x_m \circ y_m = 0 \), we have \( \theta(x_1) \circ \theta(y_1) + \cdots + \theta(x_m) \circ \theta(y_m) = 0 \). In particular, for \( x, y, u, v \in \mathcal{L} \) with \( x \circ y = u \circ v \), we have \( \theta(x) \circ \theta(y) = \theta(u) \circ \theta(v) \).

**Proof.** First we show that for any \( w \) of the forms (1)–(9) there exists a suitable idempotent \( e \) such that \( w \in e\mathcal{A}(1-e) \). In (1) set \( e = e_{ii} \). In (2) set \( e = e_{ii} + e_{kk} \). In (3) set \( e = \frac{1}{2}(e_{ii} - e_{ij} + e_{ji} + e_{jj}) \). In (4) set \( e = e_{ii} + \frac{1}{2}(e_{kk} - e_{ij} + e_{ji} + e_{jj}) \). In (5) set \( e = \frac{1}{2}(e_{ii} - e_{ij} - e_{ji} - e_{jj}) \). In (6) set \( e = \frac{1}{2}(e_{ii} + e_{jj}) \). In (7) set \( e = \frac{1}{2}(e_{ii} - e_{ij} + e_{ji} - e_{jj}) \). In (8) set \( e = \frac{1}{2}(e_{ii} + e_{jj}) \). Finally in (9) set \( e = \frac{1}{2}(e_{ii} + 2e_{jj} + e_{kk} - e_{ij} - e_{ji} - e_{jj} - e_{kk}) \).

Let \( \hat{\mathcal{E}} = \{e_{ij} \mid i \neq j \} \cup \{e_{ii} - e_{ij} \mid i \neq j \} \). Since \( \mathcal{L} \) is an \( F \)-module generated by \( \mathcal{E} \), \( \theta(\mathcal{L}) \) is an \( F \)-module generated by \( \theta(\hat{\mathcal{E}}) = \{E_{ij} \mid i \neq j \} \cup \{E_{ii} - E_{ij} \mid i \neq j \} \). Hence, to prove the last statements, we need only to consider all possible situations of \( x \circ y \) and \( \theta(x) \circ \theta(y) \) with \( x, y \in \mathcal{E} \). Examining the proof of Lemma 2.2 we see that all possible situations came from \( w \) of the forms (1)–(9). Therefore the arguments proving Lemma 2.2 work as well and complete the proof of this remark. \( \square \)

**3. Functional identities**

The proofs of the results in the sequel rely heavily on a newly developed theory on rings, namely that of *functional identities*. For introduction to some concepts as well as basic results on functional identities, readers are encouraged to consult the survey papers [8,10].

A functional identity on a ring \( R \) is, roughly speaking, an identity holding for all elements in \( R \) (or more generally, all elements from some subset of \( R \)) which involves some arbitrary set-theoretic functions on \( R \). For instance, in a ring \( R \) with center \( C \), let \( f_1, f_2, g_1, g_2 : R \rightarrow R \) be maps; then the following is an example of functional identity:

\[
f_1(x)y + f_2(y)x + xg_1(y) + yg_2(x) = 0 \quad \text{for all } x, y \in R.
\] (3.22)

As to the solution to (3.22), one natural possibility is that \( f_1, f_2, g_1, g_2 \) are of the forms

\[
\begin{align*}
f_1(x) &= xp + \mu(x), & f_2(y) &= yq + v(y), \\
g_1(y) &= -py - v(y), & g_2(x) &= -qx - \mu(x).
\end{align*}
\] (3.23)
where \( p, q \in R \) and \( \mu, \nu : R \to C \). It is obvious that the functions in (3.23) constitute a solution to the following more general functional identity:

\[
f_1(x)y + f_2(y)x + xg_1(y) + yg_2(x) \in C \quad \text{for all } x, y \in R. \tag{3.24}
\]

Both the functional identities (3.22) and (3.24) are said to be of degree 2 because two indeterminates \( x \) and \( y \) are involved. Before introducing functional identities of higher degree, we need to introduce some notation.

Let \( Q \) be a ring with center \( C \) containing 1, and \( R \) a nonempty subset of \( Q \). For a positive integer \( m \), we denote by \( R_m \) the \( m \)-th Cartesian power of \( R \). For \( x_1, x_2, \ldots, x_m \in R \), we denote by \( x_m \) the ordered \( m \)-tuple \((x_1, \ldots, x_m)\), by \( \hat{x}_m \) the ordered \((m - 1)\)-tuple obtained by dropping the \( i \)th component of \( x_m \), and by \( \hat{x}_{ij} \) for \( i \neq j \) the ordered \((m - 2)\)-tuple obtained by dropping both the \( i \)th and the \( j \)th components of \( x_m \). That is,

\[
x_m = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \quad \text{and} \quad \hat{x}_m = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m).
\]

Let \( I, J \subseteq \{1, 2, \ldots, m\} \) where \( m \geq 2 \), and for each \( i \in I, j \in J \), let \( E_i, F_j : R^{m-1} \to Q \) be arbitrary maps. The basic functional identities of degree \( m \) are

\[
\sum_{i \in I} E_i(x_m^i)x_i + \sum_{j \in J} x_j F_j(x_m^j) = 0 \quad \text{for all } x_m \in R^m, \tag{3.25}
\]

and a slightly more general one,

\[
\sum_{i \in I} E_i(x_m^i)x_i + \sum_{j \in J} x_j F_j(x_m^j) \in C \quad \text{for all } x_m \in R^m. \tag{3.26}
\]

It is understood that if the index set \( I \) or \( J \) is empty, then the corresponding sum is 0. Suppose that there exist maps \( p_{ij} : R^{m-2} \to Q, i \in I, j \in J, i \neq j \), and \( \lambda_k : R^{m-1} \to C, k \in I \cup J \), with \( \lambda_k = 0 \) for \( k \notin I \cap J \), such that

\[
E_i(x_m^i) = \sum_{j \in J, j \neq i} x_j p_{ij}(x_m^j) + \lambda_i(x_m^i) \quad \text{and} \quad F_j(x_m^j) = -\sum_{i \in I, i \neq j} p_{ij}(x_m^j)x_i - \lambda_j(x_m^j) \tag{3.27}
\]

for all \( x_m \in R^m, i \in I, \) and \( j \in J \). We make the convention that \( p_{ij} \) is just an element in \( Q \) in case \( m = 2 \). One can readily check that (3.27) implies (3.25) and it does not depend on \( R \). We shall refer to (3.27) as a standard solution of (3.25) and (3.26). Note that the functional identity \( \sum_{i \in I} E_i(x_m^i)x_i = 0 \) has only one standard solution, namely \( E_i = 0 \) for
all $i \in I$. It turns out that frequently the standard solutions are the only possible solutions. This is the reason why the following fundamental concept is introduced in [4].

For a positive integer $d$, a nonempty subset $R \subseteq Q$ is said to be $d$-free, if for any positive integer $m$ and $I, J \subseteq \{1, 2, \ldots, m\}$, both of the following two conditions are satisfied:

(a) If $\max(|I|, |J|) \leq d$, then the functional identity (3.25) has the only solution (3.27).
(b) If $\max(|I|, |J|) \leq d - 1$, then the functional identity (3.26) has the only solution (3.27).

It is obvious that a $d$-free set is $d'$-free for any positive integer $d' \leq d$. Moreover, if $R$ is a $d$-free subset of $Q$, so is any subset $S$ of $Q$ containing $R$ [4, Theorem 2.8].

Roughly speaking, $d$-free subsets are those subsets $R$ of $Q$ such that any functional identity on $R$ in “not too many” variables has only the standard solutions. Then, which sets are $d$-free after all? Algebras over fields, in particular the maximal right (or left) rings of quotients of prime rings, abound in $d$-free subsets.

For an element $x$ in an algebra $Q$ over a field $C$, we denote by $\deg_C(x)$ the degree of $x$ over $C$, or $\infty$ if $x$ is not algebraic over $C$. And for a nonempty subset $R \subseteq Q$, we set

$$\deg_C(R) = \sup \{\deg(x) \mid x \in R\}.$$ 

In case $A$ is a prime ring with maximal right quotient ring $Q$ and extended centroid $C$ (see the book [7] for definitions and basic properties), $A$ is a $d$-free subset of $Q$ if $\deg_C(A) \geq d$ [2, Theorem 1.2], a noncentral Lie ideal $L$ of $A$ is $d$-free if $\deg_C(A) \geq d + 1$ [2, Theorem 1.2], and the set $S$ of symmetric elements of $A$ and the set $K$ of skew-symmetric elements of $A$ are both $d$-free if $A$ is equipped with an involution and $\deg_C(A) \geq 2d + 2$ [4, Theorem 2.4].

For applications we need more involved functional identities than (3.25) and (3.26). Let $S$ be a set and let $\alpha : S \rightarrow Q$, $E_i, F_j : S^{m-1} \rightarrow Q$, $i \in I$, $j \in J$, be maps of sets. We are interested in the following two identities:

$$\sum_{i \in I} E_i(x^i_m)\alpha(x_i) + \sum_{j \in J} \alpha(x_j)F_j(x^j_m) = 0 \quad \text{for all } x_m \in S^m, \quad (3.28)$$

and

$$\sum_{i \in I} E_i(x^i_m)\alpha(x_i) + \sum_{j \in J} \alpha(x_j)F_j(x^j_m) \in C \quad \text{for all } x_m \in S^m. \quad (3.29)$$

It is easy to see that in case $S = R \subseteq Q$ and $\alpha$ is the identity map, the functional identities (3.28) and (3.29) are exactly identities (3.25) and (3.26). The standard solutions of the functional identities (3.28) and (3.29) are of the forms

$$E_i(x^i_m) = \sum_{j \in J, j \neq i} \alpha(x_j)p_{ij}(x^j_m) + \lambda_i(x^i_m) \quad \text{and}$$
\[ F_j(x^m) = - \sum_{i \in I, i \neq j} p_{ij}(x^m)\alpha(x_i) - \lambda_j(x^m) \] (3.30)

for all \( x_m \in S^m \), where \( p_{ij} : S^{m-2} \to Q \), \( i \in I \), \( j \in J \), \( i \neq j \), \( \lambda_k : S^{m-1} \to C \), \( k \in I \cup J \), with \( \lambda_k = 0 \) for \( k \notin I \cap J \). In light of [4, Theorem 2.6], if \( \alpha(S) \) is a \( d \)-free subset of \( Q \), then the functional identities (3.28) and (3.29) have only the standard solutions (3.30).

Another important concept in the theory of functional identities is that of Beidar polynomials, in honor of late K.I. Beidar who made an extremely important contribution to this theory. Here we give this concept in a loose manner and refer the reader to [5] for details.

Let \( S \) be a set, \( Q \) a ring with center \( C \) and \( \alpha : S \to Q \) be a map of sets. We say that a map \( E : S \to Q \) is a Beidar polynomial of degree 1 in \( \alpha \) if there exist an element \( \lambda \in C \) and a map \( \mu : S \to C \) such that

\[ E(x) = \lambda \alpha(x) + \mu(x) \] for all \( x \in S \),

where \( \lambda \) and \( \mu \) are called the coefficients of \( E \). In case when \( \mu = 0 \), \( E \) is said to be without constant coefficient.

Next, a map \( E : S^2 \to Q \) is said to be a Beidar polynomial of degree 2 in \( \alpha \) if there exist \( \lambda_1, \lambda_2 \in C \), maps \( \mu_1, \mu_2 : S \to C \) and a map \( \nu : S^2 \to C \) such that

\[ E(x, y) = \lambda_1 \alpha(x)\alpha(y) + \lambda_2 \alpha(y)\alpha(x) + \mu_1(x)\alpha(y) + \mu_2(y)\alpha(x) + \nu(x, y) \]

for all \( x, y \in S \). As before, \( \lambda_1, \lambda_2, \mu_1, \mu_2 \) and \( \nu \) are called the coefficients of \( E \), and \( E \) is said to be without constant coefficient if \( \nu = 0 \).

In this way, we can define a Beidar polynomial of degree \( m \) in \( \alpha \) which involves summands such as

\[ \lambda \alpha(x_1) \ldots \alpha(x_m), \]
\[ \mu(x_1)\alpha(x_2) \ldots \alpha(x_m), \ldots, \mu(x_m)\alpha(x_1) \ldots \alpha(x_{m-1}), \]
\[ \nu(x_1, x_2)\alpha(x_3) \ldots \alpha(x_m), \ldots, \nu(x_{m-1}, x_m)\alpha(x_1) \ldots \alpha(x_{m-2}), \]

and so on.

Now we prove a theorem which is a slight generalization of [5, Theorem 2.9] with essentially the same proof. For the sake of saving space, we write \( x^\alpha \) for \( \alpha(x) \) and \( S^\alpha \) for \( \alpha(S) \) in the statements and proofs of the following theorem and its corollary.

**Theorem 3.1.** Let \( F \) be a commutative ring with unity and \( Q \) a unital \( F \)-algebra with center \( C \) containing 1. Let \( m \geq 2 \) be an integer and \( f(x_m) \) a multilinear polynomial in indeterminates \( x_1, x_2, \ldots, x_m \) over \( F \) with at least one coefficient invertible. Let \( S \) be a nonempty subset of an \( F \)-algebra \( A \) and \( \alpha : S \to Q \) a map such that \( f(s_m) \in S \). \( f(s_m)^\alpha = \kappa f(s_1^\alpha, s_2^\alpha, \ldots, s_m^\alpha) \) for all \( s_m \in S^m \), where \( \kappa \in C \) is an invertible element, and \( S^\alpha \) is a 2m-free subset of the \( F \)-algebra \( Q \). Suppose that \( \beta : S \times S \to Q \) is a map such that
\[\beta(u, f(v_m)) = \kappa \sum_{i=1}^{m} f(v_1^\alpha, \ldots, v_{i-1}^\alpha, \beta(u, v_i), v_{i+1}^\alpha, \ldots, v_m^\alpha),\]

\[\beta(f(v_m), u) = \kappa \sum_{i=1}^{m} f(v_1^\alpha, \ldots, v_{i-1}^\alpha, \beta(v_i, u), v_{i+1}^\alpha, \ldots, v_m^\alpha),\]  

(3.31)

for all \(u, v_1, v_2, \ldots, v_m \in S\). Then there exist an element \(\gamma \in C\) and a map \(\nu : S \times S \to C\) such that

\[\beta(u, v) = \gamma[u^\alpha, v^\alpha] + \nu(u, v) \text{ for all } u, v \in S.\]

Moreover, if \(S\) is an \(F\)-submodule of \(A\), \(\alpha\) is an \(F\)-linear map and \(\beta\) is an \(F\)-bilinear map, then \(\nu\) is an \(F\)-bilinear map.

**Proof.** Without loss of generality, we may assume that the coefficient \(\gamma_0\) of the term \(x_1 x_2 \ldots x_m\) in \(f(x_m)\) is invertible. Computing \(\beta(f(u_m), f(v_m))\) in two different ways, we get

\[\sum_{i=1}^{m} \sum_{j=1}^{m} f(u_1^\alpha, \ldots, u_{i-1}^\alpha, f(v_1^\alpha, \ldots, v_{j-1}^\alpha, \beta(u_i, v_j), \ldots, v_m^\alpha), \ldots, u_m^\alpha)\]

\[= \sum_{j=1}^{m} \sum_{i=1}^{m} f(v_1^\alpha, \ldots, v_{j-1}^\alpha, f(u_1^\alpha, \ldots, u_{i-1}^\alpha, \beta(u_i, v_j), \ldots, u_m^\alpha), \ldots, v_m^\alpha)\]

for all \(u_m, v_m \in S^m\) since \(\kappa\) is invertible. Expanding both sides of the above equality, we see that the left-hand side is an \(F\)-linear combination of terms of the form

\[u_1^\alpha \ldots u_{i-1}^\alpha v_1^\alpha \ldots v_{j-1}^\alpha \beta(u_i, v_j)v_{j+1}^\alpha \ldots v_m^\alpha u_{i+1}^\alpha \ldots u_m^\alpha,\]

and the right-hand side is an \(F\)-linear combination of terms of the form

\[v_1^\alpha \ldots v_{j-1}^\alpha u_1^\alpha \ldots u_{i-1}^\alpha \beta(u_i, v_j)u_{j+1}^\alpha \ldots u_m^\alpha v_{j+1}^\alpha \ldots v_m^\alpha.\]

Note that the coefficient of the term \(\beta(u_1, v_1)v_2^\alpha \ldots v_m^\alpha u_2^\alpha \ldots u_m^\alpha\) is \(\gamma_0^2\) which is invertible. Since \(S^2\) is a \(2m\)-free subset of the ring \(Q\), it follows from [5, Theorem 1.2] that \(\beta\) is a Beidar polynomial of degree 2 in \(\alpha\), that is, there exist elements \(\lambda_1, \lambda_2 \in C\), maps \(\mu_1, \mu_2 : S \to C\) and a map \(\nu : S \times S \to C\) such that

\[\beta(u, v) = \lambda_1 u^\alpha v^\alpha + \lambda_2 v^\alpha u^\alpha + \mu_1(u)v^\alpha + \mu_2(v)u^\alpha + v(u, v)\]

for all \(u, v \in S\). Thus

\[\beta(u, f(v_m)) = \kappa \lambda_1 u^\alpha f(v_m^\alpha) + \kappa \lambda_2 f(v_m^\alpha)u^\alpha + \kappa \mu_1(u)f(v_m^\alpha)\]

\[+ \mu_2(f(v_m))u^\alpha + v(u, f(v_m))\]
for all \( u, v_1, v_2, \ldots, v_m \in S \). On the other hand,

\[
\beta(u, f(v_m)) = \kappa \sum_{i=1}^{m} f(v_i^\alpha, \ldots, v_i^{\alpha-1}, \beta(u, v_i), v_i^{\alpha+1}, \ldots, v_m^\alpha) = \kappa \sum_{i=1}^{m} f(v_i^\alpha, \ldots, \lambda_1 v_i^\alpha + \lambda_2 v_i^\alpha u^\alpha + \mu_1(u) v_i^\alpha + \mu_2(v_i) u^\alpha + v(u, v_i), \ldots, v_m^\alpha) = 0.
\]

for all \( u, v_1, v_2, \ldots, v_m \in S \). Comparing both expressions for \( \beta(u, f(v_m)) \), we get

\[
\lambda_1 u^\alpha f(v_m^\alpha) + \lambda_2 f(v_m^\alpha) u^\alpha + \mu_1(u) f(v_m^\alpha) + \kappa^{-1} \mu_2 f(v_m^\alpha) u^\alpha + \kappa^{-1} v(u, f(v_m)) = \sum_{i=1}^{m} f(v_i^\alpha, \ldots, \lambda_1 v_i^\alpha v_i^\alpha + \lambda_2 v_i^\alpha u^\alpha + \mu_1(u) v_i^\alpha + \mu_2(v_i) u^\alpha + v(u, v_i), \ldots, v_m^\alpha) = 0.
\]

for all \( u, v_1, v_2, \ldots, v_m \in S \), since \( \kappa \) is invertible. Expanding the above equality, we get a Beidar polynomial of degree \( m+1 \) in \( \alpha \) which vanishes for all \( u, v_1, v_2, \ldots, v_m \in S \). Since \( S^\alpha \) is \((m+2)\)-free, it follows from [5, Theorem 1.1] that all the coefficients of this Beidar polynomial are zero. In particular, both the coefficient \( -\gamma_0(\lambda_1 + \lambda_2) \) of the term \( v_i^\alpha v_i^\alpha \ldots v_m^\alpha \) and the coefficient \( -\gamma_0 \mu_2(v_1) \) of the term \( u^\alpha v_i^\alpha \ldots v_m^\alpha \) are zero. Hence \( \lambda_1 + \lambda_2 = 0 \) and \( \mu_2 = 0 \) since \( \gamma_0 \) is invertible. Similarly we have \( \gamma_0 = 0 \) by considering \( \beta(f(u_m, v)) \). Therefore \( \beta(u, v) = \gamma[u^\alpha, v^\alpha] + v(u, v) \) for all \( u, v \in S \), where \( \gamma = \gamma_1 = -\lambda_2 \). It is obvious that \( v \) is an \( F \)-bilinear map if \( \alpha \) is \( F \)-linear and \( \beta \) is \( F \)-bilinear. Thus the proof is complete.  

Setting \( \beta(u, v) = [u, v]^\alpha \) for \( u, v \in S \), we see that \( \beta \) satisfies the conditions (3.31) in the preceding theorem. Thus we arrive at the following corollary.

**Corollary 3.2.** Let \( F \) be a commutative ring with unity and \( Q \) a unital \( F \)-algebra with center \( C \) containing 1. Let \( m \geq 2 \) be an integer and \( f(x_m) \) a multilinear polynomial in indeterminates \( x_1, x_2, \ldots, x_m \) over \( F \) with at least one coefficient invertible. Let \( S \) be a nonempty subset of an \( F \)-algebra \( A \) and \( \alpha : S \to Q \) a map such that \( f(s_m) \in S \) for all \( s_m \in S^m \), where \( \kappa \in C \) is an invertible element, and \( S^\alpha \) is a \( 2m \)-free subset of the \( F \)-algebra \( Q \). Then there exist an element \( \gamma \in C \) and a map \( \nu : S \times S \to C \) such that \( [u, v]^\alpha = \gamma[u^\alpha, v^\alpha] + \nu(u, v) \) for all \( u, v \in S \). Moreover, if \( S \) is an \( F \)-submodule of \( A \) and \( \alpha \) is an \( F \)-linear map, then \( \nu \) is an \( F \)-bilinear map.

With all these results in hand, we are now ready to prove the following key result of this paper.

**Theorem 3.3.** Let \( F \) be a commutative ring with unity, \( A \) and \( A' \) unital \( F \)-algebras, \( \mathcal{L} \) a Lie ideal of \( A \) and \( C \) the center of \( A' \). Let \( \theta : \mathcal{L} \to A' \) be an additive map such that
θ([L, L]) \neq 0, \theta([L, L \circ L]) \neq 0 and \theta(x) \circ \theta(y) = \theta(u) \circ \theta(v) for all x, y, u, v \in L with x \circ y = u \circ v. Suppose that C contains \frac{1}{2} and \theta(L) is an 8-free subset of A'. Then

\theta([x, y \circ z]) = \lambda \theta(x) \circ \theta(y) \circ \theta(z) + v(z, x) \theta(y) + v(y, x) \theta(z)

for all x, y, z \in L, where \lambda \in C and v : L^2 \to C is a skew-symmetric bi-additive map such that \lambda v = 0. Moreover, if L is an F-submodule of A and \theta is an F-linear map, then v is an F-bilinear map. Furthermore, if C is a field, then there exists a nonzero element \gamma \in C such that \theta([x, y]) = \gamma \theta(x) \theta(y) for all x, y \in L.

Proof. Since \theta(x) \circ y + x \circ \theta(y) = [x \circ y, x \circ y] = 0, we have

\theta([x \circ y, x]) \circ \theta(y) = \theta(-x) \circ \theta([x \circ y, y]) = -\theta([x \circ y, y]) \circ \theta(x),

or equivalently,

\theta([x \circ y, x]) \circ \theta(y) + \theta([x \circ y, y]) \circ \theta(x) = 0

for all x, y \in L. Linearizing this identity, we have

(\theta([x \circ y, u]) + \theta([u \circ y, x])) \circ \theta(v) + (\theta([x \circ v, u]) + \theta([u \circ v, x])) \circ \theta(y)

+ (\theta([x \circ v, y]) + \theta([u \circ v, y])) \circ \theta(u)

+ (\theta([u \circ y, v]) + \theta([u \circ v, y])) \circ \theta(x) = 0 \quad (3.32)

for all x, y, u, v \in L. Setting

f(x_1, x_2, x_3) = \theta([x_1 \circ x_2, x_3]) + \theta([x_2 \circ x_3, x_1]) = -\theta([x_3 \circ x_1, x_2]) \quad (3.33)

for x_1, x_2, x_3 \in L, we can rewrite the identity (3.32) as

f(x, y, u) \circ \theta(v) + f(x, v, u) \circ \theta(y) + f(y, x, v) \circ \theta(u) + f(y, v, u) \circ \theta(x) = 0 \quad (3.34)

for all x, y, u, v \in L. Since \theta(L) is a 4-free subset of A', it follows from [5, Theorem 2.6] that the function f(x_1, x_2, x_3) is a Beidar polynomial of degree 3 in \theta, that is,

f(x_1, x_2, x_3) = \lambda_1 \theta(x_1) \theta(x_2) \theta(x_3) + \lambda_2 \theta(x_1) \theta(x_3) \theta(x_2) + \lambda_3 \theta(x_2) \theta(x_1) \theta(x_3)

+ \lambda_4 \theta(x_2) \theta(x_3) \theta(x_1) + \lambda_5 \theta(x_3) \theta(x_1) \theta(x_2) + \lambda_6 \theta(x_3) \theta(x_2) \theta(x_1)

+ \mu_1 (x_1) \theta(x_2) \theta(x_3) + \mu_2 (x_1) \theta(x_3) \theta(x_2) + \mu_3 (x_2) \theta(x_1) \theta(x_3)

+ \mu_4 (x_2) \theta(x_3) \theta(x_1) + \mu_5 (x_3) \theta(x_1) \theta(x_2) + \mu_6 (x_3) \theta(x_2) \theta(x_1)

+ \nu_1 (x_1, x_2) \theta(x_3) + \nu_2 (x_1, x_3) \theta(x_2) + \nu_3 (x_2, x_3) \theta(x_1)

+ \omega (x_1, x_2, x_3), \quad (3.35)
where \( \lambda_i \in C \), \( 1 \leq i \leq 6 \), \( \mu_j : \mathcal{L} \to C \), \( 1 \leq j \leq 6 \), are additive, \( v_k : \mathcal{L}^2 \to C \), \( 1 \leq k \leq 3 \), are bi-additive, and \( \omega : \mathcal{L}^3 \to C \) is tri-additive. Moreover, if \( \mathcal{L} \) is an \( F \)-submodule of \( \mathcal{A} \) and \( \theta \) is an \( F \)-linear map, then \( \mu_j \) are \( F \)-linear, \( v_k \) are \( F \)-bilinear, and \( \omega : \mathcal{L}^3 \to C \) is \( F \)-trilinear. This gives, in the left-hand side of (3.34), a Beidar polynomial of degree \( \leq 4 \) in \( \theta \) in which:

1. The coefficient of the term \( \theta(x)\theta(y)\theta(u)\theta(v) \) is \( 2\lambda_1 \).
2. The coefficient of the term \( \theta(x)\theta(u)\theta(y)\theta(v) \) is \( \lambda_2 + \lambda_3 \).
3. The coefficient of the term \( \theta(y)\theta(u)\theta(x)\theta(v) \) is \( \lambda_4 + \lambda_5 \).
4. The coefficient of the term \( \theta(u)\theta(x)\theta(y)\theta(v) \) is \( \lambda_6 + \lambda_1 \).
5. The coefficient of the term \( \theta(u)\theta(y)\theta(x)\theta(v) \) is \( \lambda_1(x) + \mu_2(x) \).
6. The coefficient of the term \( \theta(u)\theta(y)\theta(x)\theta(v) \) is \( \mu_4(y) + \mu_1(y) \).
7. The coefficient of the term \( \theta(x)\theta(y)\theta(u)\theta(v) \) is \( \mu_6(u) + \mu_3(u) \).
8. The coefficient of the term \( \theta(x)\theta(y)\theta(u)\theta(v) \) is \( -\lambda_2 \).
9. The coefficient of the term \( \theta(x)\theta(y)\theta(u)\theta(v) \) is \( 2v_2(x,u) \).
10. The coefficient of the term \( \theta(x)\theta(y)\theta(u)\theta(v) \) is \( \nu_3(y,u) + \nu_1(y,u) \).
11. The coefficient of the term \( \theta(x)\theta(y)\theta(u)\theta(v) \) is \( \lambda_1(x) + \lambda_3 \).
12. The coefficient of the term \( \theta(x)\theta(y)\theta(u)\theta(v) \) is \( \lambda_2 + \lambda_3 \).
13. The coefficient of the term \( \theta(x)\theta(y)\theta(u)\theta(v) \) is \( \lambda_4 + \lambda_5 \).
14. The coefficient of the term \( \theta(x)\theta(y)\theta(u)\theta(v) \) is \( \lambda_6 + \lambda_1 \).
15. The coefficient of the term \( \theta(x)\theta(y)\theta(u)\theta(v) \) is \( \lambda_1(x) + \mu_2(x) \).

Now, it follows from [5, Theorem 1.1] that all the coefficients of the Beidar polynomial obtained from (3.34) are 0. In particular, all the coefficients listed above are 0. Since \( \frac{1}{2} \in \mathcal{A} \), we have \( \lambda_1 = \lambda_6 = 0 \), \( \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0 \), \( \mu_i = 0 \), \( 1 \leq i \leq 6 \), \( \nu_2 = 0 \), \( \nu_1 = \nu_3 = \nu = 0 \) and \( \nu(x_1, x_2) = -\nu(x_2, x_1) \) for all \( x_1, x_2 \in \mathcal{L} \). Thus (3.35) reduces to

\[
\begin{align*}
f(x_1, x_2, x_3) & = -\lambda \left( \theta(x_1)\theta(x_3)\theta(x_2) - \theta(x_2)\theta(x_1)\theta(x_3) - \theta(x_2)\theta(x_3)\theta(x_1) + \theta(x_3)\theta(x_1)\theta(x_2) \right) \\
 & + v(x_1, x_2)\theta(x_3) - v(x_2, x_3)\theta(x_1) \\
 & = \lambda \left[ \theta(x_2), \theta(x_3) \circ \theta(x_1) \right] + v(x_1, x_2)\theta(x_3) + v(x_3, x_2)\theta(x_1) .
\end{align*}
\]

Since \( \theta([x, y \circ z]) = -\theta([y \circ z, x]) = f(z, x, y) \), the above identity can be written as

\[
\theta([x, y \circ z]) = \lambda \left[ \theta(x), \theta(y) \circ \theta(z) \right] + v(z, x)\theta(y) + v(y, x)\theta(z) \tag{3.36}
\]

for all \( x, y, z \in \mathcal{L} \). From \( [y, y \circ z] \circ z + y \circ [z, y \circ z] = 0 \), it follows that

\[
\theta([y, y \circ z]) \circ \theta(z) + \theta(y) \circ \theta([z, y \circ z]) = 0 \quad \text{and} \quad \theta([x, [y, y \circ z] \circ z]) + \theta([x, y \circ [z, y \circ z]]) = 0 \tag{3.37}
\]

for all \( x, y, z \in \mathcal{L} \). Using the expression in (3.36), the previous identity (3.38) can be written as
\[
\lambda \left[ \theta(x), \theta([y, y \circ z]) \circ \theta(z) \right] + \nu([z, y \circ z], x) \theta(z) + \nu([y, y \circ z], x) \theta(y) + \nu(y, x) \theta([z, y \circ z]) = 0
\]
and hence
\[
\nu([z, y \circ z], x) \theta(y) + \nu(y, x) \theta([z, y \circ z]) = 0 \quad (3.39)
\]
because of (3.37). By (3.36) we have
\[
\theta([y, y \circ z]) = \lambda (\theta(y)^2 \theta(z) - \theta(z) \theta(y)^2) + \nu(y, z) \theta(y),
\]
\[
\theta([z, y \circ z]) = \lambda (\theta(z)^2 \theta(y) - \theta(y) \theta(z)^2) + \nu(z, y) \theta(z).
\]
Using these to expand (3.39), we get
\[
\lambda \nu(x, y) (\theta(y)^2 \theta(z) - \theta(z) \theta(y)^2) + \lambda \nu(y, x) (\theta(z)^2 \theta(y) - \theta(y) \theta(z)^2)
+ (\nu([z, y \circ z], x) + \nu(z, x) \nu(y, z)) \theta(y) + (\nu(x, y) \nu(y, z) + \nu([y, y \circ z], x)) \theta(z) = 0
\]
for all \(x, y, z \in L\). Since \(\theta(L)\) is a 5-free subset of \(A'\) and \(\frac{1}{2} \in A'\), it follows from [5, Corollary 2.12] that all the coefficients of the Beidar polynomial in the above identity are 0. In particular, we have
\[
\lambda \nu(x, y) = 0 \quad \text{for all } x, y \in L.
\]
This together with (3.36) proves the first part of theorem.

Next suppose that \(C\) is a field. We shall show that \(\nu = 0\). Assume on the contrary that \(\nu \neq 0\). Then \(\lambda = 0\) since \(C\) is a field. Thus (3.36) reduces to
\[
\theta([x, y \circ z]) = \nu(z, x) \theta(y) + \nu(x, y) \theta(z) \quad (3.40)
\]
for all \(x, y, z \in L\). Note that
\[
[x, x \circ y] \circ y = (x \circ [x, y]) \circ y = [x^2, y] \circ y = [x^2 \circ y, y] \in L
\]
for \(x, y \in L\). We use (3.40) to expand \(\theta([[x, x \circ y] \circ y, [u, u \circ v] \circ v])\) in two different ways:
\[
\theta([[x, x \circ y] \circ y, [u, u \circ v] \circ v]) = \nu([x, x \circ y] \circ y) \theta([u, u \circ v]) + \nu([x, x \circ y] \circ y, [u \circ v, v]) \theta(v)
\]
and
\[
\theta([[x, x \circ y] \circ y, [u, u \circ v] \circ v]) = \nu([x, x \circ y] \circ y) \nu(v, u) \theta(u) + \nu([x, x \circ y] \circ y, [u \circ v, v]) \theta(v),
\]
\[\theta([x, x \circ y] \circ y, [u, u \circ v] \circ v) = -\theta([u, u \circ v] \circ v, [x, x \circ y] \circ y) = -\nu(y, [u, u \circ v] \circ v)\nu(y, x)\theta(x) - \nu([u, u \circ v] \circ v, [x \circ y, y])\theta(y).\]

Therefore,
\[\nu(v, [x, x \circ y] \circ y)\nu(v, u)\theta(u) + \nu([x, x \circ y] \circ y, [u, u \circ v] \circ v)\theta(v) = 0\]

for all \(x, y, u, v \in L\), and so
\[\nu(v, [x, x \circ y] \circ y) = 0\]

for all \(x, y, v \in L\). Note that \([x, y] \circ y = [x, y^2] \in L\) for \(x, y \in L\). We use (3.40) again to expand \(\theta([x, y] \circ y, [u, u \circ v] \circ v])\) in two ways to get, on one hand,
\[\theta([x, y] \circ y, [u, u \circ v] \circ v) = -\theta([u, u \circ v] \circ v, [x, y] \circ y) = -\nu(y, [u, u \circ v] \circ v)\nu(y, x)\theta(x) - \nu([u, u \circ v] \circ v, [x \circ y, y])\theta(y) = 0,\]

and on the other hand,
\[\theta([x, y] \circ y, [u, u \circ v] \circ v) = \nu(v, [x, y] \circ y)\nu(v, u)\theta(u) + \nu([x, y] \circ y, [u, u \circ v] \circ v)\theta(v) = 0\]

Hence,
\[\nu(v, [x, y] \circ y)\nu(v, u)\theta(u) + \nu([x, y] \circ y, [u, u \circ v] \circ v)\theta(v) = 0\]

for all \(x, y, u, v \in L\). Since \(\theta(L)\) is 8-free and \(\frac{1}{2} \in A',\) we conclude by [5, Corollary 2.12] again that
\[\nu(v, [x, y] \circ y)\nu(v, u) = 0,\]

for all \(x, y, u, v \in L\) and so \(\nu(v, [x, y] \circ y) = 0\), or equivalently,
\[\nu(v, [x, y^2]) = 0\]
for all $x, y, v \in \mathcal{L}$. Linearizing the above identity, one gets

$$
\nu(v, [x, y \circ z]) = 0
$$

for all $x, y, z, v \in \mathcal{L}$. Finally, for any $x, y, z \in \mathcal{L}$, we have

$$
0 = \theta([x, [y \circ z, y \circ z]])
= \theta([x, [y, y \circ z] \circ z]) + \theta([x, y \circ [z, y \circ z]])
= \nu(z, x) \theta([y, y \circ z]) + \nu([x, [y, y \circ z]]) \theta(z)
+ \nu([z, y \circ z], x) \theta(y) + \nu(x, y) \theta([z, y \circ z])
= \nu(z, x) \theta([y, y \circ z]) + \nu([x, [y, y \circ z]]) \theta(z)
+ \nu(z, x) \nu(z, y) \theta(y)
+ \nu(x, y) \nu(z, x) \theta(z).
$$

Since $\theta(\mathcal{L})$ is 5-free and $\frac{1}{2} \in \mathcal{A}$, it follows from [5, Corollary 2.12] that $\nu(v, x) = 0$ for all $x, y, z \in \mathcal{L}$, and so $\nu(v, x) = 0$ for all $y, z \in \mathcal{L}$, which contradicts our assumption. Therefore, we have $\nu = 0$ as claimed.

Now, (3.36) gives us the identity

$$
\theta([x, y \circ z]) = \lambda \left[ \theta(x) \circ \theta(y) \right]
$$

(3.41)

for all $x, y, z \in \mathcal{L}$. By the assumption that $\theta([\mathcal{L}, \mathcal{L} \circ \mathcal{L}]) \neq 0$, it follows that $\lambda$ is not zero. By Corollary 3.2, we have

$$
\theta([x, y]) = \gamma \left[ \theta(x), \theta(y) \right] + \phi(x, y)
$$

(3.42)

for all $x, y \in \mathcal{L}$, where $\gamma$ is an element in $C$ and $\phi$ is a map from $\mathcal{L}^2$ to $C$.

We claim that, for $x_i, y_i \in \mathcal{L}$, if $\sum_{i=1}^{m} x_i \circ y_i = 0$, then $\sum_{i=1}^{m} \theta(x_i) \circ \theta(y_i) \in C$. Indeed, for $z \in \mathcal{L}$, it follows from (3.41) that

$$
0 = \theta\left( \left[ z, \sum_{i=1}^{m} x_i \circ y_i \right] \right) = \sum_{i=1}^{m} \theta\left( [z, x_i \circ y_i] \right) = \sum_{i=1}^{m} \lambda \left[ \theta(z), \theta(x_i) \circ \theta(y_i) \right]
= \lambda \left[ \theta(z), \sum_{i=1}^{m} \theta(x_i) \circ \theta(y_i) \right].
$$

Since $\lambda \neq 0$ and $\theta(\mathcal{L})$ is a 1-free subset of $\mathcal{A}$, this implies that $\sum_{i=1}^{m} \theta(x_i) \circ \theta(y_i) \in C$.

Finally, note that

$$
[x \circ y, z, u \circ v] = [x, z, u \circ v] + [x \circ [y, z], u \circ v]
= [x, z, u \circ v] \circ y + [x, z] \circ [y, u \circ v] + [x, u \circ v] \circ [y, z]
+ x \circ [y, z, u \circ v]
$$
on one hand, and
\[ [x \circ y, z], u \circ v] = [x \circ y, z], u \circ v + u \circ [x \circ y, z], v. \]
on the other hand. Therefore,
\[ [x, z], u \circ v] \circ y + [x, z] \circ [y, u \circ v] + [x, u \circ v] \circ [y, z] + x \circ [y, z], u \circ v \]
\[ = [x \circ y, z], u] \circ v - u \circ [x \circ y, z], v = 0. \]

(3.43)

Thus, for \( x, y, z, u, v \in \mathcal{L} \), we have by the preceding paragraph that
\[
\begin{align*}
\theta([x, z], u \circ v) & \circ \theta(y) + \theta([x, z]) \circ \theta((y, u \circ v)) + \theta([x, u \circ v]) \circ \theta([y, z]) \\
& + \theta(x) \circ \theta([y, z], u \circ v) - \theta([x \circ y, z], u) \circ \theta(v) - \theta(u) \circ \theta([x \circ y, z], v]) \in C.
\end{align*}
\]

Using (3.41)–(3.43) to expand the above relation, we have
\[
-2\lambda \phi(x, z) \theta((y), \theta(u) \circ \theta(v)) - 2\lambda \phi(y, z) \theta((x), \theta(u) \circ \theta(v)) - 2\phi((x \circ y, z), u) \theta(v) - 2\phi((x \circ y, z), v) \theta(u) \in C.
\]

Since \( \theta(\mathcal{L}) \) is a 6-free subset of \( \mathcal{A}' \), it follows from [5, Theorem 1.1] that all the coefficients of the Beidar polynomial obtained above are 0. In particular, the coefficient \(-2\lambda \phi(y, z)\) of the term \( \theta(x) \theta(u) \theta(v) \) is zero. Since \( \lambda \neq 0 \) and \( \frac{1}{2} \in \mathcal{A}' \), we have \( \phi = 0 \). Therefore, we can now conclude that
\[
\theta[x, y] = \gamma[\theta(x), \theta(y)]
\]
for all \( x, y \in \mathcal{L} \). Moreover, by the assumption that \( \theta([\mathcal{L}, \mathcal{L}]) \neq 0 \), we see that \( \gamma \neq 0 \). The proof is now complete. \( \square \)

To conclude the section we recall the following result [6, Theorem 2.7] which will be used in the next section. For a subset \( S \) of an \( F \)-algebra, we use \( \langle S \rangle \) to denote the \( F \)-subalgebra generated by \( S \).

**Theorem 3.4.** Let \( F \) be a commutative ring with \( \frac{1}{2} \), \( Q \) a unital \( F \)-algebra with center \( C \) containing 1, and \( \pi : Q \to \overline{Q} \) the canonical map of \( Q \) onto the factor Lie algebra \( \overline{Q} = Q / C \). Let \( S \) be a Lie ideal of an \( F \)-algebra \( D \), and \( \alpha : S \to \overline{Q} \) an \( F \)-linear map such that
\[
\alpha([x, y]) = \lambda[\alpha(x), \alpha(y)] \quad \text{for all } x, y \in S,
\]
where \( \lambda \in C \) is an invertible element. Suppose that the inverse image \( R = \pi^{-1}(\alpha(S)) \) is a 7-free subset of \( Q \) and \( C \) is a direct summand of the \( C \)-module \( Q \). Then there exist an idempotent \( e \in C \) and an \( F \)-linear map \( \beta : \langle S \rangle \to \langle R \rangle \) such that \( \alpha(x) = \lambda^{-1}(2e - 1)\beta(x) + C \in \overline{Q} \) for all \( x \in S \) and \( \beta(xy) = e\beta(x)\beta(y) + (1 - e)\beta(y)\beta(x) \) for all \( x, y \in \langle S \rangle \).
Note that the condition on $\beta$ implies in particular that $\beta$ is a sum of a homomorphism $e\beta$ and an anti-homomorphism $(1-e)\beta$, and so $\beta$ is a Jordan homomorphism.

4. Main theorem

Throughout this section $F$ will be a commutative ring with $\frac{1}{2}$, and $\mathcal{A}$ will be an algebra over $F$ which is generated by the matrix units $\{e_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq n\}$. By [3, Corollary 5.12] the Lie ideal $\mathcal{L} = [\mathcal{A}, \mathcal{A}]$ is $(n-1)$-free. Recall that $\mathcal{L}$ is an $F$-module generated by

$$\mathcal{E} = \{e_{ij} \mid i \neq j\} \cup \{e_{ii} - e_{jj} \mid i \neq j\}.$$ 

For distinct $i, j, k$, we see that $e_{ii} = \frac{1}{2}(e_{ij} - e_{jj}) \circ (e_{ii} - e_{kk}) \in \mathcal{L} \circ \mathcal{L}$, $e_{ij} = e_{ik} \circ e_{kj} \in \mathcal{L} \circ \mathcal{L}$, $e_{ij} = [e_{ik}, e_{kj}] \in [\mathcal{L}, \mathcal{L}]$ and $e_{ii} - e_{jj} = [e_{ij}, e_{ji}] \in [\mathcal{L}, \mathcal{L}]$. Hence we have $\mathcal{L} \circ \mathcal{L} = \mathcal{A}$, $[\mathcal{L}, \mathcal{L}] = \mathcal{L}$ and so $[\mathcal{L}, \mathcal{L} \circ \mathcal{L}] = \mathcal{L}$.

Now we are in a position to prove the main theorem of this paper.

**Theorem 4.1.** Suppose that $n \geq 9$ and $\theta : \mathcal{L} \to \mathcal{L}$ is a surjective $F$-linear map such that $\theta(x)^2 = 0$ for all $x \in \mathcal{L}$ with $x^2 = 0$. Then there exist an invertible element $\lambda \in F$, an idempotent $e \in F$ and an $F$-linear map $\beta : \mathcal{A} \to \mathcal{A}$ such that $\theta(x) = \lambda \beta(x)$ for all $x \in \mathcal{L}$ and $\beta(xy) = e\beta(x)\beta(y) + (1-e)\beta(y)\beta(x)$ for all $x, y \in \mathcal{A}$.

**Proof.** By Theorem 2.3, for $x, y, u, v \in \mathcal{L}$ with $x \circ y = u \circ v$, we have $\theta(x) \circ \theta(y) = \theta(u) \circ \theta(v)$. Since $\theta([\mathcal{L}, \mathcal{L}]) = \theta([\mathcal{L}, \mathcal{L} \circ \mathcal{L}]) = \theta(\mathcal{L}) = \mathcal{L} \neq 0$, it follows from Theorem 3.3 that there exist an element $\xi \in F$ and a skew-symmetric $F$-bilinear map $\nu : \mathcal{L}^2 \to F$ such that $\xi \nu = 0$ and

$$\theta([x, y \circ z]) = \xi\left[\theta(x), \theta(y) \circ \theta(z)\right] + \nu(z, x)\theta(y) + \nu(y, x)\theta(z) \quad (4.44)$$

for all $x, y, z \in \mathcal{L}$. Note that, for $y, z \in \mathcal{L}$, we have

$$\theta([y, z^2]) = \frac{1}{2}\theta([y, z \circ z])$$

$$= \frac{1}{2}\left[\theta([y, z] \circ \theta(z)) + \nu([z, z])\theta(y) + \nu(y, z)\theta(z)\right]$$

$$= \xi\left[\theta(y), \theta(z)^2\right] + \nu(z, y)\theta(z).$$

For $x, y, z \in \mathcal{L}$, we expand $\theta([x, [y, z]])$ to get

$$\theta([x, [y, z]]) = \theta([x, y \circ [y, z]])$$

$$= \xi\left[\theta(x), \theta(y) \circ \theta([y, z])\right] + \nu([z, [y, z]], x)\theta(y) + \nu(y, x)\theta([y, z])$$

$$= \xi\left[\theta(x), \theta(y)^2, \theta(z)^2\right] + \nu([z, [y, z]], x)\theta(y) + \nu(y, x)\nu(z, y)\theta(z),$$

$$= \xi^2\left[\theta(x), [\theta(y)^2, \theta(z)^2]\right] + \nu([z, [y, z]], x)\theta(y) + \nu(y, x)\nu(z, y)\theta(z),$$

$$= \xi^2\left[\theta(x), [\theta(y)^2, \theta(z)^2]\right] + \nu([z, [y, z]], x)\theta(y) + \nu(y, x)\nu(z, y)\theta(z),$$

$$= \xi^2\left[\theta(x), [\theta(y)^2, \theta(z)^2]\right] + \nu([z, [y, z]], x)\theta(y) + \nu(y, x)\nu(z, y)\theta(z).$$
on one hand, and
\[
\theta([x, [y^2, z^2]]) = -\theta([x, [z^2, y^2]]) = -\zeta^2 \theta(x) - \theta(y^2) - \theta(z^2) - \nu([x, [y, z^2]])\theta(x) - \nu(y, z)\theta(y),
\]

on the other hand. Comparing both expressions for \(\theta([x, [y^2, z^2]])\), we get
\[
(v([y, z^2], x) + v(z, x)v(y, z)\theta(y) + (v([y^2], x) + v(y, x)v(z, y))\theta(z) = 0
\]
for all \(x, y, z \in \mathcal{L}\). Since \(\mathcal{L}\) is 5-free and \(\frac{1}{2} \in A\), it follows from [5, Corollary 2.12] that
\[
v([y, z^2], x) = v(x, z)v(y, z)
\]
for all \(x, y, z \in \mathcal{L}\) and so
\[
v([x, z^2], y) = v([y, z^2], x)
\]
for all \(x, y, z \in \mathcal{L}\).

We claim that \(\nu = 0\). By the \(F\)-bilinearity of \(\nu\), it suffices to show that
\[
v(e_{ij}, e_{kl}) = 0,
\]
\[
v(e_{ii} - e_{jj}, e_{kl}) = 0 \quad \text{and}
\]
\[
v(e_{ii} - e_{jj}, e_{kk} - e_{ll}) = 0,
\]
for all \(i, j, k, l \in \{1, 2, \ldots, n\}\) with \(i \neq j\) and \(k \neq l\). Choosing \(r \notin \{i, j, k, l\}\) and setting \(x = e_{ir}, y = e_{ri} + e_{ij}\) and \(y = e_{rl}\) in (4.46), we see that
\[
v(e_{ij}, e_{kl}) = v([e_{ir}, e_{rij}], e_{kl}) = v([e_{ir} + e_{ij}, e_{kl}] = v([e_{kl}, (e_{ri} + e_{ij})^2], e_{ir})
\]
\[
= v([e_{rl}, e_{rij}], e_{ir}) = 0
\]
provided \(j \neq k\). And so
\[
v(e_{ij}, e_{kl}) = -v(e_{kl}, e_{ij}) = 0
\]
provided \(i \neq l\). As to the case when \(j = k\) and \(i = l\), we choose two distinct integers \(r, s \notin \{i, j\}\) and set \(x = e_{ji}, y = e_{ir}\) and \(z = e_{rs} + e_{sj}\) in (4.45) to see that
\[
v(e_{ij}, e_{ji}) = v([e_{ir}, e_{ri}], e_{ji}) = v([e_{ir}, (e_{rs} + e_{sj})^2], e_{ji})
\]
\[
= v(e_{ji}, e_{rs} + e_{sj})v(e_{ir}, e_{rs} + e_{sj}) = 0.
\]
This proves (4.47). Similarly we can choose \( r \not\in \{ i, j, k, l \} \) and prove (4.48) by using (4.47) and setting \( x = e_{kl}, y = e_{ij} \) and \( z = e_{jr} + e_{ri} \) in (4.45). And, (4.49) can be proved by using (4.47) and (4.48) and setting \( x = e_{kk} - e_{ll}, y = e_{ij} \) and \( z = e_{jr} + e_{ri} \) in (4.45).

Therefore \( \nu(x, y) = 0 \) for all \( x, y \in L \) and (4.44) reduces to

\[
\theta([x, y \circ z]) = \zeta[\theta(x), \theta(y) \circ \theta(z)]
\]

(4.50)

for all \( x, y, z \in L \). Recall that \([L, L \circ L] = L \) and so

\[
\theta([L, L \circ L]) = \theta([L], \theta(L) \circ \theta(L)) = L.
\]

Thus (4.50) implies \( \zeta L = L \) and hence \( \zeta \) is an invertible element in \( F \). By Corollary 3.2 there exists \( \gamma \in F \) such that \( \theta([x, y]) = \gamma[\theta(x), \theta(y)] \) for all \( x, y \in L \). This identity implies \( \gamma L = L \) and hence \( \gamma \) is an invertible element in \( F \). Note that \( F \) is a direct summand of the \( F \)-module \( A \), and \( A \) is an \( F \)-algebra generated by \( L \). By Theorem 3.4, there exist an idempotent \( e \in F \) and an \( F \)-linear map \( \beta : A \rightarrow A \) such that \( \theta(x) = \lambda \beta(x) \) for all \( x \in L \) and \( \beta(xy) = e \beta(x) \beta(y) + (1 - e) \beta(y) \beta(x) \) for all \( x, y \in A \), where \( \lambda = \gamma^{-1}(2e - 1) \in F \) is also invertible since \((2e - 1)^2 = 1\). This completes the proof. □

Adapting the proof of Theorem 4.1 while using Remark 2.4 in place of Theorem 2.3, we obtain a slight generalization of Theorem 4.1 as follows.

**Theorem 4.2.** Suppose that \( n \geq 9 \) and \( \theta : L \rightarrow L \) is a surjective \( F \)-linear map such that \( \theta(x)^2 = 0 \) for all \( x \in L \cap f \mathcal{A}(1 - f) \) where \( f \) is an idempotent in \( \mathcal{A} \). Then there exist an invertible \( \lambda \in F \), an idempotent \( e \in F \) and an \( F \)-linear map \( \beta : \mathcal{A} \rightarrow \mathcal{A} \) such that \( \theta(x) = \lambda \beta(x) \) for all \( x \in L \) and \( \beta(xy) = e \beta(x) \beta(y) + (1 - e) \beta(y) \beta(x) \) for all \( x, y \in \mathcal{A} \).

**Proof.** By Remark 2.4, for \( x, y, u, v \in L \) with \( x \circ y = u \circ v \), we have \( \theta(x) \circ \theta(y) = \theta(u) \circ \theta(v) \). Then the rest arguments proving the preceding theorem work as well to prove this one. □

**References**