Physics Procedia
Volume 75, 2015, Pages 802–812
20th International Conference on Magnetism

Spin Glass Field Theory with Replica Fourier Transforms

Iveta R. Pimentel[1] and Cirano De Dominicis[2]

[1] Departamento de Fisica and CFTC, Universidade de Lisboa, 1749-016 Lisboa, Portugal
irpimentel@fc.ul.pt
[2] CEA Saclay Orme des Merisiers, Inst Phys Theor, F-91191 Gif Sur Yvette, France
cirano.de-dominicis@cea.fr

Abstract

We develop a field theory for spin glasses using Replica Fourier Transforms (RFT). We present the formalism for the case of replica symmetry and the case of replica symmetry breaking on an ultrametric tree, with the number of replicas n and the number of replica symmetry breaking steps R generic integers. We show how the RFT applied to the two-replica fields allows to construct a new basis which block-diagonalizes the four-replica mass-matrix, into the replicon, anomalous and longitudinal modes. The eigenvalues are given in terms of the mass RFT and the propagators in the RFT space are obtained by inversion of the block-diagonal matrix. The formalism allows to express any i-replica vertex in the new RFT basis and hence enables to perform a standard perturbation expansion. We apply the formalism to calculate the contribution of the Gaussian fluctuations around the Parisi's solution for the free-energy of an Ising spin glass.

Keywords: spin glasses, field theory, replica Fourier transforms

1 Introduction

Spin glasses are disordered magnetic systems with frustration. These systems exhibit a *freezing* transition to a low temperature phase with nontrivial properties [1]. There is still no consensus on the nature of the glassy phase. Two different pictures have been proposed. One corresponds to the Parisi solution [7, 8] of the infinite-range Sherrington-Kirkpatrick model [10], which represents the mean field theory for spin glasses and predicts a glassy phase described by an infinite number of pure states, organized in an ultrametric structure. The other one is the "droplet" model [2, 5], which claims that in the experimentally relevant short-range spin glasses the glassy phase is described by only two pure states, related by a global inversion of the spins. An important step for the understanding of spin glasses, lies in the investigation of how the fluctuations, associated into the finite-range interactions modify the mean-field picture.

 Selection and peer-review under responsibility of the Scientific Programme Committee of ICM 2015

doi:10.1016/j.phpro.2015.12.104

Edwards and Anderson [4] introduced a model for short-range spin glasses and used the replica method to perform the average over quenched disorder. A field theory is built for the spin glass with the free energy being written as a functional of replica fields Q_i^{ab}, which represent the spin glass order parameter. In mean field theory it is found that a phase transition occurs at a critical temperature, from a high-temperature phase with replica symmetry (RS) to a low-temperature phase with replica symmetry breaking (RSB). The RSB ansatz proposed by Parisi for the spin glass, represents many states in a hierarchical organization that is described by an ultrametric tree. See De Dominicis *et al* [6] for a review on the spin glass field theory, in direct replica space, with RSB. The high complexity of the theory has however inhibited the study of the glassy phase.

In this article we develop a field theory for spin glasses using Replica Fourier Transforms (RFT) [3]. We consider both the case of RS and the case of RSB on an ultrametric tree. We define a new basis in terms of the RFT of the two-replica fields which block-diagonalizes the four-replica mass-matrix, into three sets of modes, replicon, anomalous and longitudinal. The corresponding eigenvalues are given in terms of the mass RFT. The propagators in the RFT space are then readily obtained by inversion of the block-diagonal mass-matrix. The formalism allows to express any i-replica vertex in the new RFT basis, and hence enables to perform a standard perturbation expansion. We keep the number n of replicas a positive integer, the limit $n \to 0$ of the replica method can be taken on the final results. The number of RSB steps R is also considered a generic integer, the case of full RSB, proposed by Parisi, corresponding to the limit $R \to \infty$. We show that many fundamental results for the study of spin glasses, can be simply derived within the RFT formalism. We apply the formalism to calculate the contribution of the Gaussian fluctuations around the Parisi solution for the free energy of an Ising spin glass. A detailed presentation of this work is given in [9].

2 Spin Glass Model

We consider an Ising spin glass in a uniform magnetic field H, described by the Edwards-Anderson model

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} S_i S_j - H \sum_i S_i \tag{1}$$

for N spins, $S_i = \pm 1$, located on a regular d-dimensional lattice, where the bonds J_{ij}, which couple nearest-neighbor spins only, are independent random variables with a Gaussian distribution, characterized by zero mean and variance $\Delta^2 = J^2/z$, $z = 2d$ being the coordination number.

The free energy averaged over the quenched disorder is given, via the replica method, by

$$\overline{F} = -\frac{1}{\beta}\overline{\ln Z} = -\frac{1}{\beta}\lim_{n\to 0}\frac{\overline{Z^n}-1}{n} \tag{2}$$

where Z is the partition function and $\beta = 1/k_B T$.

Taking the average of n replicas of the partition function $\overline{Z^n}$, with n integer, followed by a Hubbard-Stratonovich transformation, to decouple a four-spin term, leads to

$$\overline{Z^n} = \int \prod_{(ab);i} \frac{dQ_i^{ab}}{\sqrt{2\pi}} \exp\left\{-\mathcal{L}\left[Q_i^{ab}\right]\right\} \tag{3}$$

where the fields Q_i^{ab}, with replica index $a = 1, \ldots, n$, are defined on an $n(n-1)/2$-dimensional replica space of pairs (ab) of distinct replicas, since $Q_i^{ab} = Q_i^{ba}$ and $Q_i^{aa} = 0$.

A perturbation expansion around the mean-field solution is constructed by separating the field Q_i^{ab} into $Q_i^{ab} = Q^{ab} + \phi_i^{ab}$, where Q^{ab} represents the mean field order parameter and ϕ_i^{ab} are fluctuations around it. The Lagrangian $\mathcal{L}$ is then given by

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \ldots \tag{4}$$

where, after Fourier transform into momenta space, one has, for contributions up to quadratic order in the fluctuations,

$$\mathcal{L}^{(0)} = -\frac{Nn(\beta J)^2}{4} + \frac{N(\beta J)^2}{2}\sum_{(ab)}(Q^{ab})^2 - N\ln \underset{\{S^a\}}{\mathrm{tr}} \exp\left\{(\beta J)^2 \sum_{(ab)} Q^{ab}S^aS^b + \beta H \sum_a S^a\right\} \tag{5}$$

$$\mathcal{L}^{(1)} = \sqrt{N}(\beta J)^2 \sum_{(ab)} \left[Q^{ab} - \left\langle S^aS^b\right\rangle\right] \phi^{ab}_{\mathbf{p}=\mathbf{0}} \tag{6}$$

$$\mathcal{L}^{(2)} = \frac{1}{2}\sum_{(ab)(cd)}\sum_{\mathbf{p}} \phi^{ab}_{\mathbf{p}} M^{ab,cd}(\mathbf{p})\phi^{cd}_{-\mathbf{p}} \tag{7}$$

with

$$M^{ab,cd}(\mathbf{p}) = \mathbf{p}^2\delta^{Kr}_{ab,cd} + z\left[\delta^{Kr}_{ab,cd} - (\beta J)^2\left(\left\langle S^aS^bS^cS^d\right\rangle - \left\langle S^aS^b\right\rangle\left\langle S^cS^d\right\rangle\right)\right] \tag{8}$$

where $S^a = S^a_i$, and the expectation value $\langle\cdots\rangle$ is calculated with the normalized weight $\zeta(S)/\mathrm{Tr}\zeta(S)$, where $\zeta(S) = \exp\left\{(\beta J)^2\sum_{(ab)} Q^{ab}S^aS^b + \beta H\sum_a S^a\right\}$.

The mean-field value of the order parameter Q^{ab} is determined by the stationarity condition $\mathcal{L}^{(1)} = 0$, which from (6) gives

$$Q^{ab} = \left\langle S^aS^b\right\rangle. \tag{9}$$

In zero magnetic field, $H = 0$, there is a phase transition at a critical temperature $T_c = J/k_B$ from a high-temperature RS to a low-temperature RSB phase. In a nonzero magnetic field, $H \neq 0$, the phase transition occurs along a line in the field-temperature plane, the Almeida-Thouless line.

The normal modes of the fluctuations of the order parameter are obtained by re-writing $\mathcal{L}^{(2)}$, (7), in a diagonal form. The propagators are then easily obtained by inversion of the diagonalized matrix.

3 Replica Symmetric Ansatz

Here we consider that the mean-field order parameter is replica symmetric

$$Q^{ab} = Q, \quad a \neq b. \tag{10}$$

In this case, there are three distinct masses

$$M^{ab,ab} = M_{11}, \quad M^{ab,ac} = M^{ab,bc} = M_{10}, \quad M^{ab,cd} = M_{00}. \tag{11}$$

The Lagrangian term of the fluctuations $\mathcal{L}^{(2)}$, (7), then takes the form

$$\mathcal{L}^{(2)} = \frac{1}{2}\left\{M_{11}\sum_{(ab)}\phi^2_{ab} + M_{10}\sum_{(abc)}(\phi_{ab}\phi_{ac} + \phi_{ab}\phi_{bc}) + M_{00}\sum_{(abcd)}\phi_{ab}\phi_{cd}\right\} \tag{12}$$

where the dependence on momentum $\mathbf{p}$ is implicit and the sums are restricted to distinct replicas.

One can write $\mathcal{L}^{(2)}$ in terms of sums over unrestricted replicas, with the field constraints

$$\phi_{aa} = 0, \quad a = 1, \ldots, n. \tag{13}$$

The RFT for the two-replica fields, and its inverse transformation, are defined as

$$\phi_{\hat{a}\hat{b}} = \frac{1}{n}\sum_{ab} e^{-\frac{2\pi i}{n}(a\hat{a}+b\hat{b})}\phi_{ab}, \quad \phi_{ab} = \frac{1}{n}\sum_{\hat{a}\hat{b}} e^{\frac{2\pi i}{n}(a\hat{a}+b\hat{b})}\phi_{\hat{a}\hat{b}} \tag{14}$$

with $a = 1, \ldots, n$, $\hat{a} = 0, \ldots, n-1$, and a, $\hat{a}$ considered mod(n). The fields can be written as $\phi_{\hat{a}\hat{b}} = \phi_{\hat{a},\hat{t}-\hat{a}}$, where $\hat{t} = \hat{a} + \hat{b}$.

One then obtains the expression of $\mathcal{L}^{(2)}$ in terms of the RFT of the two-replica fields, with the field constraints in (13) given by

$$\sum_{\hat{a}=0}^{n-1} \phi_{\hat{a},\hat{t}-\hat{a}} = 0, \qquad \hat{t} = 0, \ldots, n-1. \tag{15}$$

Now, one separates in $\mathcal{L}^{(2)}$ the fields with indices $\hat{0}$, (using $\hat{0}$ when $\hat{a} = 0$), and define the new fields,

$$\Phi_{\hat{a}',-\hat{a}'} = \phi_{\hat{a}',-\hat{a}'} + \frac{1}{n-1}\phi_{\hat{0},\hat{0}} \tag{16}$$

$$\Phi_{\hat{a}'',\hat{t}'-\hat{a}''} = \phi_{\hat{a}'',\hat{t}'-\hat{a}''} + \frac{1}{n-2}\left(\phi_{\hat{0},\hat{t}'} + \phi_{\hat{t}',\hat{0}}\right) \tag{17}$$

with $\hat{a}' \neq \hat{0}$, $\hat{t}' \neq \hat{0}$ and $\hat{a}'' \neq \hat{0}, \hat{t}'$.

Introducing the new fields, one obtains $\mathcal{L}^{(2)}$ in the diagonal form,

$$\mathcal{L}^{(2)} = \frac{1}{2}\left\{ M_R \sum_{\hat{a}'} \left|{}^R\Phi_{\hat{a}',-\hat{a}'}\right|^2 + M_R \sum_{\hat{t}',\hat{a}''} \left|{}^R\Phi_{\hat{a}'',\hat{t}'-\hat{a}''}\right|^2 + M_A \sum_{\hat{t}'} \left|{}^A\phi_{\hat{0},\hat{t}'}\right|^2 + M_L \left|{}^L\phi_{\hat{0},\hat{0}}\right|^2 \right\} \tag{18}$$

where

$$M_R = \hat{M}_{11}, \quad M_A = \hat{M}_{11} + \frac{1}{4}(n-2)\hat{M}_{10}, \quad M_L = \hat{M}_{11} + \frac{1}{2}(n-1)\hat{M}_{00} \tag{19}$$

are given in terms of the RFT of the original masses, which are defined as [3]

$$\hat{M}_{11} = M_{11} - 2M_{10} + M_{00}, \quad \hat{M}_{10} = 4(M_{10} - M_{00}), \quad \hat{M}_{00} = 4(M_{10} - M_{00}) + nM_{00}. \tag{20}$$

and

$${}^R\Phi_{\hat{a}',-\hat{a}'} = \frac{1}{\sqrt{2}}\Phi_{\hat{a}',-\hat{a}'}, \quad {}^R\Phi_{\hat{a}'',\hat{t}'-\hat{a}''} = \frac{1}{\sqrt{2}}\Phi_{\hat{a}'',\hat{t}'-\hat{a}''} \tag{21}$$

$${}^A\phi_{\hat{0},\hat{t}'} = \sqrt{\frac{n}{(n-2)}}\phi_{\hat{0},\hat{t}'}, \quad {}^L\phi_{\hat{0},\hat{0}} = \sqrt{\frac{n}{2(n-1)}}\phi_{\hat{0},\hat{0}} \tag{22}$$

with the fields in (21) having the properties, which follow from (15),

$$\sum_{\hat{a}'} {}^R\Phi_{\hat{a}',-\hat{a}'} = 0, \qquad \sum_{\hat{a}''} {}^R\Phi_{\hat{a}'',\hat{t}'-\hat{a}''} = 0. \tag{23}$$

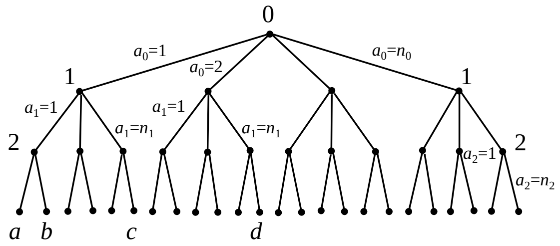

Figure 1: Tree representation for an $R = 2$ RSB ansatz

One can see from (18) that the fluctuation space is divided into three sectors, which we identify as the replicon (R) with eigenvalue M_R, the anomalous (A) with eigenvalue M_A, and the longitudinal (L) with eigenvalue M_L. The degeneracies of the eigenvalues are given by the multiplicities of the fields, $\mu_1 = (n-3)/2$ for ${}^R\Phi_{\hat{a}',-\hat{a}'}$ and $\mu_2 = (n-1)(n-3)/2$ for ${}^R\Phi_{\hat{a}'',\hat{t}'-\hat{a}''}$ leads to $\mu_R = \mu_1 + \mu_2 = n(n-3)/2$ for the replicon, $\mu_A = (n-1)$ for the anomalous and $\mu_L = 1$ for the longitudinal, so that the total number of modes is recovered $\mu_R + \mu_A + \mu_L = n(n-1)/2$.

The propagators for the longitudinal, anomalous and replicon fields, obtained from (18), are given by

$$ {}^L G_{\hat{0};\hat{0}} = \left\langle {}^L\phi_{\hat{0},\hat{0}}\, {}^L\phi_{\hat{0},\hat{0}} \right\rangle = \frac{1}{M_L} \tag{24} $$

$$ {}^A G_{\hat{t}';\hat{s}'} = \left\langle {}^A\phi_{\hat{0},\hat{t}'}\, {}^A\phi^*_{\hat{0},\hat{s}'} \right\rangle = \delta_{\hat{t}',\hat{s}'} \frac{1}{M_A} \tag{25} $$

$$ {}^R G_{\hat{a}';\hat{b}'} = \left\langle {}^R\Phi_{\hat{a}',-\hat{a}'}\, {}^R\Phi_{\hat{b}',-\hat{b}'} \right\rangle = \frac{1}{2}\left[\left(\delta_{\hat{a}',\hat{b}'} + \delta_{\hat{a}',-\hat{b}'}\right) - \frac{2}{n-1}\right]\frac{1}{M_R} \tag{26} $$

$$ {}^R G_{\hat{a}'',\hat{t}';\hat{b}'',\hat{s}'} = \left\langle {}^R\Phi_{\hat{a}'',\hat{t}'-\hat{a}''}\, {}^R\Phi^*_{\hat{b}'',\hat{s}'-\hat{b}''} \right\rangle = \delta_{\hat{t}',\hat{s}'}\frac{1}{2}\left[\left(\delta_{\hat{a}'',\hat{b}''} + \delta_{\hat{a}'',\hat{s}'-\hat{b}''}\right) - \frac{2}{n-2}\right]\frac{1}{M_R}. \tag{27} $$

The propagators in the direct replica space, $G^{ab,cd}$, can be easily obtained, in terms of their RFT expression, as shown in [9].

4 Replica Symmetry Breaking: Parisi's Ansatz

The RSB ansatz proposed by Parisi for the mean-field order parameter can be described in terms of a tree whose extremities are the n replicas $a = 1, 2, \ldots, n$, and which, for R steps of RSB, foliates at the various levels $r = 0, 1, 2, \ldots, R$ with multiplicity $n_r = p_r/p_{r+1}$, where $p_0 = n$ and $p_{R+1} = 1$, as illustrated in figure 1. Each replica is associated to a string of tree coordinates, $a : [a_0, a_1, \ldots, a_R]$, which tells the path to reach replica a. Each component takes n_r values, $a_r = 1, 2, \ldots, n_r$. The overlap of replicas a and b is defined as

$$ \begin{gathered} a \cap b = r, \qquad 0 \le r \le R+1 \\ \text{if} \quad a_0 = b_0,\ \ldots\ , a_{r-1} = b_{r-1}, \quad \text{but} \quad a_r \neq b_r \end{gathered} \tag{28} $$

with $a \cap b = R+1$ corresponding to $a = b$. At the r^{th} level of hierarchy the order parameter takes the value $Q^{ab} = Q_r$, having $Q^{aa} = Q_{R+1} = 0$. The tree displays ultrametricity, that is, given three replicas a, b, c, the overlaps between these replicas, $a \cap b = r$, $a \cap c = s$, $b \cap c = t$, either are all equal, or one is larger than the others, but then these are equal (e.g., $r = s \le t$).

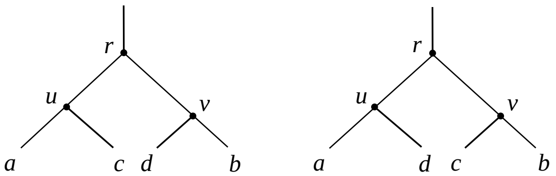

Figure 2: Tree representation for the replicon sector. The figure shows the two possible structures compatible with the replicon geometry

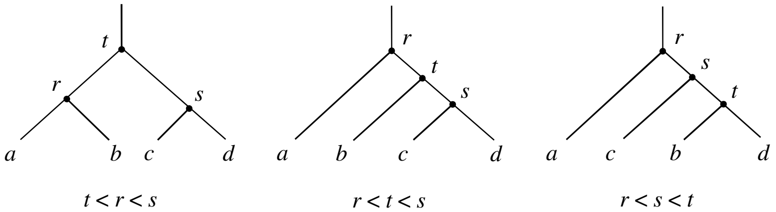

Figure 3: Tree representation for the longitudinal-anomalous sector. Exchanging r and s leads to equivalent structures

Now we consider the Lagrangian term of the fluctuations $\mathcal{L}^{(2)}$, (7). The fields are characterized by the overlap of the replicas and depend on the tree coordinates

$$\phi_{ab} = \phi_r = \begin{bmatrix} a_0 \dots a_{r-1} a_r \dots a_R \\ a_0 \dots a_{r-1} b_r \dots b_R \end{bmatrix} = \begin{bmatrix} a_0 \dots a_{r-1} & a_r \dots a_R \\ & b_r \dots b_R \end{bmatrix} \qquad a_r \neq b_r \tag{29}$$

with $r = a \cap b$ and $\phi_{aa} = \phi_{R+1} = 0$.

The mass-matrix depends only on the overlaps of the replicas, and can be parametrized as follows,

$$M^{ab,cd} = M^{r,s}_{u,v} \tag{30}$$

with $r = a \cap b$, $s = c \cap d$, $u = \max(a \cap c, a \cap d)$ and $v = \max(b \cap c, b \cap d)$. Ultrametricity implies that with four replicas there are generically three overlaps, i.e., among the overlaps r, s, u, v at least two are equal; r, s are direct-overlaps and u, v are cross-overlaps.

The Lagrangian $\mathcal{L}^{(2)}$ is then written as

$$\mathcal{L}^{(2)} = \sum_{r,s;u,v} \sum_{\{a,b,c,d\}} \phi_r M^{r,s}_{u,v} \phi_s \tag{31}$$

with $0 \leq r, s \leq R$, $0 \leq u, v \leq R+1$, and where the sum over the set $\{a,b,c,d\}$ depends on the overlaps r, s, u, v; again the dependence on momentum space $\mathbf{p}$ is implicit. The possible geometries of the tree representation of the mass-matrix, (30), are presented in figures 2 and 3. We distinguish two sets of contributions:

- the replicon (R) configurations, in figure 2, are characterized by two identical upper indices, $r = s$, and two lower indices u, $v \geq r+1$, $M^{r,s}_{u,v} = M^{r,r}_{u,v}$;

- the longitudinal-anomalous (LA) configurations, in figure 3, are characterized by a single lower index, $t = \max(u,v)$ (the other lower index is r, s, or t) and two upper indices r, s (where it may happen, accidently, that $r = s$), $M^{r,s}_{u,v} = M^{r,s}_t$.

We now generalize the RFT introduced in (14). To RFT with respect to replica a on a tree, we RFT each of the $[a_r]$ coordinates of a on the tree. Focusing on $[a_r]$, one defines

$$\phi\left[\hat{a}_r\right] = \frac{1}{\sqrt{n_r}} \sum_{a_r} e^{-\frac{2\pi i}{n_r} a_r \hat{a}_r} \phi\left[a_r\right], \quad \phi\left[a_r\right] = \frac{1}{\sqrt{n_r}} \sum_{\hat{a}_r} e^{\frac{2\pi i}{n_r} a_r \hat{a}_r} \phi\left[\hat{a}_r\right] \tag{32}$$

where $\hat{a}_r$ takes $n_r = p_r/p_{r+1}$ values on the circle, $\hat{a}_r = 0, 1, 2, \ldots, n_r - 1$, $\mathrm{mod}(n_r)$.

Let us then carry out the steps needed to accomplish the diagonalization of the Lagrangian $\mathcal{L}^{(2)}$:

1. Write the expression for the various contributions to $\mathcal{L}^{(2)}$;
2. For the cross-overlaps t, u, v, transform the restricted sums over the tree coordinates into unrestricted sums, and regroup the terms among the various contributions;
3. Perform the RFT on all the tree coordinates a, b, c, d of the replicas;
4. Separate in the sums over the tree coordinates of the replicas, the $\hat{0}$ components.

The restrictions on the tree coordinates associate with the direct-overlaps r, s are incorporated in the RFT of $\left[\begin{array}{c} a_r \\ b_r \end{array}\right]_{a_r \neq b_r}$, which defines the *marker*

$$\left(\begin{array}{c} \hat{\mu}_r \\ \hat{\gamma}_r - \hat{\mu}_r \end{array}\right) = \left[\begin{array}{c} \hat{\mu}_r \\ \hat{\gamma}_r - \hat{\mu}_r \end{array}\right] - \frac{1}{n_r}\left[\hat{\gamma}_r\right] \tag{33}$$

with the notation $\left[\hat{\gamma}_r\right] = \sum_{\hat{a}_r} \left[\begin{array}{c} \hat{a}_r \\ \hat{\gamma}_r - \hat{a}_r \end{array}\right]$. The *marker* has the property

$$\sum_{\hat{\mu}_r} \left(\begin{array}{c} \hat{\mu}_r \\ \hat{\gamma}_r - \hat{\mu}_r \end{array}\right) = 0. \tag{34}$$

We define a set of new fields,

$$\left\{\begin{array}{c} \hat{\mu}'_r \\ \hat{\gamma}_r - \hat{\mu}'_r \end{array}\right\} = \left(\begin{array}{c} \hat{\mu}'_r \\ \hat{\gamma}_r - \hat{\mu}'_r \end{array}\right) + \frac{1}{n_r - 1}\left(\begin{array}{c} \hat{0}_r \\ \hat{\gamma}_r \end{array}\right) \tag{35}$$

$$\left\{\begin{array}{c} \hat{\mu}''_r \\ \hat{\gamma}'_r - \hat{\mu}''_r \end{array}\right\} = \left(\begin{array}{c} \hat{\mu}''_r \\ \hat{\gamma}'_r - \hat{\mu}''_r \end{array}\right) + \frac{1}{n_r - 2}\left[\left(\begin{array}{c} \hat{0}_r \\ \hat{\gamma}'_r \end{array}\right) + \left(\begin{array}{c} \hat{\gamma}'_r \\ \hat{0}_r \end{array}\right)\right] \tag{36}$$

with $\hat{\mu}'_r \neq \hat{0}_r$, $\hat{\gamma}'_r \neq 0$ and $\hat{\mu}''_r \neq \hat{0}_r, \hat{\gamma}'_r$.

The complete replicon contribution, provided explicitly in [9], can be written in the generic form

$$\mathcal{L}_{\mathcal{R}} = \frac{1}{2} \sum_{r=0}^{R} \sum_{u,v=r+1}^{R+1} \sum_{\{\hat{\gamma},\hat{\mu},\hat{\nu}\}} \hat{M}^{r,r}_{u,v} \left|{}^{\mathcal{R}}\Phi^{r}_{u,v}\right|^2 \tag{37}$$

where $\hat{M}^{r,r}_{u,v}$ is the replicon mass, which is given by the mass double RFT [3],

$$\hat{M}^{r,r}_{u,v} = \sum_{k=u}^{R+1} \sum_{l=v}^{R+1} p_k p_l \left(M^{r,r}_{k,l} - M^{r,r}_{k-1,l} - M^{r,r}_{k,l-1} + M^{r,r}_{k-1,l-1}\right) \tag{38}$$

and ${}^{\mathcal{R}}\Phi^r_{u,v}$ are the replicon fields, defined as:

$$ {}^{\mathcal{R}}\Phi^r_{u,v}\begin{pmatrix} \hat{\mu}_r \\ \hat{\gamma}_r-\hat{\mu}_r \end{pmatrix} = \mathcal{N}_1\left[\hat{\gamma}_0\ldots\hat{\gamma}_{r-1}\begin{pmatrix} \hat{\mu}_r \\ \hat{\gamma}_r-\hat{\mu}_r \end{pmatrix} \begin{matrix} \hat{\mu}_{r+1}\ldots\hat{\mu}'_{u-1}\hat{0}_u\ldots\hat{0}_{v-1}\hat{0}_v\ldots\hat{0}_R \\ \hat{\nu}_{r+1}\ldots\hat{\nu}_{u-1}\hat{\nu}_u\ldots\hat{\nu}'_{v-1}\hat{0}_v\ldots\hat{0}_R \end{matrix}\right]_{SN} \tag{39} $$

for $u,\, v > r+1$, with multiplicity

$$ \mu(r;u,v) = \frac{1}{2}p_0\left(p_r-p_{r+1}\right)\left(\frac{1}{p_u}-\frac{1}{p_{u-1}}\right)\left(\frac{1}{p_v}-\frac{1}{p_{v-1}}\right); \tag{40} $$

$$ {}^{\mathcal{R}}\Phi^r_{r+1,v}\begin{pmatrix} \hat{\mu}'_r \\ \hat{\gamma}_r-\hat{\mu}'_r \end{pmatrix} = \mathcal{N}_2\left[\hat{\gamma}_0\ldots\hat{\gamma}_{r-1}\begin{Bmatrix} \hat{\mu}'_r \\ \hat{\gamma}_r-\hat{\mu}'_r \end{Bmatrix} \begin{matrix} \hat{0}_{r+1}\ldots\hat{0}_{v-1}\hat{0}_v\ldots\hat{0}_R \\ \hat{\nu}_{r+1}\ldots\hat{\nu}'_{v-1}\hat{0}_v\ldots\hat{0}_R \end{matrix}\right]_{SN} \tag{41} $$

for $u=r+1$, $v>r+1$ (or $v=r+1$, $u>r+1$), with multiplicity

$$ \mu(r;r+1,v) = \frac{1}{2}p_0\left(\frac{p_r}{p_{r+1}}-2\right)\left(\frac{1}{p_v}-\frac{1}{p_{v-1}}\right); \tag{42} $$

$$ {}^{\mathcal{R}}\Phi^r_{r+1,r+1}\begin{pmatrix} \hat{\mu}'_r \\ -\hat{\mu}'_r \end{pmatrix} = \mathcal{N}_3\left[\hat{\gamma}_0\ldots\hat{\gamma}_{r-1}\begin{Bmatrix} \hat{\mu}'_r \\ -\hat{\mu}'_r \end{Bmatrix} \begin{matrix} \hat{0}_{r+1}\ldots\hat{0}_R \\ \hat{0}_{r+1}\ldots\hat{0}_R \end{matrix}\right]_{SN} \tag{43} $$

with multiplicity

$$ \mu_1(r;r+1,r+1) = \frac{1}{2}p_0\left(\frac{p_r}{p_{r+1}}-3\right)\frac{1}{p_r}; \tag{44} $$

$$ {}^{\mathcal{R}}\Phi^r_{r+1,r+1}\begin{pmatrix} \hat{\mu}''_r \\ \hat{\gamma}'_r-\hat{\mu}''_r \end{pmatrix} = \mathcal{N}_4\left[\hat{\gamma}_0\ldots\hat{\gamma}_{r-1}\begin{Bmatrix} \hat{\mu}''_r \\ \hat{\gamma}'_r-\hat{\mu}''_r \end{Bmatrix} \begin{matrix} \hat{0}_{r+1}\ldots\hat{0}_R \\ \hat{0}_{r+1}\ldots\hat{0}_R \end{matrix}\right]_{SN} \tag{45} $$

with multiplicity

$$ \mu_2(r;r+1,r+1) = \frac{1}{2}p_0\left(\frac{p_r}{p_{r+1}}-3\right)\left(\frac{1}{p_{r+1}}-\frac{1}{p_r}\right). \tag{46} $$

All the fields are symmetrized (S) at the *marker* and normalized (N), the constants $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{N}_4$ can be found in [9].

The replicon fields have the properties, which follow from (34),

$$ \sum_{\hat{\mu}_r} {}^{\mathcal{R}}\Phi^r_{u,v}\begin{pmatrix} \hat{\mu}_r \\ \hat{\gamma}_r-\hat{\mu}_r \end{pmatrix} = 0, \quad \sum_{\hat{\mu}'_r} {}^{\mathcal{R}}\Phi^r_{r+1,v}\begin{pmatrix} \hat{\mu}'_r \\ \hat{\gamma}_r-\hat{\mu}'_r \end{pmatrix} = 0, \tag{47} $$

$$ \sum_{\hat{\mu}'_r} {}^{\mathcal{R}}\Phi^r_{r+1,r+1}\begin{pmatrix} \hat{\mu}'_r \\ -\hat{\mu}'_r \end{pmatrix} = 0, \quad \sum_{\hat{\mu}''_r} {}^{\mathcal{R}}\Phi^r_{r+1,r+1}\begin{pmatrix} \hat{\mu}''_r \\ \hat{\gamma}'_r-\hat{\mu}''_r \end{pmatrix} = 0. \tag{48} $$

The propagators for the replicon fields, obtained from (37), are given by:

$$ {}^{\mathcal{R}}G^r_{u,v}\left(\hat{\mu}_r,\hat{\gamma}_r;\hat{\eta}_r,\hat{\lambda}_r\right) = \delta_{\gamma,\lambda}\left[\delta_{\mu,\eta}-\frac{1}{n_r}\right]\frac{1}{\hat{M}^{r,r}_{u,v}} \tag{49} $$

$$ {}^{\mathcal{R}}G^r_{r+1,v}\left(\hat{\mu}'_r,\hat{\gamma}_r;\hat{\eta}'_r,\hat{\lambda}_r\right) = \delta_{\gamma,\lambda}\left[\delta_{\mu',\eta'}-\frac{1}{n_r-1}\right]\frac{1}{\hat{M}^{r,r}_{r+1,v}} \tag{50} $$

$$
{}^{\mathcal{R}}G^{r}_{r+1,r+1}\left(\hat{\mu}'_r,\hat{0}_r;\hat{\eta}'_r,\hat{0}_r\right)=\left[\frac{1}{2}(\delta_{\mu',\eta'}+\delta_{\mu',-\eta'})-\frac{1}{n_r-1}\right]\frac{1}{\hat{M}^{r,r}_{r+1,r+1}} \tag{51}
$$

$$
{}^{\mathcal{R}}G^{r}_{r+1,r+1}\left(\hat{\mu}''_r,\hat{\gamma}'_r;\hat{\eta}''_r,\hat{\lambda}'_r\right)=\delta_{\gamma',\lambda'}\left[\frac{1}{2}\left(\delta_{\mu'',\eta''}+\delta_{\gamma'-\mu'',\eta''}\right)-\frac{1}{n_r-2}\right]\frac{1}{\hat{M}^{r,r}_{r+1,r+1}}. \tag{52}
$$

The complete longitudinal-anomalous contribution, provided explicitly in [9], can be written in the generic form

$$
\mathcal{L}_{\mathcal{LA}}=\frac{1}{2}\sum_{t=0}^{R+1}\sum_{r,s=0}^{R}\sum_{\{\hat{\gamma}\}}{}^{\mathcal{LA}}\Psi^{r}_{t}\left[\delta^{Kr}_{r,s}\hat{\Lambda}^{r}_{t}+\frac{1}{4}\sqrt{\delta^{(t-1)}_{r}}\hat{M}^{r,s}_{t}\sqrt{\delta^{(t-1)}_{s}}\right]{}^{\mathcal{LA}}\Psi^{s*}_{t} \tag{53}
$$

where $\delta^{(l)}_r=p^{(l)}_r-p^{(l)}_{r+1}$, with $p^{(l)}_r=p_r$ for $r\le l$, $2p_r$ for $r>l$, $\hat{\Lambda}^r_t$ is defined as

$$
\hat{\Lambda}^{r}_{t}=\begin{cases}\hat{M}^{r,r}_{t,r+1} & t>r+1\\ \hat{M}^{r,r}_{r+1,r+1} & t\le r+1\end{cases} \tag{54}
$$

with $\hat{M}^{r,r}_{u,v}$ given by (38) and $\hat{M}^{r,s}_{t}$ is the mass RFT [3],

$$
\hat{M}^{r,s}_{t}=\sum_{k=t}^{R+1}p^{(r,s)}_{k}\left(M^{r,s}_{k}-M^{r,s}_{k-1}\right) \tag{55}
$$

with $p^{(r,s)}_k=p_k$ for $k\le r\le s$, $2p_k$ for $r<k\le s$, $4p_k$ for $r\le s<k$.

The longitudinal-anomalous fields ${}^{\mathcal{LA}}\Psi^r_t$ have the generic form:

$$
{}^{\mathcal{LA}}\Psi^{r}_{t}\left(\begin{array}{c}\hat{0}_r\\ \hat{\gamma}_r\end{array}\right)=\frac{1}{\sqrt{2}}\left[\hat{\gamma}_0\ldots\hat{\gamma}_{r-1}\left(\begin{array}{c}\hat{0}_r\\ \hat{\gamma}_r\end{array}\right)\begin{array}{c}\hat{0}_{r+1}\ldots\hat{0}_{t-1}\hat{0}_t\ldots\hat{0}_R\\ \hat{\gamma}_{r+1}\ldots\hat{\gamma}'_{t-1}\hat{0}_t\ldots\hat{0}_R\end{array}\right]_{SN} \tag{56}
$$

with multiplicity

$$
\mu(t)=p_0\left(\frac{1}{p_t}-\frac{1}{p_{t-1}}\right),\quad \mu(0)=1. \tag{57}
$$

The propagators for the longitudinal-anomalous fields, obtained from (53), are given by:

$$
{}^{\mathcal{LA}}\hat{G}^{r,s}_{t}=\delta^{Kr}_{r,s}\frac{1}{\hat{\Lambda}^{r}_{t}}+\frac{1}{4}\sqrt{\delta^{(t-1)}_{r}}\hat{F}^{r,s}_{t}\sqrt{\delta^{(t-1)}_{s}} \tag{58}
$$

with

$$
\hat{F}^{r,s}_{t}=-\frac{1}{\hat{\Lambda}^{r}_{t}}\hat{M}^{r,s}_{t}\frac{1}{\hat{\Lambda}^{s}_{t}}-\frac{1}{\hat{\Lambda}^{r}_{t}}\sum_{k=0}^{R}\hat{M}^{r,k}_{t}\frac{\delta^{(t-1)}_{k}}{4}\hat{F}^{k,s}_{t}. \tag{59}
$$

A fully explicit form for the solution of $\hat{F}^{r,s}_t$ can be found in [6] .

From (37) and (53) one sees that the Lagrangian, $\mathcal{L}^{(2)}=\mathcal{L}_{\mathcal{LA}}+\mathcal{L}_{\mathcal{R}}$, breaks up into a string of $(R+1)\times(R+1)$ blocks followed by a string of 1×1 "blocks" along the diagonal. The $(R+1)\times(R+1)$ blocks correspond to the longitudinal–anomalous sector, they contain the matrix elements $\hat{M}^{r,s}_t$ with $r,s=0,\ldots,R$, and are labelled by the index $t=0,1,\ldots,R+1$, ($t=0$ is the longitudinal and $t\neq0$ are the anomalous). The 1×1 "blocks" correspond to the replicon sector, they are the elements $\hat{M}^{r,r}_{u,v}$ with $r=0,\ldots,R$ and $u,v=r+1,\ldots,R+1$. The total number of longitudinal-anomalous modes is $\mu_{\mathcal{LA}}=(R+1)n$, and the total number of replicon modes is $\mu_{\mathcal{R}}=n(n-1)/2-(R+1)n$, so that the total number of modes is $\mu=\mu_{\mathcal{LA}}+\mu_{\mathcal{R}}=n(n-1)/2$.

One can easily obtain the propagators in the direct replica space, $G^{ab,cd}$, for general R, in terms of their RFT expression, as shown in [9].

5 Spin Glass Free Energy with Fluctuations

The spin glass free energy (2) can be written as

$$\overline{F} = F_{mf} + F_{fluct} \tag{60}$$

where

$$F_{mf} = \frac{1}{\beta} \lim_{n\to 0} \frac{\mathcal{L}^{(0)}}{n} \tag{61}$$

provides the mean field value of the free energy, with $\mathcal{L}^{(0)}$ given by (5), and

$$F_{fluct} = -\frac{1}{\beta} \lim_{n\to 0} \frac{\ln \left[\overline{Z^n}\right]_{fluct}}{n} \tag{62}$$

provides the contribution of the fluctuations. For fluctuations up to the quadratic order,

$$\left[\overline{Z^n}\right]_{fluct} = \int \mathcal{D}\left({}^{\mathcal{LA}}\Psi\right) \mathcal{D}\left({}^{\mathcal{R}}\Phi\right) \exp\left\{-\mathcal{L}^{(2)}\right\} \tag{63}$$

where $\mathcal{L}^{(2)} = \mathcal{L}_{\mathcal{LA}} + \mathcal{L}_{\mathcal{R}}$, with $\mathcal{L}_{\mathcal{LA}}$ given by (53) and $\mathcal{L}_{\mathcal{R}}$ given by (37), the replicon fields verifying the constraints given by (47) and (48). Performing the integration over the longitudinal-anomalous and the replicon fields in (63) considering the replicon constraints, one obtains for the free-energy fluctuations,

$$\begin{aligned} F_{fluct} =& \frac{1}{\beta} \lim_{n\to 0} \frac{1}{2n} \left\{ \sum_{t=0}^{R+1} \mu(t) \ln\det \hat{\Delta}_t + \sum_{r=0}^{R} \sum_{u,v=r+1}^{R+1} \bar{\mu}(r;u,v) \ln \hat{M}_{u,v}^{r;r} \right. \\ & \left. - \frac{1}{2} \sum_{r=0}^{R} p_0 \left[\left(p_{r+1} + \frac{1}{p_{r+1}}\right) \ln\left(p_{r+1}\right) + \left(1 - \frac{1}{p_{r+1}}\right)\left(\frac{p_r}{p_{r+1}} + p_r - p_{r+1} - 3\right) \ln 2 \right] \right\} \end{aligned} \tag{64}$$

where $\hat{\Delta}_t$ is

$$\hat{\Delta}_t^{r,s} = \delta_{r,s}^{Kr} + \frac{1}{4} \sqrt{\delta_r^{(t-1)}}\, \frac{\hat{M}_t^{r,s}}{\hat{\Lambda}_t^r} \sqrt{\delta_s^{(t-1)}} \tag{65}$$

the longitudinal-anomalous multiplicity $\mu(t)$ is given by (57) and

$$\bar{\mu}(r;u,v) = \frac{1}{2} p_0 \left(p_r - p_{r+1}\right) \left(\frac{1}{p_u} - \frac{1}{p_{u-1}}\right) \left(\frac{1}{p_v} - \frac{1}{p_{v-1}}\right), \quad u,v > r+1 \tag{66}$$

$$\bar{\mu}(r;r+1,v) = \frac{1}{2} p_0 \left(p_r - p_{r+1}\right) \left(\frac{1}{p_v} - \frac{1}{p_{v-1}}\right) \frac{1}{p_{r+1}}, \quad v > r+1 \tag{67}$$

$$\bar{\mu}(r;r+1,r+1) = \frac{1}{2} p_0 \left(p_r - p_{r+1}\right) \frac{1}{p_{r+1}^2}. \tag{68}$$

We note that in (64) there is a cancellation of terms between the longitudinal-anomalous and the replicon contributions, which leads to the proper multiplicities $\bar{\mu}$ as remarked in [6].

For $R = 0$, (64) reduces to

$$F_{fluct} = \frac{1}{\beta} \lim_{n\to 0} \frac{1}{2n} \left\{ \ln M_L + (n-1) \ln M_A + \frac{1}{2} n(n-3) \ln M_R \right\}. \tag{69}$$

6 Conclusion

The field theory in RFT space provides a new tool to investigate the behaviour of spin glasses. We have a field theory that is directly defined in terms of the replicon, anomalous and longitudinal fields, in RFT space. We applied the formalism to calculate the contribution of the Gaussian fluctuations around the Parisi mean field solution for the free energy of an Ising spin glass. The propagators in the direct replica space can be simply related to the propagators in the RFT space, which enables to calculate important physical quantities, such as the spin glass and the nonlinear susceptibilities. It is of interest to evaluate the contribution of the different fluctuation sectors, replicon, anomalous and longitudinal, to the various quantities. We expect that the RFT field theory will allow to study the properties of the glassy phase, and hence contribute for the understanding of spin glasses.

Acknowledgments

IRP acknowledges the support from Fundação para a Ciência e a Tecnologia, UI0618, Portugal.

References

[1] K. Binder and A. P. Young. Spin glasses: Experimental facts, theoretical concepts, and open questions. *Rev. Mod. Phys.*, 58:801–976, 1986.

[2] A. J. Bray and A. Moore. Heidelberg colloquium on glassy dynamics. *Lecture Notes in Physics*, 275:121–153, 1986.

[3] C. De Dominicis D. M. Carlucci and T. Temesvari. Replica fourier tansforms on ultrametric trees, and block-diagonalizing multi-replica matrices. *J. Phys. I France*, 7:105–115, 1997.

[4] S. F. Edwards and P. W. Anderson. Theory of spin glasses. *J. Phys. F: Met. Phys.*, 5:965, 1975.

[5] D. S. Fisher and D. A. Huse. Ordered phase of short-range ising spin-glasses. *Phys. Rev. Lett.*, 56:1601–1604, 1986.

[6] C. De Dominicis I. Kondor and T. Temesvari. *Spin Glasses and Random Fields*. World Scientific, 1998.

[7] G. Parisi. Infinite number of order parameters for spin-glasses. *Phys. Rev. Lett.*, 43:1754–1756, 1979.

[8] G. Parisi. The order parameter for spin glasses: a function on the interval 0-1. *J. Phys. A: Math. Gen.*, 13:1101, 1980.

[9] I. R. Pimentel and C. De Dominicis. Spin glass field theory with replica fourier transforms. *J. Phys. A: Math. Theor.*, 47:455001 (34pp), 2014.

[10] D. Sherrington and S. Kirkpatrick. Solvable model of a spin-glass. *Phys. Rev. Lett.*, 35:1792–1796, 1975.