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ON FLOWCHART THEORIES: PART II. THE NONDETERMINISTIC CASE

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Abstract. We give a calculus for nondeterministic flowchart schemes similar to the calculus of polynomials. It uses a formal representation of flowchart schemes and a natural equivalence on these formal representations. The algebraic structure involved is a matrix theory endowed with an axiomatized repetition.

Introduction

This paper deals with semantics of nondeterministic flowchart algorithms. The interesting flowchart schemes involve a kind of iteration. Frequently, the behaviours of such programs are specified by using fixed-point theorems in ordered algebras (for example in [2, 12, 13]), formal power series [1], or other devices modeling the behaviour of a program as a limit of its finite approximations (for example, a categorical colimit, as in [11]). We prefer the viewpoint of Elgot [8], namely to find the *pure* algebraic structure of flowchart schemes behaviours. For this, axiomatized operations (particularly, an axiomatized iteration) have to be used.

In a research briefly presented in [18] we have attempted to find calculi for some types of flowchart schemes that have the calculus of polynomials as a model. In this second part we shall give such a calculus for nondeterministic flowchart schemes.

The multi-input multi-output flowchart scheme in Fig. 1(a) may be ordered as in Fig. 1(b), and is represented by

$$((1_2 \oplus x \oplus y) \cdot f)\uparrow^3.$$

Figure 2 illustrates the meaning of the symbols "$\oplus$", "$\cdot$", and "$\uparrow$". An (abstract) *flowchart theory* $\mathrm{Fl}_{X,T}$ is a theory of such formal representations of flowchart schemes; it is based on

- a double indexed set X of variables for atomic flowchart schemes;
- a 'support theory' T consisting of a family of sets $T(m, n)$, $m, n \in \mathbb{N}$, whose elements are used for connections f (an element $f \in T(m, n)$ is considered to be a known computation process with m inputs and n outputs; for example, a trivial one which only redirects the flow of control).

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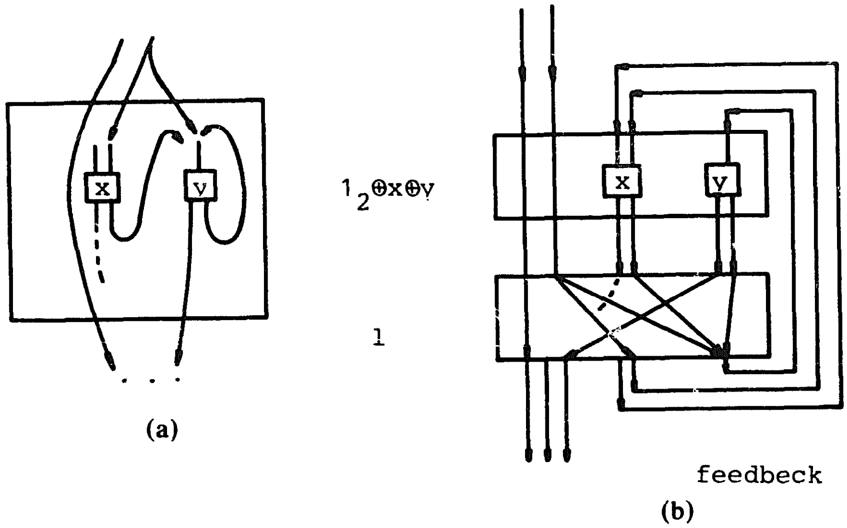

Fig. 1. The standard form of a flowchart scheme.

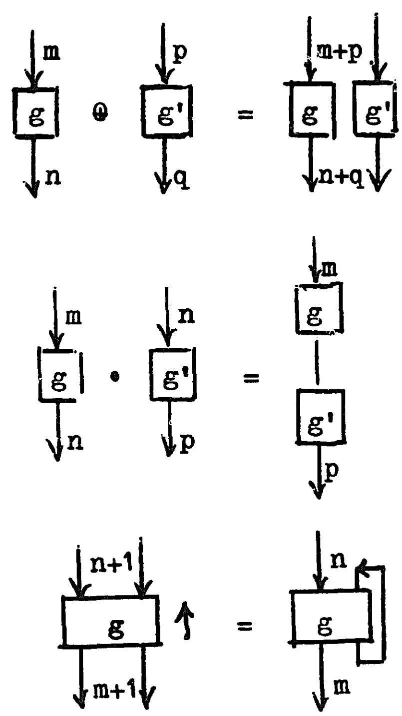

Fig. 2. "⊕" denotes separated sum; "·" denotes composition; "↑" denotes feedback.

The type of T agrees with the type of flowchart schemes we want to consider. For example, if T has only elements for redirecting flow of control, such a correspondence is as shown in Table 1. In our nondeterministic case, the basic support theory is that of relations.

For a calculus similar to that of polynomials we need to find an algebra *struc* for T such that the classes of 'naturally' equivalent flowcharts in $\mathrm{Fl}_{X,T}$ form the coproduct of T and the *struc* structure freely generated by the double indexed set X. (Such a 'coproduct' is also used for M-construction in [2], namely the coproduct

Table 1

Elements of $T(m, n)$	Flowchart schemes modeled by $\mathbf{Fl}_{X,T}$
functions	total, deterministic
surjective functions	accessible, total, deterministic
partial functions	partial, deterministic
relations	nondeterministic

of an ω-continuous algebraic theory and one freely generated by a ranked set.) By 'natural' we mean the following:

(1) The deterministic case corresponds to the physical world: physical processes have deterministic evolutions; two processes may be considered equivalent if they evolve in the same way. This shows that two deterministic flowcharts may be considered 'naturally' equivalent if 'they have the same set of finite and infinite computation paths'. (This is Elgot's strong equivalence; it is generalized to abstract flowcharts as the deterministic equivalence $\equiv_{\rm d}$, cf. [19].)

(2) The nondeterministic case corresponds to the living world. Living processes are based on physical processes, their characteristic feature consisting in the ability of fixing aims. In view of its aim, a living process can choose between several variants, or can stop if it feels that it cannot reach its aim and restart trying another way. Two living processes with the same starting state and the same aim may be considered equivalent if they reach the aim in the same way. This shows that two nondeterministic flowcharts may be considered 'naturally' equivalent if 'they have the same set of successful paths'. (Note that it is an open problem whether our nondeterministic equivalence $\equiv_{\rm nd}$ generalizes this—see Section 9.)

In fact, our calculus for nondeterministic flowcharts is based on the hypothesis that analytic processes for reaching aims (the passing from future to present) obey the same laws as physical processes (the passing from present to future). To be more specific, we define the *dual flowchart* F° (with n inputs and m outputs) of a flowchart F (with m inputs and n ouputs) as the flowchart obtained by reversing the arrows of F. The hypothesis is then formulated in the following way.

Hypothesis. *Nondeterminism = determinism + dual determinism.*

This means that all valid deterministic laws and the corresponding dual laws are to be considered as valid nondeterministic laws.

Example. Consider the widely accepted fact that (behaviours of) deterministic flowchart schemes form an algebraic theory—in the sense of Lawvere—i.e., a category in which each morphism has a unique source-splitting into components [1, 2, 8, 12, 15]. By using the Hypothesis this yields that (behaviours of) nondeterministic flowchart schemes form a category—this notion is selfdual—in which each morphism has a unique source-splitting into components and a unique target-splitting into co-components; namely they form a matrix theory, cf. [9, p. 395].

Consequently, if the Hypothesis is accepted, then matrix theories are to be considered as a basic algebraic structure for (noniterative) nondeterministic flowchart schemes; this agrees with [1, 15].

In Section 1, applying the Hypothesis to our basic algebra for deterministic flowchart schemes, namely to strong iteration algebraic theory [18, 19], will yield our algebra for nondeterministic flowchart schemes, called a *repetition theory.* (In a very quaint and exciting way, this algebra is closely related to the regular Kleene algebra, cf. [5]—see Section 8.) Section 2 will give the basic definitions of flowcharts and in Section 3 we shall define our nondeterministic equivalence $\equiv_{nd}$ as that obtained by applying the Hypothesis to the deterministic equivalence in [18, 19]. In Sections 4, 5 and 6 the support theory is supposed to obey a technical condition ('with intersection'). The main technical result is the characterization theorem for $\equiv_{nd}$ given in Section 4 (and Appendix A); in Section 5 we shall prove that the classes of $\equiv_{nd}$-equivalent flowcharts over a repetition theory with intersection form a repetition theory; and in Section 6 we shall prove a universality property. These show that, for a wide class of repetition theories (including all repetition theories of interest $\mathbf{Rel}_D$), our calculus is similar to the calculus of polynomials. An application (Section 7), some connections with related algebraic structures (Section 8), and some conclusions (Section 9) will conclude the paper.

1. Repetition theories

In this section we shall introduce our algebra for nondeterministic flowchart schemes. It is obtained by applying the Hypothesis in the Introduction to our algebra for deterministic flowchart schemes, namely to strong iteration algebraic theory [18, 19].

First, (cf. Introduction) the Hypothesis applied to an algebraic theory yields a *matrix theory.* It is known (cf. [9, Corollary 10.2]) that a matrix theory can equivalently be presented as *a theory of (finite) matrices over a semiring* (cf. [14]; a semiring A is a commutative monoid with respect to addition with neutral element 0, a monoid with respect to multiplication with neutral element 1, with multiplication distributing over addition and such that $a \cdot 0 = 0 \cdot a = 0$ for each $a \in A$). The free matrix theory is that of matrices over the set of natural numbers $\mathbb{N}$. An *idempotent* matrix theory is a matrix theory over a semiring in which $1 + 1 = 1$. The free idempotent matrix theory is that of matrices over the Boolean semiring $\{0, 1\}$.

Second, iteration $^\dagger: T(m, m+n) \to T(m, n)$ for $m, n \in \mathbb{N}$ used in a (strong) iteration theory T is replaced by a 'selfdual' operation $^*: T(n, n) \to T(n, n)$ for $n \in \mathbb{N}$, which intuitively means to repeat nondeterministically the application of a morphism zero or more times. (A comparison between repetition * and iteration † will be made in Section 8.) Some axioms for repetition are listed below (there "+" stands for ordinary addition of matrices).

(R1)	$A^* = AA^* + 1_n$,	for $A \in T(n, n)$;
(R2)	$(A+B)^* = (A^*B)^*A^*$,	for $A, B \in T(n, n)$;
(R3)	$A(BA)^* = (AB)^*A$,	for $A \in T(m, n)$ and $B \in T(n, m)$;
(R4)	if $A\rho = \rho B$, then $A^*\rho = \rho B^*$,	for $A \in T(m, m)$, $B \in T(n, n)$ and ρ an $(m \times n)$-matrix over $\mathbb{N}$.

Some subcases of (R4) are of interest, namely when ρ is a particular $\{0, 1\}$-matrix. These are listed in Table 2, where ρ^{-1} denotes the transpose of the matrix ρ. The axioms satisfied by iteration in a strong iteration theory can be rewritten in terms of * as (R1), (R2), (R3), (R4f). (More about this rewriting may be found in Section 8.) By the Hypothesis, the dual axioms of (R1)-(R3), (R4f) also have to be considered as valid axioms for this nondeterministic case. (R3) is selfdual and, under (R3), the axioms (R1), (R2) are selfdual, too. The dual of (R4f) is (R4f^{-1}).

Table 2.

Restriction	ρ = function	ρ = surjection	ρ = bijection	ρ^{-1} = function	ρ^{-1} = surjection:
Notation	(R4f)	(R4s)	(R4b)	(R4f^{-1})	(R4s^{-1})

These lead to our basic algebra (i.e., repetition theory) defined as follows.

Definition 1.1. A *prerepetition theory* T is a matrix theory in which a repetition $^*: T(n, n) \to T(n, n)$ for $n \in \mathbb{N}$ is given. A *repetition theory* is a prerepetition theory in which the axioms (R1), (R2), (R3), (R4f), and (R4f^{-1}) hold.

Basic example (*semantical theories for nondeterministic flowchart schemes, cf.* [18]). Fix a set of memory states, D say, and consider the set $\mathbf{Rel}_D(1, 1)$ consisting of all relations over D. The theory $\mathbf{Rel}_D$ of matrices over $\mathbf{Rel}_D(1, 1)$, endowed with the usual operations (addition = union of relations, multiplication = composition of relations and $A^* = 1_n + A + A^2 + \cdots$ for $A \in \mathbf{Rel}_D(n, n)$), is a repetition theory.

Another example (*more general example*). The theory of matrices over a normal Kleene algebra (cf. [5, p. 34]) is a repetition theory.

In order to compare this algebra with that used in the first version of this paper [20] (also in [18]), we prove that, under a 'divisibility' condition, (R4) is equivalent to (R4f) & (R4f^{-1}).

A monoid (denoted additive) is said *divisible* if $\sum_i a_i = \sum_j b_j$ implies that there exist elements denoted $a_i \,\&\, b_j$ such that $\sum_i a_i \,\&\, b_j = b_j$ and $\sum_j a_i \,\&\, b_j = a_i$, where i, j vary in finite index sets (in fact, this has to be verified only for index sets with two elements; the other cases follow); equidivisible monoids (cf. [16]) are divisible.

A semiring is divisible if its additive monoid is divisible. A (pre)repetition theory is divisible if its supporting semiring is divisible.

In the sequel only the following consequence of divisibility will be used.

Fact 1.2. *If A, B are matrices over a divisible semiring S and if σ, τ are $\{0,1\}$-matrices such that σ^{-1}, τ represent functions and $A\sigma\tau=\sigma\tau B$, then there exists a matrix Z over S such that $A\sigma=\sigma Z$ and $Z\tau=\tau B$.*

Proof. Obviously, A, B are square. Suppose that $A=(a_{i,j})_{i,j\in[m]}$ [1], $B=(b_{i,j})_{i,j\in[n]}$, σ^{-1} represents the function $f:[p]\to[m]$, and τ the function $g:[p]\to[n]$. The equality $A\sigma\tau=\sigma\tau B$ says that

$$\sum_{s\in g^{-1}(j)} a_{i,f(s)}=\sum_{r\in f^{-1}(i)} b_{g(r),j} \quad \text{for } i\in[m], j\in[n];$$

hence, by divisibility, there are elements $a_{i,f(s)}\,\&\,b_{g(r),j}$ for $r\in f^{-1}(i)$, $s\in g^{-1}(j)$ such that

$$\sum_r a_{i,f(s)}\,\&\,b_{g(r),j}=a_{i,f(s)} \quad \text{and} \quad \sum_s a_{i,f(s)}\,\&\,b_{g(r),j}=b_{g(r),j}.$$

Now, $Z=(a_{f(r),f(s)}\,\&\,b_{g(r),g(s)})_{r,s\in[p]}$ fulfils $A\sigma=\sigma Z$ and $Z\tau=\tau B$. □

Proposition 1.3. *In a divisible repetition theory* (R4) *holds.*

Proof. Every matrix over $\mathbb{N}$, ρ say, has a decomposition $\rho=\sigma\tau$, where σ, τ are $\{0,1\}$-matrices such that σ^{-1}, τ represent functions. By Fact 1.2, (R4) follows from (R4f) and (R4f^{-1}). □

A semiring is *zerosum-free* if $a+b=0$ implies $a=b=0$ (cf. [14]). A (pre)repetition theory is zerosum-free if its supporting semiring is zerosum-free.

An easily provable consequence is the following fact.

Fact 1.4. *Let S be a zerosum-free semiring, A, B be matrices over S and ρ be a matrix over $\mathbb{N}$.*

(a) *If $A\rho=0$ and ρ has no rows containing only zeros, then $A=0$.*
(b) *If $\rho A=0$ and ρ has no columns containing only zeros, then $A=0$.*

We conclude this section with rewriting two technical results from the deterministic case [19, Propositions B.1 and B.3]: Consider a particular case of (R3), namely, (R3i) which is (R3) applied only to A's of the form $[1_m\ 0_{m,p}]$. The first result rewrites the 'pairing axiom' and seems to be well known (the oldest reference we have found is in [5], namely P.J. Cleave, 1961); however, all the proofs known to us (i.e., [5, Theorem 4, p. 27] and [14, Theorem 4.21]) are given in a stronger context, namely

[1] Typically, $[m]$ denotes the set $\{1,\ldots,m\}$.

where the star of an $(n\times n)$-matrix is *computed* by using the formula $A^* = 1_n + A + A^2 + \cdots$.

Proposition 1.5. *If* (R1), (R2), (R3i), *and* (R4b) *hold in a prerepetition theory T, then*

$$\text{(P)}\qquad \begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} A^* + A^*BWCA^* & A^*BW \\ WCA^* & W \end{bmatrix}, \quad \textit{where } W = (CA^*B + D)^*$$

holds in T.

Sketch of a direct proof. By using (R3i) and (R1), first prove the formula for $B=0$, $D=0$. Use this and (R4b) for an adequate permutation to prove the formula for $A=0$, $C=0$. Finally, apply (R2) and these to

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ 0 & D \end{bmatrix}. \qquad \square$$

Proposition 1.6. *A prerepetition theory T in which* (R1), (R2), (R3i), (R4s), *and* (R4s^{-1}) *hold is a repetition theory.*

Sketch of a direct proof. By Proposition 1.5, (P) holds in T. By (R4b) the lower-left corner in $[\begin{smallmatrix} 0 & B \\ A & 0 \end{smallmatrix}]^*$ equals the upper-right corner in $[\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}]^*$. By (P), this gives (R3). By (P), the axioms (R4s), (R4s^{-1}) imply (R4) for all $\rho = [\begin{smallmatrix} \sigma & 0 \\ 0 & 0 \end{smallmatrix}]$ such that σ or σ^{-1} is a surjective function. By (R4b) this covers (R4f), (R4f^{-1}). $\square$

2. Flowchart theories

According to the Introduction, a *flowchart theory* is a certain $\mathbf{Fl}_{X,T}$ with a suitable choice of a double indexed set X of variables for atomic flowchart schemes and with a support theory T for connections. Initially, the support theory is a prerepetition theory, the basic model being the theory of matrices over $\{0, 1\}$. Since, in a matrix theory, every morphism equals the matrix of its components, we may restrict ourselves to the particular case of X having variables only for flowchart schemes with one input and one exit. In this case, a generic element in $\mathbf{Fl}_{X,T}(m, n)$ is

$$F = ((1_m \oplus x_1 \oplus \cdots \oplus x_k) \cdot f)\uparrow^k$$

for $f \in T(m+k, n+k)$. We shall frequently use the following, more compact, representation based on $f = [\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]$, i.e.,

$$F = \begin{array}{c|c} & n \\ \begin{matrix} m \\ e \end{matrix} & \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \end{array}, \quad \text{where } e \text{ is the string } x_1 \ldots x_k.$$

Here, the A-part gives the visible connections of the flowchart F (external connections), the D-part gives the nonvisible connections of F (internal connections), the

B-part gives the input from the exterior into the box representing F, and the C-part gives the outputs from the box representing F to the exterior. (Note that this representation has some similarities with Conway's general linear mechanism, cf. [5, p. 45]. Roughly speaking, the main difference is due to the fact that we use variables—compare Conway's semantics $A+BD^*C$ to ours, given in the beginning of Section 6.)

For example, in this framework, the flowchart from Fig. 1 may be represented as in Fig. 3, where x_j^i denotes the restriction of the double ranked variable x to its i-input and j-output. Further, the flowchart for Fig. 3 may be represented as follows:

$$
\begin{array}{c}
\begin{array}{cccccccccc}
 & 1 & 2 & 3 & x_1^1 & x_2^1 & x_1^2 & x_2^2 & y_1^1 & y_2^1
\end{array}\\
\begin{array}{c} 1\\ 2\\ x_1^1\\ x_2^1\\ x_1^2\\ x_2^2\\ y_1^1\\ y_2^1 \end{array}
\left[\begin{array}{ccc|cccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right]
\end{array}
.
$$

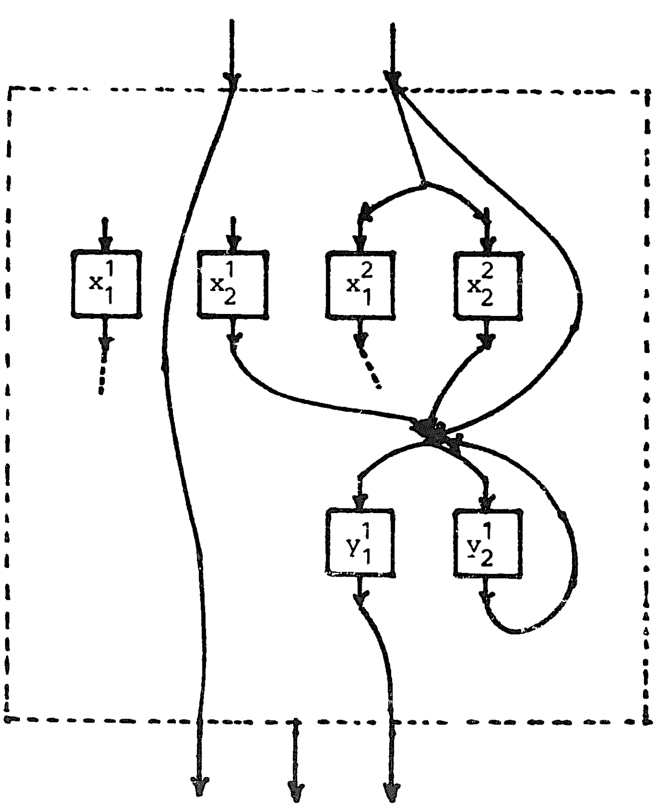

Fig. 3.

The operations in T can be naturally extended to flowcharts. *Sum* (or *union*) is defined by the formula

$$\begin{array}{cc} & \begin{array}{cc} n & \end{array} \\ \begin{array}{c} m \\ e \end{array} & \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \end{array} + \begin{array}{cc} & \begin{array}{cc} n & \end{array} \\ \begin{array}{c} m \\ e' \end{array} & \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right] \end{array} = \begin{array}{cc} & \begin{array}{cc} n & \end{array} \\ \begin{array}{c} m \\ e \\ e' \end{array} & \left[\begin{array}{c|cc} A+A' & B & B' \\ \hline C & D & 0 \\ C' & 0 & D' \end{array}\right] \end{array}.$$

Composition (or *sequential composition*) is defined by the formula

$$\begin{array}{cc} & \begin{array}{cc} n & \end{array} \\ \begin{array}{c} m \\ e \end{array} & \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \end{array} \cdot \begin{array}{cc} & \begin{array}{cc} p & \end{array} \\ \begin{array}{c} n \\ e' \end{array} & \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right] \end{array} = \begin{array}{cc} & \begin{array}{cc} p & \end{array} \\ \begin{array}{c} m \\ e \\ e' \end{array} & \left[\begin{array}{c|cc} AA' & B & AB' \\ \hline CA' & D & CB' \\ C' & 0 & D' \end{array}\right] \end{array}.$$

Repetition is defined by the formula

$$\begin{array}{cc} & \begin{array}{cc} n & \end{array} \\ \begin{array}{c} n \\ e \end{array} & \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]^* \end{array} = \begin{array}{cc} & \begin{array}{cc} n & \end{array} \\ \begin{array}{c} n \\ e \end{array} & \left[\begin{array}{c|c} A^* & A^*B \\ \hline CA^* & CA^*B+D \end{array}\right] \end{array}.$$

Note that the sum can be obtained from *separated sum* (or *parallel composition*) defined as follows:

$$\begin{array}{cc} & \begin{array}{cc} n & \end{array} \\ \begin{array}{c} m \\ e \end{array} & \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \end{array} \oplus \begin{array}{cc} & \begin{array}{cc} n' & \end{array} \\ \begin{array}{c} m' \\ e' \end{array} & \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right] \end{array} = \begin{array}{cc} & \begin{array}{cccc} n & n' & & \end{array} \\ \begin{array}{c} m \\ m' \\ e \\ e' \end{array} & \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & A' & 0 & B' \\ \hline C & 0 & D & 0 \\ 0 & C' & 0 & D' \end{array}\right] \end{array}$$

by using the formula $F+F'=[1_m\ 1_m](F\oplus F')\left[\begin{smallmatrix}1_n\\1_n\end{smallmatrix}\right]$. Also note that these definitions are compatible with those used in the deterministic case (in [18, 19]).

3. Equivalent flowcharts

Here we introduce a natural equivalence relation on flowcharts (denoted $\equiv_{\mathrm{nd}}$) and prove that the quotient theory $\mathrm{Fl}_{X,T}/\equiv_{\mathrm{nd}}$ preserves the prerepetition-theory structure of T.

Our flowchart schemes are protected, i.e., we have access only to their inputs and outputs. Hence, we allow a flowchart to be changed (for example, in order to minimize its vertex number) as far as its input-output behaviour does not change. The natural changes used here are: deleting or adding nonaccessible or noncoaccessible parts and folding or unfolding some vertices.

For e and e', strings over X, let us denote by $\boldsymbol{R}_X(e, e')$ the set of all $(|e| \times |e'|)$-matrices $(y_{i,i})$ over $\mathbb{N}$ ($|e|$ stands for the length of e) which preserve letters; that is,

$$\text{if } y_{ij} \neq 0, \quad \text{then } e_i = e'_j$$

where, typically, e_i stands for the ith letter of e. (Note that if e or e' is the empty string, then $\boldsymbol{R}_X(e, e')$ has exactly one element.) We allow 1_e to stand for the identity $(|e| \times |e|)$-matrix. Since the theory of matrices over $\mathbb{N}$ is the free matrix theory, morphisms in $\boldsymbol{R}_X$ may be considered as elements in T; moreover, this embedding preserves composition.

Definition 3.1 (*Basic relation* $\rightarrow$). Two X-flowcharts over T

$$F = \begin{array}{c c} & n \\ \begin{array}{c} m \\ e \end{array} & \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \end{array} \quad \text{and} \quad F' = \begin{array}{c c} & n \\ \begin{array}{c} m \\ e' \end{array} & \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right] \end{array}$$

are *in simulation* if there exists a $y \in \boldsymbol{R}_X(e, e')$ such that

$$\begin{bmatrix} A & By \\ C & Dy \end{bmatrix} = \begin{bmatrix} A' & B' \\ yC' & yD' \end{bmatrix};$$

in such a case, we write $F \rightarrow_y F'$ (or $F \rightarrow F'$ when y is of no importance).

In the case of flowcharts over $\{0, 1\}$-matrices one may think about $F \rightarrow_y F'$ as 'via y, every path in F' (going from an input) has a corresponding path in F, *and* every reverse path in F (going from an output) has a corresponding reverse path in F'.' The simulation used in automata theory (Eilenberg's 'state-mappings' [7, p. 38]) takes into account only the first half of this rewriting. Therefore, one should read $F \rightarrow_y F'$ as 'via y, F simulates F' and $(F')^{\circ}$ simulates F°' rather than 'F simulates F' via y', but we prefer the shorter name 'F and F' are in simulation via y'.

The simulation and all its particular cases in Table 3 are reflexive and transitive relations, but (except for $\approx$) *not* symmetrical. The *deterministic equivalence* $\equiv_{\mathrm{d}}$ is the congruence generated by $\rightarrow^{\mathrm{f}}$; in the first part of this paper we proved that

$$\equiv_{\mathrm{d}} = \overset{\mathrm{s}}{\longrightarrow} \cdot \overset{\mathrm{i}}{\longleftarrow} \cdot \overset{\mathrm{i}}{\longrightarrow} \cdot \overset{\mathrm{s}}{\longleftarrow}$$

(cf. [19, Theorem 3.8]); in particular, this shows that $\rightarrow^{\mathrm{f}}$ and $\rightarrow^{\mathrm{s}} \cup \rightarrow^{\mathrm{i}}$ (i.e., folding + adding) generate the same congruence. Similarly, the *codeterministic equivalence* is the congruence relation generated by ${}^{\mathrm{f}^{-1}}\!\leftarrow$, or equivalently, by ${}^{\mathrm{s}^{-1}}\!\leftarrow \cup {}^{\mathrm{i}^{-1}}\!\leftarrow$.

Definition 3.2. The *nondeterministic equivalent* $\equiv_{\mathrm{nd}}$ is the equivalence relation generated by the union of deterministic and codeterministic equivalences.

Table 3.
Some particular simulation relation (y^{-1} is the transpose matrix of y).

Restriction: y is a matrix over $\{0,1\}$ representing	Notation	Name	Notations and name for the inverse relation
$y=$ function	$\xrightarrow{\mathrm{f}}$	deterministic simulation	
$y=$ surjective function	$\xrightarrow{\mathrm{s}}$	folding	$\xleftarrow{\mathrm{s}}=$ unfolding
$y=$ injective function	$\xrightarrow{\mathrm{i}}$	adding	$\xleftarrow{\mathrm{i}}=$ deleting
$y^{-1}=$ function	$\xrightarrow{\mathrm{f}^{-1}}$		$\xleftarrow{\mathrm{f}^{-1}}=$ codeterministic simulation
$y^{-1}=$ surjective function	$\xrightarrow{\mathrm{s}^{-1}}$	counfolding	$\xleftarrow{\mathrm{s}^{-1}}=$ cofolding
$y^{-1}=$ injective function	$\xrightarrow{\mathrm{i}^{-1}}$	codeleting	$\xleftarrow{\mathrm{i}^{-1}}=$ coadding
$y=$ bijective function	$\approx$	isomorphism	

To be more specific, we note that $F\equiv_{\mathrm{nd}}F'$ iff there exist $F_1,F_2,\ldots$ such that

$$F\xrightarrow{\mathrm{f}}F_1\xleftarrow{\mathrm{f}}F_2\xleftarrow{\mathrm{f}^{-1}}F_3\xrightarrow{\mathrm{f}^{-1}}F_4\xrightarrow{\mathrm{f}}\cdots\xrightarrow{\mathrm{f}^{-1}}F'.$$

By the above observation $\equiv_{\mathrm{nd}}$ also should be defined as the equivalence generated by

$$\xrightarrow{\mathrm{s}}\cup\xrightarrow{\mathrm{i}}\cup\xleftarrow{\mathrm{s}^{-1}}\cup\xleftarrow{\mathrm{i}^{-1}}.$$

Examples. In order to have some intuitive understanding of what $\equiv_{\mathrm{nd}}$ means, we give generic examples for its basic generators $\rightarrow^{\mathrm{s}}$, $\rightarrow^{\mathrm{i}}$, ${}^{\mathrm{s}^{-1}}\!\leftarrow$ and ${}^{\mathrm{i}^{-1}}\!\leftarrow$.

(i) For folding, note that

$$\begin{array}{c} \\ e \\ e \end{array}\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & D_{11} & D_{12} \\ C & D_{21} & D_{22} \end{array}\right]\xrightarrow[\left[\begin{smallmatrix}1_e\\1_e\end{smallmatrix}\right]]{\mathrm{s}}\begin{array}{c} \\ e \end{array}\left[\begin{array}{c|c} A & B_1+B_2 \\ \hline C & D_{11}+D_{12} \end{array}\right]$$

iff $D_{11}+D_{12}=D_{21}+D_{22}$; generally, $F\rightarrow^{\mathrm{s}}F'$ means that F' can be obtained from F by identifying vertices which have the same label and whose output connections (i.e., whose corresponding rows in the matrix of F) are equal, after identification.

(ii) For adding, note that

$$\begin{array}{c} \\ e_1 \end{array}\left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right]\xrightarrow[\left[\begin{smallmatrix}1_{e_1}\\0\end{smallmatrix}\right]]{\mathrm{i}}\begin{array}{c} \\ e_1 \\ e_2 \end{array}\left[\begin{array}{c|cc} A & B_1 & 0 \\ \hline C_1 & D_{11} & 0 \\ C_2 & D_{21} & D_{22} \end{array}\right];$$

generally, $F\rightarrow^{\mathrm{i}}F'$ means that F' can be obtained from F by adding nonaccessible vertices.

(iii) For cofolding, note that

$$\begin{array}{c} \\ e \\ e \end{array}\left[\begin{array}{c|cc} A & B & B \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \xleftarrow[\left[\begin{smallmatrix}1_e\\1_e\end{smallmatrix}\right]]{\mathrm{s}^{-1}} \begin{array}{c} \\ e \end{array}\left[\begin{array}{c|c} A & B \\ \hline C_1+C_2 & D_{11}+D_{21} \end{array}\right]$$

iff $D_{11}+D_{21}=D_{12}+D_{22}$; generally, $F \overset{\mathrm{s}^{-1}}{\longleftarrow} F'$ means that F' can be obtained from F by identifying vertices which have the same label and whose input connections (i.e., whose corresponding columns in the matrix of F) are equal, after identification.

(iv) For coadding, note that

$$\begin{array}{c} \\ e_1 \end{array}\left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right] \xleftarrow[\left[\begin{smallmatrix}1_{e_1}\\0\end{smallmatrix}\right]]{\mathrm{i}^{-1}} \begin{array}{c} \\ e_1 \\ e_2 \end{array}\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ 0 & 0 & D_{22} \end{array}\right];$$

generally, $F \overset{\mathrm{i}^{-1}}{\longleftarrow} F'$ means that F' can be obtained from F by adding noncoaccessible vertices.

(v) *Exercise*: use Fig. 1 and the 'decodification'

$$\begin{array}{c} \\ e \end{array}\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \to \left((1\oplus x_1\oplus\cdots\oplus x_k)\cdot\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)\uparrow^k$$

where $e=x_1\ldots x_k$, in order to obtain graphical representations of the simulations in (i)–(iv).

An easy computation shows that all the simulation relations are compatible with sum, composition and repetition (in the sense expressed by the following lemma).

Lemma 3.3. *Suppose that* $F_1 \underset{y}{\overset{\alpha}{-}} F_1'$ *and* $F_2 \underset{z}{\overset{\beta}{-}} F_2'$, *for arbitrary* $\overset{\alpha}{-}, \overset{\beta}{-} \in \{\to, \to^{\mathrm{f}}, {}^{\mathrm{f}}\!\leftarrow, {}^{\mathrm{f}^{-1}}\!\leftarrow, \to^{\mathrm{f}^{-1}}, \leftarrow$, *etc.*$\}$; *denote*

$$y' = \begin{bmatrix} y & 0 \\ 0 & 1_{e_2} \end{bmatrix} \quad \textit{and} \quad z' = \begin{bmatrix} 1_{e_1'} & 0 \\ 0 & z \end{bmatrix}.$$

Then the following relations hold, whenever the operations make sense:

(i) $F_1+F_2 \underset{y'}{\overset{\alpha}{-}} F_1'+F_2 \underset{z'}{\overset{\beta}{-}} F_1'+F_2'$;

(ii) $F_1\cdot F_2 \underset{y'}{\overset{\alpha}{-}} F_1'\cdot F_2 \underset{z'}{\overset{\beta}{-}} F_1'\cdot F_2'$;

(iii) $F_1^* \underset{y}{\overset{\alpha}{-}} F_1'^*$.

Proposition 3.4. *The equivalence relation* $\equiv_{\mathrm{nd}}$ *is compatible with sum, composition and repetition.*

Proof. The proof directly follows from Lemma 3.3. □

This proposition shows that $+$, $\cdot$, and $*$ are well defined in $\mathbf{Fl}_{X,T}/\equiv_{\mathrm{nd}}$.

Theorem 3.5. *If T is a prerepetition theory, then $\mathbf{Fl}_{X,T}/\equiv_{\mathrm{nd}}$ is a prerepetition theory.*

Indirect proof. Since $\equiv_{\mathrm{nd}}$ contains $\equiv_{\mathrm{d}}$ and T is a para-iteration theory, by [19, Theorem 3.11], $\mathbf{Fl}_{X,T}/\equiv_{\mathrm{nd}}$ is an algebraic theory. This and the dual result prove the theorem. $\square$

Sketch of a direct proof. In $\mathbf{Fl}_{X,T}(1,1)$ the sum is associative with neutral element 0 and the composition is associative with neutral element 1; moreover, $F+F' \approx F'+F$, $F\cdot(F'+F'') \rightarrow^{\mathrm{s}^{-1}} F\cdot F'+F\cdot F''$, $(F'+F'')\cdot F \,{}^{\mathrm{s}}\!\!\leftarrow F'\cdot F+F''\cdot F$, $F\cdot 0 \rightarrow^{\mathrm{i}^{-1}} 0$ and $0\cdot F \,{}^{\mathrm{i}}\!\!\leftarrow 0$. Hence $\mathbf{Fl}_{X,T}(1,1)/\equiv_{\mathrm{nd}}$ is a semiring. Denote by x_i^n the $(1\times n)$-matrix $(\delta_{i,k})_k$, where $\delta_{i,k} =$ "**if** $i=k$ **then** 1 **else** 0". Further, denote by y_i^n the $(n\times 1)$-matrix $(\delta_{k,i})_k$. An $(m\times n)$-matrix over $\mathbf{Fl}_{X,T}(1,1)$ may be defined as

$$(F_{ij})_{i,j} = \sum_{i\in[m]} \sum_{j\in[n]} y_i^m \cdot F_{ij} \cdot x_j^n.$$

Since

$$(x_i^m \cdot F\cdot y_j^n)_{i,j} \xleftarrow{\mathrm{s}^{-1}} \cdot \xrightarrow{\mathrm{s}} F \quad\text{and}\quad x_i^m \cdot (F_{ij})_{i,j} \cdot y_i^n \xleftarrow{\mathrm{i}^{-1}} \cdot \xrightarrow{\mathrm{i}} F_{ij},$$

$\mathbf{Fl}_{X,T}(m,n)/\equiv_{\mathrm{nd}}$ is isomorphic with the set of all $(m\times n)$-matrices over $\mathbf{Fl}_{X,T}(1,1)/\equiv_{\mathrm{nd}}$. Since

$$(F_{ij})_{i,j} + (F'_{ij})_{i,j} \approx (F_{ij}+F'_{ij})_{i,j}$$

and

$$(F_{ij})_{i,j}\cdot (F'_{jk})_{j,k} \xleftarrow{\mathrm{s}} \cdot \xrightarrow{\mathrm{s}^{-1}} \left(\sum_j F_{ij}\cdot F'_{jk}\right)_{i,k}$$

sum and composition in $\mathbf{Fl}_{X,T}/\equiv_{\mathrm{nd}}$ are ordinary matrix addition and multiplication. $\square$

On the other hand, we define $\sim$*-equivalence* as the equivalence relation generated by $\rightarrow$. We conclude this section by pointing to a case in which $\sim\, = \,\equiv_{\mathrm{nd}}$.

Lemma 3.6. *If T is zerosum-free and divisible, then*

$$\rightarrow \subseteq \xrightarrow{\mathrm{i}^{-1}} \cdot \xrightarrow{\mathrm{s}^{-1}} \cdot \xrightarrow{\mathrm{s}} \cdot \xrightarrow{\mathrm{i}}.$$

Proof. Suppose that $F' \rightarrow_y F''$. Using isomorphic representations for F', F'' we may suppose

$$F' = \begin{array}{c} \\ e_1 \\ e_3 \end{array}\!\!\left[\begin{array}{c|cc} A & B_1 & B_3 \\ \hline C_1 & D_{11} & D_{13} \\ C_3 & D_{31} & D_{33} \end{array}\right], \qquad F'' = \begin{array}{c} \\ e_2 \\ e_4 \end{array}\!\!\left[\begin{array}{c|cc} A & B_2 & B_4 \\ \hline C_2 & D_{22} & D_{24} \\ C_4 & D_{42} & D_{44} \end{array}\right]$$

and $y = \left[\begin{smallmatrix} z & 0 \\ 0 & 0 \end{smallmatrix}\right]$ for a matrix $z \in \boldsymbol{R}_X(e_1, e_2)$ without rows or columns containing only zeros. The simulation says that

$$\begin{bmatrix} A & B_1 z & 0 \\ C_1 & D_{11} z & 0 \\ C_3 & D_{31} z & 0 \end{bmatrix} = \begin{bmatrix} A & B_2 & B_4 \\ zC_2 & zD_{22} & zD_{24} \\ 0 & 0 & 0 \end{bmatrix}$$

Consequently, $B_4 = 0$ and $C_3 = 0$; moreover, by applying Fact 1.4 to $zD_{24} = 0$ and $D_{31}z = 0$, we obtain $D_{24} = 0$, $D_{31} = 0$. Take

$$F^- = {}_{e_1}\left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right], \qquad F^\bullet = {}_{e_2}\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right],$$

$$z' = \begin{bmatrix} 1_{e_1} \\ 0 \end{bmatrix}, \quad \text{and} \quad z'' = [1_{e_2} \;\; 0];$$

it follows that

$$F' \xrightarrow[z']{\mathrm{i}^{-1}} F^- \xrightarrow[z]{} F^\bullet \xrightarrow[z'']{\mathrm{i}} F''.$$

Let $z = z_1 z_2$ be a decomposition where $z_1^{-1} \in \boldsymbol{R}_X(e^\sim, e_1)$, $z_2 \in \boldsymbol{R}_X(e^\sim, e_2)$ are $\{0, 1\}$-matrices representing surjective functions (obviously, there is one!). Since $D_{11}z = zD_{22}$, by Fact 1.2, there is a matrix Z such that $D_{11}z_1 = z_1 Z$ and $Zz_2 = z_2 D_{22}$. Take

$$F^\sim = {}_{e^\sim}\left[\begin{array}{c|c} A & B_1 z_1 \\ \hline z_2 C_2 & Z \end{array}\right];$$

it then follows that

$$F^- \xrightarrow[z_1]{\mathrm{s}^{-1}} F^\sim \xrightarrow[z_2]{\mathrm{s}} F^\bullet. \qquad \square$$

Corollary 3.7. *If T is zerosum-free and divisible, then* $\sim \; = \; \equiv_{\mathrm{nd}}$.

4. A characterization theorem for the flowchart equivalence

In this section a characterization theorem for the nondeterministic equivalence will be given, namely

nondeterministic equivalence
= (deleting · unfolding · codeleting · counfolding)
· (codeleting · counfolding · deleting · unfolding)$^{-1}$.

Roughly speaking, this shows that two flowcharts are nondeterministically equivalent iff, except for nonaccessible and noncoaccessible parts, they can be *refined* by unfolding the counfolding to the same flowchart. (Note that the situation is completely different from the deterministic case: two flowcharts are deterministically equivalent iff by folding and deleting they can be *reduced* to the same flowchart.)

Technically, the support theory T is supposed to be a prerepetition theory *with intersection*, namely a zerosum-free and 'uniform' divisible prerepetition theory; by 'uniform divisible' we mean that an operation $\&: T(1,1)\times T(1,1)\to T(1,1)$ is given which fulfils the following axioms:

(As) $x \& (y \& z) = (x \& y) \& z$;

(C) $x \& y = y \& x$;

(D) $(x+y) \& z = (x \& z) + (y \& z)$;

(A) $(x+y) \& x = x$.

The component-wise extension of "&" to arbitrary $T(m,n)$ is also denoted by &. (Clearly, uniform divisibility implies divisibility. Question: can every divisible monoid be made uniform divisible?)

Table 4. Is $x\cdot y \subseteq y \cdot x$?

x \ y	$\xleftarrow{i}$	$\xleftarrow{s}$	$\xrightarrow{i^{-1}}$	$\xrightarrow{s^{-1}}$	$\xrightarrow{s}$	$\xrightarrow{i}$	$\xleftarrow{s^{-1}}$	$\xleftarrow{i^{-1}}$
deleting $\xleftarrow{i}$	A-yes	N—yes	D—yes	E—yes	O—yes	P—yes	Q—no	R—yes
unfolding $\xleftarrow{s}$	B—yes	C—yes	F—no	G—no	S—yes	O—yes	T—no	Q—no
codeleting $\xrightarrow{i^{-1}}$	D—yes	E—yes	A—yes	N—yes	Q—no	R—yes	O—yes	P—yes
counfolding $\xrightarrow{s^{-1}}$	F—no	G—no	B—yes	C—yes	T—no	Q—no	S—yes	O—yes
folding $\xrightarrow{s}$	H—yes	I—yes	J—yes	K—yes	C—yes	N—yes	G—no	E—yes
adding $\xrightarrow{i}$	L—yes	H—yes	M—yes	J—yes	B—yes	A—yes	F—no	D—yes
cofolding $\xleftarrow{s^{-1}}$	J—yes	K—yes	H—yes	I—yes	G—no	E—yes	C—yes	N—yes
coadding $\xleftarrow{i^{-1}}$	M—yes	J—yes	L—yes	H—yes	F—no	D—yes	B—yes	A—yes

(1) By an occurrence of a letter, Z say, in front of an answer, we mean that this answer directly follows from Lemma Z of Appendix A, by duality (i.e., interchange deleting—codeleting and so on) or inversion (i.e., $x\cdot y\subseteq y\cdot x$ implies $y^{-1}\cdot x^{-1}\subseteq x^{-1}\cdot y^{-1}$).

(2) Only the facts in the lower-left triangle (i.e., Lemmas A-M) will be used in the proof of Theorem 4.1.

Example. For an arbitrary set D the repetition theory $\mathbf{Rel}_D$ is naturally with intersection (the intersection of relations stands for &).

Theorem 4.1 (characterization theorem for the flowchart equivalence). *If T is a* *prerepetition theory with intersection, then, in* $\mathbf{Fl}_{X,T}$ *the following equalities hold*:

$$\equiv_{\mathrm{nd}} = \xleftarrow{\mathrm{i}} \cdot \xleftarrow{\mathrm{s}} \cdot \xrightarrow{\mathrm{i}^{-1}} \cdot \xrightarrow{\mathrm{s}^{-1}} \cdot \xrightarrow{\mathrm{s}} \cdot \xrightarrow{\mathrm{i}} \cdot \xleftarrow{\mathrm{s}^{-1}} \cdot \xleftarrow{\mathrm{i}^{-1}} = \leftarrow \cdot \rightarrow \cdot \leftarrow.$$

For a proof of Theorem 4.1, beside the simple commutations pointed out in Table 4, we will need the following facts.

Lemma Fe

$$\xrightarrow{\mathrm{s}^{-1}} \cdot \xleftarrow{\mathrm{i}} \subseteq \xleftarrow{\mathrm{i}} \cdot \xrightarrow{\mathrm{s}^{-1}} \cdot \quad \xrightarrow{\mathrm{i}};$$

$$(d/i)\ \xleftarrow{\mathrm{i}^{-1}} \cdot \xrightarrow{\mathrm{s}} \subseteq \xrightarrow{\mathrm{i}^{-1}} \cdot \quad \xrightarrow{\mathrm{s}} \cdot \xleftarrow{\mathrm{i}^{-1}}.$$

Lemma Ge

$$\xrightarrow{\mathrm{s}^{-1}} \cdot \xleftarrow{\mathrm{s}} \subseteq \xleftarrow{\mathrm{s}} \cdot \xrightarrow{\mathrm{s}^{-1}} \cdot \quad \xrightarrow{\mathrm{s}};$$

$$(d/i)\ \xleftarrow{\mathrm{s}^{-1}} \cdot \xrightarrow{\mathrm{s}} \subseteq \xrightarrow{\mathrm{s}^{-1}} \cdot \quad \xrightarrow{\mathrm{s}} \cdot \xleftarrow{\mathrm{s}^{-1}}.$$

The proofs of the Lemmas Fe, Ge and A–M (for Table 4) will be given in Appendix A.

Proof of Theorem 4.1 assuming Lemmas A–M, Fe, Ge. By Lemma 3.6, we have to prove only the first equality. The correctness of the answers in the lower-left half of Table 4 directly follows from the Lemmas A–M. Let ρ be the relation

$$\xleftarrow{\mathrm{i}} \cdot \xleftarrow{\mathrm{s}} \cdot \xrightarrow{\mathrm{i}^{-1}} \cdot \xrightarrow{\mathrm{s}^{-1}} \cdot \xrightarrow{\mathrm{s}} \cdot \xrightarrow{\mathrm{i}} \cdot \xleftarrow{\mathrm{s}^{-1}} \cdot \xleftarrow{\mathrm{i}^{-1}}.$$

We know that

$$F \equiv_{\mathrm{nd}} F' \text{ iff } \exists n \in \mathbb{N} \text{ with } F \rho^n F'.$$

The proof will be concluded if we can prove that ρ is transitive, i.e., the relations $\rho \cdot {}^{\mathrm{i}}\!\leftarrow$, $\rho \cdot {}^{\mathrm{s}}\!\leftarrow$, $\rho \cdot \rightarrow^{\mathrm{i}^{-1}}$, $\rho \cdot \rightarrow^{\mathrm{s}^{-1}}$, $\rho \cdot \rightarrow^{\mathrm{s}}$, $\rho \cdot \rightarrow^{\mathrm{i}}$, $\rho \cdot {}^{\mathrm{s}^{-i}}\!\leftarrow$, and $\rho \cdot {}^{\mathrm{i}^{-1}}\!\leftarrow$ are included in ρ. All the inclusions, except for $\rho \cdot {}^{\mathrm{i}}\!\leftarrow$, $\rho \cdot {}^{\mathrm{s}}\!\leftarrow$ and $\rho \cdot \rightarrow^{\mathrm{s}}$, directly follow from (the reflexivity of ${}^{\mathrm{i}}\!\leftarrow, \ldots, {}^{\mathrm{i}^{-1}}\!\leftarrow$ and) the facts in the lower-left half of Table 4. In the excepted cases we can use Lemmas Fe, Ge to correct the negative answers of F, G in Table 4; these lemmas give the needed commutations, but add supplementary terms; since, by Lemmas A–M in Table 4, these supplementary terms can be deleting, in the excepted cases, the inclusions also hold. $\square$

We conclude this section with an example showing that minimal flowcharts with respect to the nondeterministic equivalence need not be unique (up to an isomorphism). Clearly,

$$F = \begin{array}{c} \\ x \\ x \end{array}\!\left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 0 \end{array}\right] \xleftarrow{s} \begin{array}{c} \\ x \\ x \\ x \end{array}\!\left[\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right] \xleftarrow{s^{-1}} \begin{array}{c} \\ x \\ x \end{array}\!\left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ 1 & 0 & 1 \end{array}\right] = F'.$$

On the other hand, F and F' are minimal, but nonisomorphic flowcharts.

5. $\mathbf{Fl}_{X,T}/\equiv_{\mathrm{nd}}$ preserves the repetition-theory structure of T

In this section we shall show that under the condition 'with intersection', the axioms for repetition imply themselves for flowcharts.

By Proposition 1.3, (R4) holds in T and, by Corollary 3.7, $\equiv_{\mathrm{nd}} = \sim$. We shall prove that (R1)-(R4) hold in $\mathrm{Fl}_{X,T}/\sim$. The axioms (R1)-(R3) are directly extensible to flowcharts. For the last axiom which, in fact, is a rule 'if $A\rho \sim \rho B$, then ...' we need to know when two flowcharts are $\sim$-equivalent. The partial answer given by Theorem 4.1 says that $\sim\, = \leftarrow \cdot \rightarrow \cdot \leftarrow$, and under this condition we can prove that (R4) implies itself for flowcharts.

Theorem 5.1. *If T is a repetition theory satisfying* (R4) *and T, X are such that in* $\mathrm{Fl}_{X,T}$ $\sim\, = \leftarrow \cdot \rightarrow \cdot \leftarrow$, *then* $\mathrm{Fl}_{X,T}/\sim$ *is a repetition theory that satisfies* (R4).

Proof. By Theorem 3.5, only the axioms of repetition remain to be proved. We shall verify them in the simpler, equivalent form given by Proposition 1.6.

For (R1), note that the flowchart $1+F\cdot F^*$ is

$$\begin{array}{c} \\ e \\ e \end{array}\!\left[\begin{array}{c|cc} 1+AA^* & B & AA^*B \\ \hline CA^* & D & CA^*B \\ CA^* & 0 & D+CA^*B \end{array}\right]$$

Now, by using (R1) in T we have $1+F\cdot F^* \rightarrow_y F^*$, for $y=[{}^{1}_{1}{}^{e}_{e}]$, hence the flowcharts F^* and $1+F\cdot F^*$ are $\sim$-equivalent. Consequently, (R1) implies itself for flowcharts.

For (R2), set $Y=(A+A')^*$ and $Z=(A^*A')^*A^*$. By (R1) and (R2) in T, we have $Y=Z=A^*+A^*A'Z$. The flowchart $(F+F')^*$ is

$$\begin{array}{c} \\ e \\ e' \end{array}\!\left[\begin{array}{c|cc} Y & YB & YB' \\ \hline CY & D+CYB & CYB' \\ C'Y & C'YB & D'+C'YB' \end{array}\right]$$

and $(F^*F')^*F^*$ is

$$\begin{array}{c} \\ e \\ e' \\ e \end{array}\!\left[\begin{array}{c|ccc} Z & ZB & ZB' & ZB \\ \hline CA^*A'Z & D+CA^*B+CA^*A'ZB & CA^*B'+CA^*A'ZB' & CA^*A'ZB \\ C'Z & C'ZB & D'+C'ZB' & C'ZB \\ CA^* & 0 & 0 & D+CA^*B \end{array}\right].$$

Now, $(F+F')^* \rightarrow_y (F^*F')^*F^*$, for

$$y=\begin{bmatrix} 1_e & 0 & 1_e \\ 0 & 1_{e'} & 0 \end{bmatrix},$$

hence these flowcharts are $\sim$-equivalent. Consequently, under (R1), the axiom (R2) implies itself for flowcharts.

For (R3i), take for F a scalar flowchart $[\frac{a}{\ }|\ ]$ (more exactly, only the cases $a=[1_m\ \ 0_{m,p}]$ are necessary). The flowcharts $F(F'F)^*$ and $(FF')^*F$ are

$$\begin{array}{c} \\ e' \end{array}\left[\begin{array}{c|c} a(A'a)^* & a(A'a)^*B' \\ \hline C'a(A'a)^* & D'+C'a(A'a)^*B' \end{array}\right] \quad \text{and}$$

$$\begin{array}{c} \\ e' \end{array}\left[\begin{array}{c|c} (aA')^*a & (aA')^*aB' \\ \hline C(aA')^*a & D'+C(aA')^*aB \end{array}\right].$$

Hence, $F(F'F)^*=(FF')^*F$. Consequently, (R3i) implies itself for flowcharts.

For the last axiom (R4) we have to show that $F\rho \sim \rho F'$ implies $F^*\rho \sim \rho F'^*$. Using the hypothesis that $\sim\ =\ \leftarrow \cdot \rightarrow \cdot \leftarrow$, we may suppose that the equivalence $F\rho \sim \rho F'$ is given by the following chain of simulations

$$F\rho \underset{a}{\leftarrow} F_1 \underset{b}{\rightarrow} F_1' \underset{c}{\leftarrow} \rho F'.$$

This means that

$$\begin{bmatrix} A\rho & B \\ aC\rho & aD \end{bmatrix} = \begin{bmatrix} A_1 & B_1a \\ C_1 & D_1a \end{bmatrix}, \begin{bmatrix} A_1 & B_1b \\ C_1 & D_1b \end{bmatrix} = \begin{bmatrix} A_1' & B_1' \\ bC_1' & bD_1' \end{bmatrix} \quad \text{and}$$

$$\begin{bmatrix} A_1' & B_1' \\ cC_1' & cD_1' \end{bmatrix} = \begin{bmatrix} \rho A' & \rho B'c \\ C' & D'c \end{bmatrix}. \tag{α}$$

Particularly, this gives $A\rho=\rho A'$; hence, by (R4) in T, $A^*\rho=\rho A'^*$. The above chain of simulations between $F\rho$ and $\rho F'$ can be translated into a chain of simulations between $F^*\rho$ and $\rho F'^*$, namely

$$F^*\rho \underset{a}{\leftarrow} F_2^*\rho \underset{b}{\rightarrow} \rho F_2'^* \underset{c}{\leftarrow} \rho F'^*,$$

where

$$F_2 = \begin{array}{c} \\ e_1 \end{array}\left[\begin{array}{c|c} A & B_1 \\ \hline aC & D_1 \end{array}\right] \quad \text{and} \quad F_2' = \begin{array}{c} \\ e_1' \end{array}\left[\begin{array}{c|c} A' & B'c \\ \hline C_1' & D_1' \end{array}\right].$$

A routine computation based on (α) shows that these are simulations, indeed. □

Using Theorem 4.1 we obtain the following corollary.

Corollary 5.2. *For every set D and X, the classes of $\equiv_{\mathrm{nd}}$-equivalent X-flowcharts over* $\mathbf{Rel}_D$ *form a repetition theory.*

6. The main result

Here we are in a position to state the main theorem which implies that, (in all interesting cases, i.e., $\mathbf{Rel}_D$), our calculus for nondeterministic flowcharts is similar to the calculus of polynomials; more exactly, the classes of $\equiv_{\mathrm{nd}}$-equivalent X-flowcharts over a repitition theory with intersection T form the repetition theory freely generated by adding the double indexed set X to T. Before this proof, we define the interpretation of flowcharts in arbitrary repetition theories.

As it was seen in the Introduction, a flowchart

$$F=\begin{array}{c c}
 & n \\
\begin{array}{c} m \\ e \end{array} & \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]
\end{array},$$

where $e=x_1\ldots x_k\in X^*$ and $f=[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}]\in T(m+k, n+k)$, represents the scheme

$$((1_m\oplus x_1\oplus\cdots\oplus x_k)\cdot f)\uparrow^k.$$

For its meaning in a (pre)repetition theory Q, we have to endow Q with the parallel composition $\oplus$ given by $I\oplus J=[\begin{smallmatrix} I & 0 \\ 0 & J\end{smallmatrix}]$, and with the k-right feedback $\uparrow^k$ given by $[\begin{smallmatrix} I & J \\ K & L\end{smallmatrix}]\uparrow^k=I+JL^*K$, for L a $(k\times k)$-matrix. Consequently, suppose we have an interpretation of the variables in the (pre)repetition theory Q given by a function $\varphi_X: X\to Q(1, 1)$ and an interpretation of the morphisms of Q by a *(pre)repetition-theory morphism* $\varphi_T: T\to Q$ (that is, a family of functions $\varphi_{m,n}: T(m, n)\to Q(m, n)$ which preserve the distinguished morphisms $0, 1$, the matrix-building and the operations "$\cdot$", "$+$", "$*$"); then we define the *interpretation* $\varphi^{\#}:\mathbf{Fl}_{X,T}\to Q$ as

(α) $\quad \varphi^{\#}(F)=\varphi_T(A)+\varphi_T(B)(\varphi^{\bullet}_X(e)\varphi_T(D))^*\varphi^{\bullet}_X(e)\varphi_T(C),$

where

$$F=\begin{array}{c c} \begin{array}{c} \\ e \end{array} & \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]\end{array} \quad\text{and}\quad \varphi^{\bullet}_X(e)=\varphi_X(x_1)\oplus\cdots\oplus\varphi_X(x_k),$$

for $e=x_1\ldots x_k$ ($x_i\in X$). The right-hand side expression shows what can be known at the outside of the machine that implements the algorithm F, after zero (i.e., $\varphi_T(A)$), one (i.e., $\varphi_T(B)\varphi^{\bullet}_X(e)\varphi_T(C)$), or more repetitions.

Theorem 6.1. *For a repetition theory T, the theory $\mathbf{Fl}_{X,T}/\equiv_{\mathrm{nd}}$ has the following universality property: there exists a function $I_X: X\to \mathbf{Fl}_{X,T}/\equiv_{\mathrm{nd}}(1, 1)$ and a prerepetition-theory morphism $I_T: T\to\mathbf{Fl}_{X,T}/\equiv_{\mathrm{nd}}$ such that, for every repetition theory Q, for every function $\varphi_X: X\to Q(1, 1)$ and every repetition-theory morphism $\varphi_T: T\to Q$, there exists a unique prerepetition-theory morphism $\varphi^{\wedge}:\mathbf{Fl}_{X,T}/\equiv_{\mathrm{nd}}\to Q$ such that $I_T\varphi^{\wedge}=\varphi_T$ and $I_X\varphi^{\wedge}=\varphi_X$.*

$$\begin{tikzcd}
T \arrow[r, "I_T"] \arrow[dr, "\varphi_T"'] & \mathbf{Fl}_{X,T}/\equiv_{\mathrm{nd}} \arrow[d, dashed, "\varphi^{\wedge}"] & X \arrow[l, "I_X"'] \arrow[dl, "\varphi_X"] \\
 & Q &
\end{tikzcd}$$

Proof. Consider the following embeddings of T and X in $\mathbf{Fl}_{X,T}$:

$$I'_T(A)=\left[\begin{array}{c|c} A & \\ \hline & \end{array}\right] \quad \text{and} \quad I'_X(x)=\begin{array}{c} \\ x \end{array}\!\left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right].$$

The necessary embeddings in $\mathbf{Fl}_{X,T}/\equiv_{\mathrm{nd}}$ are $I_T=I'_T\cdot\mathrm{pr}$ and $I_X=I'_X\cdot\mathrm{pr}$, where $\mathrm{pr}:\mathbf{Fl}_{X,T}\to\mathbf{Fl}_{X,T}/\equiv_{\mathrm{nd}}$ is the canonical projection. By using the formula (α) above, the interpretation (φ_X,φ_T) extends itself for flowcharts, i.e., $\varphi^*:\mathbf{Fl}_{X,T}\to Q$; moreover, below, for the sake of simplicity, we shall drop the writing of φ_T, φ_X and use the following typical notation: a stands for $\varphi_T(A)$, b for $\varphi_T(B)$, c for $\varphi^{\cdot}_X(e)\varphi_T(C)$, and d for $\varphi^{\cdot}_X(e)\varphi_T(D)$.

The first problem is to show that this extension is also good for $\mathbf{Fl}_{X,T}/\equiv_{\mathrm{nd}}$, i.e., all $\equiv_{\mathrm{nd}}$-equivalent flowcharts have the same interpretation. In fact, it is enough to prove this for a simulation, i.e., for y or y^{-1} a function:

$$\text{if } F\underset{y}{\to}F', \text{ then } \varphi^*(F)=\varphi^*(F').$$

For this, by definition, the simulation $F\to_y F'$ says that $A=A'$, $By=B'$, $C=yC'$, and $Dy=yD'$. Since, for the sorted relation $y\in\mathcal{R}_X(e,e')$, we have $\varphi^{\cdot}_X(e)y=y\varphi^{\cdot}_X(e')$, the equality $Dy=yD'$ gives $dy=\varphi^{\cdot}_X(e)Dy=\varphi^{\cdot}_X(e)yD'=y\varphi^{\cdot}_X(e')D'=yd'$. By applying the axiom (R4f) or (R4f^{-1}) in Q, this yields $d^*y=yd'^*$. Now, it easily follows that

$$\varphi^*(F)=a+bd^*c=a'+b'd'^*c'=\varphi^*(F').$$

Clearly, the function $\varphi^{\wedge}:\mathbf{Fl}_{X,T}/\equiv_{\mathrm{nd}}\to Q$, induced by φ^*, extends φ_T and φ_X, i.e., $I_T\varphi^{\wedge}=\varphi_T$ and $I_X\varphi^{\wedge}=\varphi_X$.

The second problem is to show that the function $\varphi^{\wedge}$ is a prerepetition-theory morphism. For this, we have to show that φ^* preserves the distinguished morphisms 0, 1, the matrix-building, and the aforementioned operations; this will be done by using the formula for the star of matrices given in Proposition 1.5. Since T contains all $\{0,1\}$-matrices and these are preserved by $\varphi_T=\varphi^*|_T$, we obtain that φ^* preserves all $\{0,1\}$-matrices. Moreover, since every matrix can be obtained by the composition of a diagonal matrix with two $\{0,1\}$-matrices while preserving composition, it is enough to prove that φ^* preserves diagonal-matrix building, namely the paraliel tion $\oplus$. This is easily shown as follows:

$$\begin{aligned}\varphi^*(F\oplus F')&=\begin{bmatrix} a & 0\\ 0 & a'\end{bmatrix}+\begin{bmatrix} b & 0\\ 0 & b'\end{bmatrix}\cdot\begin{bmatrix} d & 0\\ 0 & d'\end{bmatrix}^*\cdot\begin{bmatrix} c & 0\\ 0 & c'\end{bmatrix}\\ &=\begin{bmatrix} a+bd^*c & 0\\ 0 & a'+b'd'^*c'\end{bmatrix}\\ &=\varphi^*(F)\oplus\varphi^*(F').\end{aligned}$$

Now, the preservation of choice "+" follows using the equality

$$A+B=[1\ \ 1](A\oplus B)\begin{bmatrix}1\\1\end{bmatrix}.$$

Composition is also preserved since

$$\varphi^*(F\cdot F')=aa'+[b\ \ ab']\cdot\begin{bmatrix} d & cb' \\ 0 & d'\end{bmatrix}^*\cdot\begin{bmatrix} ca' \\ c'\end{bmatrix}$$

$$=aa'+[b\ \ ab']\cdot\begin{bmatrix} d^* & d^*cb'd'^* \\ 0 & d'\end{bmatrix}\cdot\begin{bmatrix} ca' \\ c'\end{bmatrix}$$

$$=(a+bd^*c)\cdot(a'+b'd'^*c')=\varphi^*(F)\cdot\varphi^*(F').$$

Repetition is preserved since, by (R2), (R3), (R1), and again (R2) we have

$$\varphi^*(F^*)=a^*+a^*b\cdot(d+ca^*b)^*ca^*=a^*+a^*bd^*(ca^*bd^*)^*ca^*$$

$$=a^*(1+bd^*ca^*(bd^*ca^*)^*)=a^*(bd^*ca^*)^*=(a+bd^*c)^*$$

$$=\varphi^*(F)^*.$$

The third problem is to show that the extension $\varphi\hat{}$ is the unique prerepetition-theory morphism such that $I_T\varphi\hat{}=\varphi_T$ and $I_X\varphi\hat{}=\varphi_X$. It is enough to note that every flowchart F has an equivalent representation as

$$F':=I_T(A)+I_T(B)(I^\bullet_X(e)I_T(D))^*I^\bullet_X(e)I_T(C).$$

Since

$$I^\bullet_X(e)=\begin{array}{c} \\ e\end{array}\left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0\end{array}\right],\qquad (I^\bullet_X(e)I_T(D))^*=\begin{array}{c} \\ e\end{array}\left[\begin{array}{c|c} 1 & 1 \\ \hline D & D\end{array}\right]\quad\text{and}$$

$$I^\bullet_X(e)I_T(C)=\begin{array}{c} \\ e\end{array}\left[\begin{array}{c|c} 0 & 1 \\ \hline C & 0\end{array}\right],$$

the flowchart F' is

$$\begin{array}{c} \\ e \\ e\end{array}\left[\begin{array}{c|cc} A & B & B \\ \hline 0 & D & D \\ C & 0 & 0\end{array}\right].$$

Now, it is easy to see that $F'\to_{[1_e\ 1_e]}F$; hence $F\equiv_{\mathrm{nd}}F'$ indeed. □

We can now state the main result given by Theorem 4.1, Theorem 5.1 and Theorem 6.1. Note that all the above results can easily be extended to arbitrary double ranked sets X (we have restricted ourselves to the case $X(m,n)=\emptyset$ for $m\neq 1$ or $n\neq 1$ only to avoid some complications in writing).

As usual with partial structures, T will be called a partial repetition theory if the repetition-theory operations are partially defined in T. A morphism of partial repetition theories must preserve definedness and the operations. We say that the repetition theory T^c is the *free repetition-theory completion* of the partial repetition theory T if there exists a partial repetition-theory morphism $I:T\to T^c$ such that, for every repetition theory T' and every partial repetition-theory morphism $\varphi:T\to T'$, there exists a unique repetition-theory morphism $\varphi^c:T^c\to T'$ such that $\varphi=I\cdot\varphi^c$.

$$\begin{tikzcd} T \arrow[r, "I"] \arrow[dr, "\varphi"'] & T^c \arrow[d, dashed, "\varphi^c"] \\ & T' \end{tikzcd}$$

By the *partial repetition theory obtained by adding the double ranked set X to the repetition theory T* we mean the partial repetition theory given by the family of disjoint unions $\{T(m, n) \sqcup X(m, n)\}_{m,n}$ and having the operations defined on elements in T only and in accordance with the corresponding operations in T.

Main Theorem 6.2. (i) *If T is a repetition theory with intersection, then the theory $\mathbf{Fl}_{X,T}/\equiv_{\mathrm{nd}}$ of the classes of $\equiv_{\mathrm{nd}}$-equivalent X-flowcharts over T is a repetition theory.*

(ii) *If $\mathbf{Fl}_{X,T}/\equiv_{\mathrm{nd}}$ is a repetition theory, then this is even the free repetition-theory completion of the partial repetition theory obtained by adding the double ranked set X to the repetition theory T.*

Corollary 6.3. *For every set D and double ranked set X the theory $\mathbf{Fl}_{X,\mathbf{Rel}_D}/\equiv_{\mathrm{nd}}$ is the free repetition-theory completion of the partial repetition theory obtained by adding X to $\mathbf{Rel}_D$.*

Corollary 6.4. *The classes of nondeterministically equivalent X-flowcharts over the theory of $\{0, 1\}$-matrices form the repetition theory freely generated by the double ranked set X in the category of repetition theories obeying $1^* = 1$.*

It remains an open problem, of theoretical interest, to decide whether the condition 'with intersection' in our Main Theorem 6.2 can be eliminated.

7. An application

Consider the following ADA-like program:

```
P :: integer n;
   begin
     «IN» select
              n := 1; go to OUT;
          or
              n := 1; go to 2;
          end select;
     «2»  x;
          select
             when not 6 divide n ⇒ n := n + 1; go to OUT;
          or
             when n ≠ 0 ⇒ n := n + 1; go to 2;
```

```
        or
          when n = 0 ⇒ n := 1; go to 3;
        or
          when n = 0 ⇒ n := 1; go to 4;
        end select;
  《3》 x;
        select
          when not 3 divide n ⇒ n := n + 1; go to OUT;
        or
          when 2 divide n ⇒ n := n + 1; go to 4;
        end select;
  《4》 x;
        select
          when not 2 divide n ⇒ n := n + 1; go to OUT;
        or
          n := n + 1; go to 3;
        or
          when not 2 divide n ⇒ n := n + 1; go to 4;
        end select;
end.
```

Here, x is a variable procedure.

This program can be represented by the following flowchart over $\mathbf{Rel}_{\mathbb{N}}$:

$$\begin{array}{c} \\ \rightarrow \\ x \\ x \\ x \end{array}\begin{array}{c}\uparrow\\ \left[\begin{array}{c|ccc} a & a & 0 & 0 \\ b & c & d & d \\ e & 0 & 0 & f \\ g & 0 & h & g \end{array}\right]\end{array}$$

where the involved relations are

$$a = \{(k, 1) \mid k \in \mathbb{N}\}, \qquad b = \{(k, k+1) \mid k \in \mathbb{N}\backslash 6\mathbb{N}\},$$

$$c = \{(k, k+1) \mid k \in \mathbb{N}\backslash\{0\}\}, \qquad d = \{(0, 1)\},$$

$$e = \{(k, k+1) \mid k \in \mathbb{N}\backslash 3\mathbb{N}\}, \qquad f = \{(k, k+1) \mid k \in 2\mathbb{N}\},$$

$$g = \{(k, k+1) \mid k \in \mathbb{N}\backslash 2\mathbb{N}\}, \qquad h = \{(k, k+1) \mid k \in \mathbb{N}\}.$$

Note that $b = e \cup g$, $h = f \cup g = c \cup d$. Hence,

$$\begin{array}{c} \\ x \\ x \\ x \end{array}\left[\begin{array}{c|ccc} a & a & 0 & 0 \\ \hline b & c & d & d \\ e & 0 & 0 & f \\ g & 0 & h & g \end{array}\right] \xleftarrow[\left[\begin{smallmatrix}1&0&0\\0&1&1\end{smallmatrix}\right]]{s^{-1}} \begin{array}{c} \\ x \\ x \end{array}\left[\begin{array}{c|cc} a & a & 0 \\ \hline b & c & d \\ e \cup g & 0 & h \end{array}\right] \xrightarrow[\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]]{s} \begin{array}{c} \\ x \end{array}\left[\begin{array}{c|c} a & a \\ \hline b & h \end{array}\right].$$

Consequently, the given program is nondeterministically equivalent to the following one:

```
Q :: integer n;
   begin
     «IN» select
              n := 1; go to OUT;
           or
              n := 1; go to 2;
           end select;
     «2»   x;
           select
              when not 6 divide n ⇒ n := n + 1; go to OUT;
           or
              n := n + 1; go to 2;
           end select;
   end.
```

When x is interpreted as a relation $\rho \subseteq \mathbb{N} \times \mathbb{N}$ the relation denoted by Q is $a \cup a(\rho h)^* \rho b$ (cf. Section 6). Some evaluations for P can now be computed and are shown in Table 5.

Table 5.

x	the associated relation	the relation computed by P
$n := 2 * n - 1$	$\{(k, 2k-1) \mid k \in \mathbb{N}\}$	$\{(n, 2^k) \mid n, k \in \mathbb{N}\}$
$n := n - 1$	$\{(k+1, k) \mid k \in \mathbb{N}\}$	$\{(n, 1) \mid n \in \mathbb{N}\}$ (i.e., a)
null	$\{(k, k) \mid k \in \mathbb{N}\}$	$\{(n, k+1) \mid n \in \mathbb{N},\ k \in \mathbb{N} \backslash 6\mathbb{N}\} \cup a$

8. Repetition theories, strong iteration theories and regular algebras

In this section we shall give some connections between our algebra for deterministic flowchart schemes (i.e., strong iteration theories cf. [18, 19]), our algebra for nondeterministic flowchart schemes (i.e., repetition theories, cf. Section 1), Kleene algebras (cf. [5, p. 34]) and Salomaa's axiomatization of the algebra of regular events (cf. [5, 17]).

A *strong iteration theory* is an iteration theory (defined by Bloom, Elgot and Wright [3] and axiomatized by Ésik [see 10]) which obeys the functorial dagger implication (I4) below.

A particular axiomatization of iteration is:

(I0) $f^\dagger g = (f(1_m \oplus g))^\dagger$ for $f \in T(m, m+n)$, $g \in T(n, p)$;

(I1) $f^\dagger = f\langle f^\dagger, 1_n\rangle$ for $f \in T(m, m+n)$;

(I2) $(f(\langle 1_m, 1_m\rangle \oplus 1_n))^\dagger = f^{\dagger\dagger}$ for $f \in T(m, m+m+n)$;

(I3) $g(f(g \oplus 1_p))^\dagger = (gf)^\dagger$ for $f \in T(m, n+p)$, $g \in T(n, m)$;

(I4) if $f(\rho \oplus 1_p) = \rho g$, then $f^\dagger = \rho g^\dagger$ for $f \in T(m, m+p)$, $g \in T(n, n+p)$ and a function $\rho : [m] \to [n]$.

Here $\langle\ ,\ \rangle$ denotes (binary) tupling.

Let T be a matrix theory, St be the set of all *-operations on T and It be the set of all †-operations on T, satisfying the parameter axiom (I0). Consider the mappings $a : \mathrm{St} \to \mathrm{It}$ and $b : \mathrm{It} \to \mathrm{St}$, defined as follows:

$$[f]^{b(\dagger)} = [f \quad 1_m]^\dagger \quad \text{for } f \in T(m, m);$$

$$[f \quad g]^{a(*)} = [f]^* g \quad \text{for } [f \quad g] \in T(m, m+n) \text{ (such that } [f] \in T(m, m)).$$

Using a routine computation we can prove that

(1) the pair a, b gives an one-to-one correspondence between St and It;

(2) via this correspondence, the axioms (I1)–(I4) are rewritten in terms of * as (R1), (R2), (R3), (R4f) respectively; this means that † fulfils (I1) in T iff $^{a(\dagger)}$ fulfils (R1) in T, and so on. (The fact that each axiom of iteration alone is equivalent to the corresponding axiom of repetition was made clear by Căzănescu.)

In a very pleasant, but quaint way repetition theories are strongly connected with regular Kleene algebras, cf. [5, p. 34]. (Recently, Klop has given me this work, but, unfortunately, I couldn't find the exact relation, yet.) More precisely, repetition theories obeying $1^* = 1$ are classical Kleene algebras, i.e., all classical laws (C1)–(C14) (cf. [5, p. 25]) hold in such a repetition theory ((C1)-(C12) are common axioms; (C13) is equivalent with $1^* = 1$; and, for (C14), note that, under (R4f), the conditions (C14, n^*) in [5, p. 111] hold; hence, (C14) follows by [5, Theorem 3, p. 111]). On the other hand, all theories of matrices over normal Kleene algebras are repetition theories in which $1^* = 1$ holds. Perhaps, repetition theories obeying $1^* = 1$ and regular Kleene algebras are the same (this seems to be related to a conjecture of Conway, see [5, p. 103]).

Salomaa's rule "if $X = AX + B$ and $A + 1 \neq A$ then $X = A^*B$" is stronger than our rule (R4). More exactly, "if $A\rho = \rho B$ and $A + 1 \neq A$, then $A^*\rho = \rho B^*$" follows by applying Salomaa's rule to $\rho B^* = \rho(BB^* + 1) = A\rho B^* + \rho$. Otherwise, note that Salomaa's rule is similar to the condition used by Elgot in iterative algebraic theories in [8] (i.e., the recursive equation $x = f\langle x, 1\rangle$ has a unique solution for ideal morphisms) and our strong iteration theories extend these theories.

9. Conclusions

We have a partial abstract type (cf. [4]) for the nondeterministic flowchart algorithms, i.e., a repetition theory. Both syntactic and semantic models have this type. The given calculus is similar to that of polynomials, namely it naturally puts together known and unknown computation processes.

However, some important problems are still open. The first one is to decide if our syntactical nondeterministic equivalence $\equiv_{\mathrm{nd}}$ generalizes 'having the same set of successful computation paths'. (More exactly, consider the interpretation of X-flowcharts over $\{0, 1\}$-matrices induced by $\varphi_X(x) = \{(w, wx) \mid w \in X^*\}$ in $\mathbf{Rel}_{X^*}$. We conjecture that $F \equiv_{\mathrm{nd}} F'$ iff $\varphi^{\#}(F) = \varphi^{\#}(F')$.)

Another open question is the technical one pointed out in Section 6: can the condition 'with intersection' be eliminated (or weakened)?

The last open question was pointed out in Section 8: find the exact relation between repetition theories and regular algebras.

Also, it would be interesting to compare our transformation system to the transformation systems in the beautiful paper of Courcelle [6].

Appendix A

Here we give the proofs of Lemmas Fe, Ge in Section 4, and of Lemmas A–M needed for the answers in the lower-left half of Table 4. The lemmas will be ordered in the lexicographical order.

The support theory T is supposed to be a *prerepetition theory with intersection*. In fact, the zerosum-free condition is used for Lemmas B, Fe, H, J and the divisibility condition for Lemmas I and K. We only have a proof for Lemma Ge when T is uniform divisible. The other lemmas hold for arbitrary prerepetition theories.

Convention. Fix an (arbitrary) infinite flowchart

$$F^{\infty} = \begin{array}{c} \\ e_1 \\ \vdots \\ e_k \\ \vdots \end{array} \left[\begin{array}{c|c} A & B_1 \cdots B_k \cdots \\ \hline C_1 & D_{11} \cdots D_{1k} \cdots \\ \vdots & \vdots \quad\quad \vdots \\ C_k & D_{k1} \cdots D_{kk} \cdots \\ \vdots & \vdots \quad\quad \vdots \end{array}\right].$$

We say that a flowchart F is the *standard flowchart* given by $e = e_{i_1} \ldots e_{i_r}$ ($i_1, \ldots, i_r$ distinct indices) if F is the 'subflowchart' of F^{∞} uniquely determined by e; for instance, e_2e_5 gives the standard flowchart

$$\begin{array}{c} \\ e_2 \\ e_5 \end{array} \left[\begin{array}{c|cc} A & B_2 & B_5 \\ \hline C_2 & D_{22} & D_{25} \\ C_5 & D_{52} & D_{55} \end{array}\right],$$

and so on.

Lemma A. $\overset{i}{\longleftarrow} \cdot \overset{i}{\longleftarrow} = \overset{i}{\longleftarrow} \cdot \overset{i}{\longleftarrow} = \overset{i}{\longleftarrow}$.

Lemma B. $\overset{s}{\leftarrow}\cdot\overset{i}{\leftarrow}\subseteq\overset{i}{\leftarrow}\cdot\overset{s}{\leftarrow}$.

Proof. Suppose that $F' \,{}^{s}_{y'}\!\!\leftarrow F\,{}^{i}_{y''}\!\!\leftarrow F''$. Using isomorphic representations of F', F, F'', without loss of generality (briefly, w.l.o.g.), we may suppose that

(i) these flowcharts are the standard flowcharts given by $e'=e_3e_4$, $e=e_1e_2$ and $e''=e_1$;

(ii) $y'=\left[\begin{smallmatrix} y_1 & 0\\ y_2 & y_3\end{smallmatrix}\right]$, where $y_1: e_1\to e_3$ is a surjective function, and $y''=[1_{e_1}\ \ 0]$.

The second simulation gives $B_2=0$, $D_{12}=0$; hence, by the first simulation, we have

$$\begin{bmatrix} A & B_3 & B_4\\ y_1C_3 & y_1D_{33} & y_1D_{34}\\ \cdot\cdot & \cdot\cdot & \cdot\cdot\end{bmatrix}=\begin{bmatrix} A & B_1y_1 & 0\\ C_1 & D_{11}y_1 & 0\\ \cdot\cdot & \cdot\cdot & \cdot\cdot\end{bmatrix}.$$

Since y_1 is a surjective function, by Fact 1.4, $y_1D_{34}=0$ implies $D_{34}=0$. Now, take for $F^\bullet$ the standard flowchart given by $e^\bullet=e_3$ and set $z'=[1_{e_3}\ \ 0]$; it follows that

$$F' \xleftarrow[z']{\ \mathrm{i}\ } F^\bullet \xleftarrow[y_1]{\ \mathrm{s}\ } F''. \qquad \square$$

Lemma C. $\overset{s}{\leftarrow}\cdot\overset{s}{\leftarrow}=\overset{s}{\leftarrow}\cdot\overset{s}{\leftarrow}=\overset{s}{\leftarrow}$.

Lemma D. $\xrightarrow{\mathrm{i}^{-1}}\cdot\xleftarrow{\ \mathrm{i}\ }\subseteq\xleftarrow{\ \mathrm{i}\ }\cdot\xrightarrow{\mathrm{i}^{-1}}$.

Proof. Suppose that $F' \rightarrow^{\mathrm{i}^{-1}}_{y'} F\,{}^{\mathrm{i}}_{y''}\!\!\leftarrow F''$. W.l.o.g., we may suppose that these flowcharts are the standard flowcharts given by $e'=e_1e_2e_3$, $e=e_1e_2$ and $e''=e_1$;

$$y'=\begin{bmatrix} 1_{e_1} & 0\\ 0 & 1_{e_2}\\ 0 & 0\end{bmatrix}$$

and $y''=[1_{e_1}\ 0]$. The two simulations give $C_3=0$, $D_{31}=0$, $D_{32}=0$, respectively $B_2=0$, $D_{12}=0$. Take for $F^\bullet$ the standard flowchart given by $e^\bullet=e_1e_3$ and for z', z'' the relations

$$\begin{bmatrix} 1_{e_1} & 0 & 0\\ 0 & 0 & 1_{e_3}\end{bmatrix}\quad\text{respectively}\quad\begin{bmatrix} 1_{e_1}\\ 0\end{bmatrix};$$

it follows that $F'\,{}^{\mathrm{i}}_{z'}\!\!\leftarrow F^\bullet \rightarrow^{\mathrm{i}^{-1}}_{z''} F''$. $\square$

Lemma E. $\xrightarrow{\mathrm{i}^{-1}}\cdot\xleftarrow{\ \mathrm{s}\ }\subseteq\xleftarrow{\ \mathrm{s}\ }\cdot\xrightarrow{\mathrm{i}^{-1}}$.

Proof. Suppose that $F' \rightarrow^{\mathrm{i}^{-1}}_{y'} F\,{}^{\mathrm{s}}_{y''}\!\!\leftarrow F''$. W.l.o.g., we may suppose that

(i) these flowcharts are the standard flowcharts given by $e'=e_1e_2$, $e=e_1$ and $e''=e_3$;

(ii) $y'=\begin{bmatrix}1_{e_1}\\ 0\end{bmatrix}$. The first simulation gives $C_2=0$, $D_{21}=0$ and the second one gives

$$\begin{bmatrix} A & B_1 \\ y''C_1 & y''D_{11}\end{bmatrix}=\begin{bmatrix} A & B_3y'' \\ C_3 & D_{33}y''\end{bmatrix}.$$

Set

$$F^\bullet=\begin{array}{c}\\ e_3\\ e_2\end{array}\left[\begin{array}{c|cc} A & B_3 & B_2\\ \hline C_3 & D_{33} & y''D_{12}\\ 0 & 0 & D_{22}\end{array}\right]$$

and take for z', z'' the relations

$$\begin{bmatrix} y'' & 0\\ 0 & 1_{e_2}\end{bmatrix},\quad \text{respectively}\quad \begin{bmatrix}1_{e_3}\\ 0\end{bmatrix};$$

it follows that $F'\ {}_{z'}^{s}\!\!\leftarrow F^\bullet \to_{z''}^{i^{-1}} F''$. □

Lemma F. $\xrightarrow{s^{-1}}\cdot\xleftarrow{i}\ \not\subseteq\ \xleftarrow{i}\cdot\xrightarrow{s^{-1}}$.

Proof. The flowcharts

$$F'=\begin{array}{c}\\ x\end{array}\left[\begin{array}{c|c} 0 & 0\\ \hline 0 & a+b\end{array}\right],\quad F=\begin{array}{c}\\ x\\ x\end{array}\left[\begin{array}{c|cc} 0 & 0 & 0\\ \hline 0 & a & 0\\ 0 & b & a+b\end{array}\right]\quad\text{and}\quad F''=\begin{array}{c}\\ x\end{array}\left[\begin{array}{c|c} 0 & 0\\ \hline 0 & a\end{array}\right]$$

fulfil $F'\to_{[1\ 1]}^{s^{-1}} F\ {}_{[1\ 0]}\!\!\overset{i}{\leftarrow} F''$. On the other hand, there is no $F^\bullet\neq F''$ such that $F^\bullet \to^{s^{-1}} F''$; thus, for $a\neq a+b$,

$$(F', F'')\not\in\ \xleftarrow{i}\cdot\xrightarrow{s^{-1}}.\quad □$$

Lemma Fe. $\xrightarrow{s^{-1}}\cdot\xleftarrow{i}\ \subseteq\ \xleftarrow{i}\cdot\xrightarrow{s^{-1}}\cdot\ \xrightarrow{i}$;

$$(d/i):\ \xleftarrow{i^{-1}}\cdot\xrightarrow{s}\ \subseteq\ \xrightarrow{i^{-1}}\cdot\ \xrightarrow{s}\cdot\xleftarrow{i^{-1}}.$$

Proof. Suppose that $F'\to_{y'}^{s^{-1}} F\ {}_{y''}\!\!\overset{i}{\leftarrow} F''$. W.l.o.g., we may suppose that

(i) these flowcharts are the standard flowcharts given by $e'=e_5e_6e_7$, $e=e_1e_2e_3e_4$ and $e''=e_1e_2$;

(ii) e_6 is the intersection of the images by y'^{-1} of e_1e_2 and e_3e_4, i.e.,

$$y'=\begin{bmatrix} y_1 & 0 & 0 & 0\\ 0 & y_2 & y_3 & 0\\ 0 & 0 & 0 & y_4\end{bmatrix},$$

where $y_2^{-1}:e_2\to e_6$ and $y_3^{-1}:e_3\to e_6$ are surjective functions; and the relation y'' is

$$\begin{bmatrix} 1_{e_1} & 0 & 0 & 0\\ 0 & 1_{e_2} & 0 & 0\end{bmatrix}.$$

The second simulation shows that B_3, B_4, D_{13} and D_{14} are zero matrices, and the first one shows that

$$\begin{bmatrix} A & B_5y_1 & B_6y_2 & B_6y_3 & B_7y_4 \\ C_5 & D_{55}y_1 & D_{56}y_2 & D_{56}y_3 & D_{57}y_4 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 & 0 & 0 \\ y_1C_1 & y_1D_{11} & y_1D_{12} & 0 & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \end{bmatrix}.$$

Since y_3, y_4 are total relations, using Fact 1.4 we obtain that $B_6 = 0$, $B_7 = 0$, $D_{56} = 0$ and $D_{57} = 0$; applying one more time Fact 1.4 (y_1 is a surjective relation) we obtain $D_{12} = 0$. Take for $F^{\cdot}, F^{-}$ the standard flowcharts given by $e^{\cdot} = e_5$ respectively by $e^{-} = e_1$ and, for z', z'', the relations $[1_{e_5}\ 0\ 0]$ respectively $[1_{e_1}\ 0]$; it follows that

$$F' \underset{z'}{\overset{\mathrm{i}}{\longleftarrow}} F^{\cdot} \underset{y_1}{\overset{\mathrm{s}^{-1}}{\longrightarrow}} F^{-} \underset{z''}{\overset{\mathrm{i}}{\longrightarrow}} F''.$$

The second part directly follows from the first one by *d*uality and *i*nversion. □

Lemma G. $\overset{\mathrm{s}^{-1}}{\longrightarrow} \cdot \overset{\mathrm{s}}{\longleftarrow} \not\subseteq \overset{\mathrm{s}}{\longleftarrow} \cdot \overset{\mathrm{s}^{-1}}{\longrightarrow}$.

Proof. Clearly, for $y' = [1\ 1]$,

$$y'' = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad F' = \begin{array}{c} \\ x \end{array}\!\!\left[\begin{array}{c|c} 0 & c+d \\ \hline a+b & 0 \end{array}\right],$$

$$F = \begin{array}{c} \\ x \\ x \end{array}\!\!\left[\begin{array}{c|cc} 0 & c+d & c+d \\ \hline a & 0 & 0 \\ b & 0 & 0 \end{array}\right], \qquad F'' = \begin{array}{c} \\ x \\ x \\ x \end{array}\!\!\left[\begin{array}{c|ccc} 0 & c & d & c+d \\ \hline a & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & 0 & 0 & 0 \end{array}\right]$$

we have $F' \to_{y'}^{\mathrm{s}^{-1}} F\ {}_{y''}^{\mathrm{s}}\!\leftarrow F''$. Since, for c and d fulfilling $c \neq c+d \neq d$ and $c \neq d$, there is no $F^{\cdot} \not\approx F''$ such that $F^{\cdot} \to^{\mathrm{s}^{-1}} F''$, it follows that $(F', F'') \notin {}^{\mathrm{s}}\!\leftarrow \cdot \to^{\mathrm{s}^{-1}}$. □

Lemma Ge. $\overset{\mathrm{s}^{-1}}{\longrightarrow} \cdot \overset{\mathrm{s}}{\longleftarrow} \subseteq \overset{\mathrm{s}}{\longleftarrow} \cdot \overset{\mathrm{s}^{-1}}{\longrightarrow} \cdot \overset{\mathrm{s}}{\longrightarrow}$;

$(d/i)\ \overset{\mathrm{s}^{-1}}{\longleftarrow} \cdot \overset{\mathrm{s}}{\longrightarrow} \subseteq \overset{\mathrm{s}^{-1}}{\longrightarrow} \cdot \overset{\mathrm{s}}{\longrightarrow} \overset{\mathrm{s}^{-1}}{\longleftarrow}$.

Proof. Suppose that $F' \to_{y'}^{\mathrm{s}^{-1}} F\ {}_{y''}^{\mathrm{s}}\!\leftarrow F''$, where:

(i) F', F, F'' are the standard flowcharts whose vertices are respectively indexed by $i \in [n]$; (i, j) for $i \in [n]$, $j \in [n_i]$; and (i, j, k) for $i \in [n]$, $j \in [n_i]$ and $k \in [n_{i,j}]$;

(ii) with respect to these indices $y' = \{(i, (i, j))\}$ and $y'' = \{((i, j, k), (i, j))\}$.

The second simulation shows that

$$(\alpha) \qquad A = A''; \qquad B_{(s,t)} = \sum_r B''_{(s,t,r)}; \qquad C_{(i,j)} = C''_{(i,j,k)}, \quad \text{for all } k;$$

$$D_{(i,j),(s,t)} = \sum_r D''_{(i,j,k),(s,t,r)}, \quad \text{for all } k.$$

The first simulation shows that

$$(\beta) \qquad A'=A; \qquad B'_s=B_{(s,t)}, \quad \text{for all } t;$$

$$C'_i=\sum_j C_{(i,j)}; \qquad D'_{(i,s)}=\sum_j D_{(i,j),(s,t)} \quad \text{for all } t.$$

We shall construct a flowchart F^- such that $F' \;{}^{s}\!\!\leftarrow \cdot \rightarrow^{s^{-1}} F^-$ and $F^- \rightarrow^{s} F''$. As indices for F^- we shall take all the sequences

$$\langle (i,j_0,k_0),(i,1,k_1),\ldots,(i,n_i,k_{n_i})\rangle, \quad \text{where } k_{j_0}=k_0;$$

(the label associated with such an index is, for example, the ith letter of e'); the connections in F^- are given by

$$A^-=A;$$

$$B^-_{\langle (s,t_0,r_0),(s,1,r_1),\ldots,(s,n_s,r_{n_s})\rangle}=B''_{(s,1,r_1)}\ \&\cdots\&\ B''_{(s,n_s,r_{n_s})};$$

$$C^-_{\langle (i,j_0,k_0),(i,1,k_1),\ldots,(i,n_i,k_{n_i})\rangle}=C''_{(i,j_0,k_0)};$$

$$D^-_{\langle (i,j_0,k_0),(i,1,k_1),\ldots,(i,n_i,k_{n_i})\rangle,\langle (s,t_0,r_0),(s,1,r_1),\ldots,(s,n_s,r_{n_s})\rangle}$$

$$= D''_{(i,j_0,k_0),(s,t_0,r_0)}\ \&\left(\sum_{j\geq 1} D''_{(i,j,k_j),(s,1,r_1)}\right)\&\cdots\&\left(\sum_{j\geq 1} D''_{(i,j,k_j),(s,n_s,r_{n_s})}\right).$$

First, we have the simulation $F^- \rightarrow^{s}_{z''} F''$, where the surjection z'' is

$$z'':\langle (i,j_0,k_0),(i,1,k_1),\ldots\rangle \mapsto (i,j_0,k_0).$$

Indeed, using (α), (β) and the axioms (C), (D), and (A) for intersection we obtain

$$A^-=A'';$$

$$\sum_{\text{on } z''^{-1}(s,t_0,r_0)} B^-_{\langle (s,t_0,r_0),(s,1,r_1),\ldots\rangle}$$

$$= \sum_{r_1,\ldots,r_{n_s} \text{ with } r_{t_0}=r_0} B''_{(s,1,r_1)}\ \&\cdots\&\ B''_{(s,n_s,r_{n_s})}$$

$$= B_{(s,1)}\ \&\cdots\&\ B''_{(s,t_0,r_0)}\ \&\cdots\&\ B_{(s,n_s)}$$

$$= B'_s\ \&\cdots\&\ B''_{(s,t_0,r_0)}\ \&\cdots\&\ B'_s$$

$$= B''_{(s,t_0,r_0)}.$$

On $z''^{-1}(i,j_0,k_0)$ (i.e., for any $k_1,\ldots,k_{n_i}$ with $k_{j_0}=k_0$), we have

$$C^-_{\langle (i,j_0,k_0),(i,1,k_1),\ldots\rangle}=C''_{(i,j_0,k_0)};$$

$$\sum_{\text{on } z''^{-1}(s,t_0,r_0)} D^-_{\langle (i,j_0,k_0),(i,1,k_1),\ldots\rangle,\langle (s,t_0,r_0),(s,1,r_1),\ldots\rangle}$$

$$= \sum_{r_1,\ldots,r_{n_s} \text{ with } r_{t_0}=r_0} D''_{(i,j_0,k_0),(s,t_0,r_0)}\ \&\left(\sum_{j\geq 1} D''_{(i,j,k_j),(s,1,r_1)}\right)\&\cdots$$

$$= D''_{(i,j_0,k_0),(s,t_0,r_0)} \& \left(\sum_{j\geq 1}\sum_{r_1} D''_{(i,j,k_j),(s,1,r_1)}\right) \& \cdots \& \left(\sum_{j\geq 1} D''_{(i,j,k_j),(s,t_0,r_0)}\right) \& \cdots$$

$$= D''_{(i,j_0,k_0),(s,t_0,r_0)} \& D'_{(i,s)} \& \cdots \& \left(\sum_{j\geq 1} D''_{(i,j,k_j),(s,t_0,r_0)}\right) \& \cdots$$

$$= D''_{(i,j_0,k_0),(s,t_0,r_0)}.$$

Second, there exists a flowchart, $F^{\bullet}$ say, such that $F^{\bullet} \rightarrow_{z}^{s^{-1}} F^{-}$ for the surjection z^{-1} given by

$$z^{-1}: \langle (i, j_0, k_0), (i, 1, k_1), \ldots \rangle \mapsto \langle (i, 1, k_1), \ldots \rangle.$$

In order to find F' we note that we can prove that z generates a simulation for F^{-}. Indeed, on $z(\langle (s, 1, r_1), \ldots \rangle)$ (i.e., for any $(s, t_0, r_0) \in \{(s, 1, r_1), \ldots\}$) we have

$$B^{-}_{\langle (s,t_0,r_0),(s,1,r_1),\ldots\rangle} = B''_{(s,1,r_1)} \& \cdots \& B''_{(s,n_s,r_{n_s})} \quad (=\text{constant})$$
$$=: B^{\bullet}_{\langle (s,1,r_1),\ldots\rangle};$$

$$\sum_{\text{on } z(\langle (i,1,k_1),\ldots\rangle)} D^{-}_{\langle (i,j_0,k_0),(i,1,k_1),\ldots\rangle,\langle (s,t_0,r_0),(s,1,r_1),\ldots\rangle}$$

$$= \sum_{(i,j_0,k_0)\in\{(i,1,k_1),\ldots\}} D''_{(i,j_0,k_0),(s,t_0,r_0)} \& \left(\sum_{j\geq 1} D''_{(i,j,k_j),(s,1,r_1)}\right) \& \cdots$$

$$= \left(\sum_{j\geq 1} D''_{(i,j,k_j),(s,t_0,r_0)}\right) \& \left(\sum_{j\geq 1} D''_{(i,j,k_j),(s,1,r_1)}\right) \& \cdots$$

$$= \left(\sum_{j\geq 1} D''_{(i,j,k_j),(s,1,r_1)}\right) \& \cdots \quad (=\text{constant})$$

$$=: D^{\bullet}_{\langle (i,1,k_1),\ldots\rangle,\langle (s,1,r_1),\ldots\rangle}.$$

Moreover, we have to take $A^{\bullet} := A^{-}$ and

$$C^{\bullet}_{\langle (i,1,k_1),\ldots\rangle} =: \sum_{(i,j_0,k_0)\in\{(i,1,k_1),\ldots\}} C^{-}_{\langle (i,j_0,k_0),(i,1,k_1),\ldots\rangle} = \sum_j C''_{(i,j,k_j)}.$$

Lastly, let us observe that $F' \overset{s}{\underset{z'}{\leftarrow}} F^{\bullet}$ for the surjection z' given by

$$z': \langle (i, 1, k_1), \ldots \rangle \mapsto i.$$

Indeed, $A' = A^{\bullet}$;

$$\sum_{\text{on } z'^{-1}(s)} B^{\bullet}_{\langle (s,1,r_1),\ldots\rangle} = \sum_{r_1,\ldots,r_{n_s}} B''_{(s,1,r_1)} \& \cdots$$
$$= B_{(s,1)} \& \cdots \& B_{(s,n_s)} = B'_s;$$

$$C^{\bullet}_{\langle (i,1,k_1),\ldots\rangle} = \sum_j C''_{(i,j,k_j)} = \sum_j C_{(i,j)} = C'_i \quad \text{for all } k_1, \ldots, k_{n_i};$$

$$\sum_{\text{on } z'^{-1}(s)} D^{\bullet}_{\langle(i,1,k_1),\ldots\rangle,\langle(s,1,r_1),\ldots\rangle} = \left(\sum_{r_1}\sum_{j} D''_{(i,j,k_j),(s,1,r_1)}\right) \& \cdots$$

$$= \left(\sum_{j} D_{(i,j),(s,1)}\right) \& \cdots = D'_{(i,s)} \& \cdots \& D'_{(i,s)}$$

$$= D'_{(i,s)} \quad \text{for all } k_1, \ldots, k_{n_i}.$$

The second part directly follows from the first part by duality and inversion. □

Lemma H. $\xrightarrow{s} \cdot \xleftarrow{i} \subseteq \xleftarrow{i} \cdot \xrightarrow{s}$.

Proof. Suppose that $F' \rightarrow^{s}_{y'} F \,{}_{y''}\!\!\xleftarrow{i} F''$. W.l.o.g., we may suppose that

(i) these flowcharts are the standard flowcharts given by $e' = e_3 e_4$, $e = e_1 e_2$, and $e'' = e_1$;

(ii) $y' = [\begin{smallmatrix} y_1 & 0 \\ 0 & y_2 \end{smallmatrix}]$, where $y_1 : e_3 \rightarrow e_1$, and $y'' = [1_{e_1} \ 0]$. The second simulation gives $B_2 = 0$, $D_{12} = 0$. Hence, the first one says that

$$\begin{bmatrix} A & B_3 y_1 & B_4 y_2 \\ C_3 & D_{33} y_1 & D_{34} y_2 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot \end{bmatrix} = \begin{bmatrix} A & B_1 & 0 \\ y_1 C_1 & y_1 D_{11} & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot \end{bmatrix}.$$

Fact 1.4 applied to $B_4 y_2 = 0$, $D_{34} y_2 = 0$ for the total relation y_2 yields $B_4 = 0$, $D_{34} = 0$. For $F^{\bullet}$ take the standard flowchart given by $e^{\bullet} = e_3$ and set $z' = [1_{e_3} \ 0]$; it follows that $F' \,{}_{z'}\!\!\xleftarrow{i} F^{\bullet} \rightarrow^{s}_{y_1} F''$. □

Lemma I. $\xrightarrow{s} \cdot \xleftarrow{s} \subseteq \xleftarrow{s} \cdot \xrightarrow{s}$.

Proof. Suppose that $F' \rightarrow^{s}_{y'} F \,{}_{y''}\!\!\xleftarrow{s} F''$, where

(i) F', F, F'' are the standard flowcharts whose vertices are respectively indexed by (i,j) for $i \in [n]$, $j \in [n'_i]$; $i \in [n]$; and (i,k) for $i \in [n]$, $k \in [n''_i]$;

(ii) $y' : (i,j) \mapsto i$ and $y'' : (i,k) \mapsto i$.

The first simulation says that

$$(\alpha) \quad A = A'; \qquad B_r = \sum_s B'_{(r,s)}; \qquad C_i = C'_{(i,j)}, \quad \text{for all } j;$$

$$D_{i,r} = \sum_s D'_{(i,j),(r,s)} \quad \text{for all } j.$$

The second simulation gives similar equalities, (β) say.

We shall construct a flowchart $F^{\bullet}$ such that $F' \,{}^{s}\!\!\leftarrow F^{\bullet} \rightarrow^{s} F''$. As indices for $F^{\bullet}$, we take all the pairs $\langle(i,j),(i,k)\rangle$ (and with such an index we associate the ith letter of e). The connections of $F^{\bullet}$ are given by

$$A^{\bullet} = A; \qquad B^{\bullet}_{\langle(r,s),(r,t)\rangle} = B'_{(r,s)} \& B''_{(r,t)};$$

$$C^{\bullet}_{\langle(i,j),(i,k)\rangle} = C_i; \qquad D^{\bullet}_{\langle(i,j),(i,k)\rangle,\langle(r,s),(r,t)\rangle} = D'_{(i,j),(r,s)} \& D''_{(i,k),(r,t)}.$$

By using the equalities from (α) and (β), it easily follows that

$$F' \underset{z'}{\overset{s}{\leftarrow}} F^{\bullet} \underset{z''}{\overset{s}{\rightarrow}} F''$$

for the surjections $z':\langle(i,j),(i,k)\rangle\mapsto(i,j)$ and $z'':\langle(i,j),(i,k)\rangle\mapsto(i,k)$. □

Lemma J. $\xrightarrow{s}\cdot\xrightarrow{i^{-1}}\subseteq\xrightarrow{i^{-1}}\cdot\xrightarrow{s}$.

Proof. The proof is similar to the proof of Lemma H. □

Lemma K. $\xrightarrow{s}\cdot\xrightarrow{s^{-1}}\subseteq\xrightarrow{s^{-1}}\cdot\xrightarrow{s}$.

Proof. The proof of this lemma is similar to the second part of the proof of Lemma 3.6. □

Lemma L. $\xrightarrow{i}\cdot\xleftarrow{i}\subseteq\xleftarrow{i}\cdot\xrightarrow{i}$.

Proof. Suppose that $F'\rightarrow^{i}_{y'}F\,{}_{y''}^{i}\!\leftarrow F''$. W.l.o.g., we may suppose that

(i) these flowcharts are the standard flowcharts given by $e'=e_1e_2$, $e=e_1e_2e_3e_4$ and $e''=e_1e_4$;

(ii) $$y'=\begin{bmatrix}1_{e_1} & 0 & 0 & 0\\ 0 & 1_{e_2} & 0 & 0\end{bmatrix} \quad\text{and}\quad y''=\begin{bmatrix}1_{e_1} & 0 & 0 & 0\\ 0 & 0 & 0 & 1_{e_4}\end{bmatrix}.$$

Particularly, the first simulation says that $B_4=0$ and $D_{14}=0$, and the second one says that $B_2=0$ and $D_{12}=0$. Take for $F^{\bullet}$ the standard flowchart given by e_1 and, for z', z'', the relations $[1_{e_1}\ 0]$ respectively $[1_{e_1}\ 0]$; it follows that $F'\,{}_{z'}^{i}\!\leftarrow F^{\bullet}\rightarrow^{i}_{z''}F''$. □

Lemma M. $\xrightarrow{i}\cdot\xrightarrow{i^{-1}}\subseteq\xrightarrow{i^{-1}}\cdot\xrightarrow{i}$.

Proof. The proof is similar to the proof of Lemma L. □

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