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OBSTRUCTION THEORY IN ALGEBRAIC CATEGORIES, I *

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§ 0. Introduction

In this paper and a sequel, we generalize Barr's results on obstruction theory for commutative algebras [1]. The gist of our work is that if C is a suitably restricted "category of interest", then we can formulate questions about the existence and nature of nonsingular extensions and answer these questions in terms of criteria having to do with the second cohomology group.

Our first concern is to give a general definition of extensions. In [5], Beck described extensions in the context of a tripleable adjoint pair. We restrict the setting so that a more general definition can be given with Beck's extensions by modules corresponding to the singular extensions.

Further conditions are imposed on the categories we consider in order to insure the existence of centers (in the sense of [2]) and to show that centralizers of ideals are ideals.

In § 1 we describe categories of interest. In § 2 we discuss notions of module and extension and relate them to Beck's definitions. In §§ 3 and 4 we give some constructions needed in §§ 5 and 6 which correspond to §§ 2 and 3 of [1].

The concept of triple and the associated notation and terminology are treated in the introduction to [6]. We follow the conventions established there, except that we write morphisms on the left.

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§ 1. Categories of interest

Let C be a category which satisfies

(1). There is a triple $\mathbf{T} = (T, \eta, \mu)$ on S (the category of sets) such that $T(\emptyset) = \{p\}$

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(a one-point set) and C is equivalent to $S^{\mathbf{T}}$.

Remark 1.1. A category which satisfies (1) is complete and cocomplete, pointed, and is tripleable over S_*, the category of pointed sets with basepoint preserving maps.

We use the following notation: objects of $S^{\mathbf{T}}$ are pairs (A,α) with A a set and $\alpha: TA \to A$, while objects of S_* are pairs $\langle A, a\rangle$ with A a set and a the basepoint in A.

Proof. Identify C and $S^{\mathbf{T}}$.

In [12] it is shown that categories tripleable over sets have $\varprojlim$'s and coequalizers. Linton has shown that if K is any category with coproducts and $K^{\mathbf{T}}$ has coequalizers of reflexive pairs, then $K^{\mathbf{T}}$ has $\varinjlim$'s. Thus, categories tripleable over S are complete and cocomplete.

It is easy to see that C is pointed, with $(T(\emptyset), \mu_\emptyset)$ as zero object.

That C is tripleable over S_* follows from a result of Beck. Any functor U: $S^{\mathbf{T}_1} \to S^{\mathbf{T}_2}$ which commutes with underlying functors is tripleable (see [3]).

Let $\mathbf{T}'$ be the triple on S_* such that C is equivalent to $S_*^{\mathbf{T}'}$.

Remark 1.2. $\mathbf{T}'$ is a pointed triple. That is, $T'(\langle\{p\}, p\rangle) = \langle\{p\}, p\rangle$.

Next, we place two more restrictions on C:

(2). U: $C \to S_*$ factors through the category of groups.

(3). All operations in C are finitary.

Axiom (2) enables us to view the objects of C as groups with extra structure. We will denote the group operation by + although it need not be abelian.

Axiom (2) and the previously mentioned theorem of Beck mean that C is tripleable over G (the category of groups). We write $C = G^{\hat{\mathbf{T}}}$.

A theorem of Barr [2, Theorem 3.3] shows that in a category satisfying (1)–(3) each object has a subobject with special properties, called its center. If A is an object in C, let ZA denote the center of A.

Proposition 1.3. *If $u: (A_1,\alpha_1) \to (A_2,\alpha_2)$ is a morphism in C and is onto on underlying sets, then*

$$K = \{a \in A_1 \mid u(a) = 0 \text{ in the group underlying } A_2\}$$

is an object in C, $v: K \to A_1$ is the kernel of u, and u is the cokernel of v.

Proof. $U^{\hat{\mathbf{T}}}: C \to G$ creates $\varprojlim$'s [12, 2.4]. Therefore, K is an object in C and $v: K \to (A_1,\alpha_1)$ is the kernel of u.

To see that $u = \operatorname{coker} v$, note that $uv = 0$ and suppose u': $(A_1,\alpha_1) \to (A_3,\alpha_3)$ with $u'v = 0$. In G, $u = \operatorname{coker} v$ since u is onto, and so there is a unique group homomorphism $w: A_2 \to A_3$ such that $wu = u'$. It is easy to see that w is also the unique $\mathbf{T}'$-morphism with this property.

Definition 1.4. Let R and A be objects in $\mathcal{C}$. An *extension* of R by A is a sequence

$$0 \to A \xrightarrow{i} E \xrightarrow{p} R \to 0$$

in which p is surjective and i is the kernel of p.

Definition 1.5. Let T be an object in $\mathcal{C}$. A subobject A of T is called an *ideal* if it is the kernel of some morphism. We write $A < T$ when this is the case.

To formulate a criterion for a subobject to be an ideal, we assume that the operations in $\mathcal{C}$ can be generated by a set Ω which satisfies several conditions. For a discussion of operations in the language of triples see [12].

Let Ω_i be the set of i-ary operations in Ω.

In addition to (1) – (3) we assume that $\mathcal{C}$ satisfies:

(4). There is a generating set Ω for the operations in $\mathcal{C}$ and

$$\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2.$$

There is no harm in assuming that Ω contains the operations identity, inverse and +, associated with the group structure.

Let

$$\Omega_2' = \Omega_2 \setminus \{+\},$$
$$\Omega_1' = \Omega_1 \setminus \{-\},$$

and assume that if $* \in \Omega_2'$, then $*^0$ defined by

$$x *^0 y = y * x$$

is also in Ω_2'.

Remark 1.6. Ω_0 contains only one element, the group identity, since null-ary operations in $\mathcal{C}$ correspond to $T(\emptyset) = \{p\}$.

Further conditions on $\mathcal{C}$ are that Ω can be chosen satisfying:

(5). If $* \in \Omega_2'$, then

$$a * (b + c) = a * b + a * c.$$

(6). If $\omega \in \Omega_1'$, then ω is a homomorphism with respect to +, and if $* \in \Omega_2'$, then

$$\omega(a * b) = \omega(a) * b.$$

The morphisms in C can be thought of as precisely the operation preserving maps of the underlying sets.

Theorem 1.7. *Let A be a subobject of B. Then $A < B$ iff the following conditions hold:*

(i). *A is a normal subgroup of B.*
(ii). *For any $a \in A$, $b \in B$ and $* \in \Omega_2'$, we have $a * b \in A$.*

Proof. That $A < B$ implies (i) and (ii) is easy to see.

For the converse, let B/A be the quotient group and define

$$\omega(b_1 + A) = \omega(b_1) + A \qquad \text{for } \omega \in \Omega_1',$$
$$(b_1 + A) * (b_2 + A) = b_1 * b_2 + A \quad \text{for } * \in \Omega_2'.$$

These are easily seen to be well-defined and preserved by the canonical projection for groups.

Definition 1.8. An object A in C is *singular* if it is abelian as a group and if $A * A = 0$ for each $* \in \Omega_2'$.

The following is an easy consequence of Theorem 1.7 and Barr's definition of center [2].

Theorem 1.9. *If C is a category satisfying (1)–(6) and A is an object in C, then*

$$ZA = \{z \in A \mid \textit{for all } a \in A \textit{ and } * \in \Omega_2', a + z = z + a \textit{ and } a * z = 0\}.$$

Furthermore, ZA is singular and $ZA < A$.

Definition 1.10. If $A < B$, then

$$Z(B, A) = \{b \in B \mid \text{for all } a \in A \text{ and } * \in \Omega_2', a + b = b + a \text{ and } a * b = 0\}$$

is called the *centralizer* of A in B.

Although it is clear that $Z(B, A)$ is a subset of B, it is not necessarily the case that $Z(B, A) < B$.

Example 1.11. Let C be the category of real Jordan algebras. Then C satisfies (1)–(6).

Let B be the algebra of 2×2 upper triangular real matrices. If juxtaposition denotes ordinary matrix multiplication, then B, with ordinary matrix addition and $*$ defined by

$$X * Y = \tfrac{1}{2}(XY + YX),$$

is a Jordan algebra. However,

$$A = \{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \mid x \text{ is a real number}\}$$

is an ideal in B, while

$$Z(B, A) = \{\begin{pmatrix} u & v \\ 0 & -u \end{pmatrix} \mid u, v \text{ are real numbers}\}$$

is not, since $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}$ is not in $Z(B, A)$.

Example 1.12. Let C be a category satisfying (1)–(6) such that $\Omega_1' = \emptyset$ and $\Omega_2' = \{*, *^0\}$. Let B be a free group on generators X_1, X_2, X_3 and X_4, and assume that X_2 and X_3 commute with the other symbols. Let the operation $*$ on B be defined by the table

$*$	X_1	X_2	X_3	X_4
X_1	0	0	0	0
X_2	0	0	0	0
X_3	0	0	X_3	X_4
X_4	0	0	X_4	X_4

and distributivity. $*$ is well-defined and associative.

Let A be the normal subgroup generated by X_1. A is clearly closed under $*$ and in fact, $A * B = 0$. Thus, $A < B$. However, $Z(B, A) \nless B$ because X_3 is in $Z(B, A)$ but $X_3 * X_4 \notin Z(B, A)$ since

$$X_1 + (X_3 * X_4) = X_1 + X_4 \neq X_4 + X_1 = (X_3 * X_4) + X_1.$$

We require that $A < B$ implies $Z(B, A) < B$. To insure this we formulate two more axioms. Let C be a category satisfying (1)–(6). If X is an object in C and $x_1, x_2, x_3 \in X$, then

(7). $x_1 + (x_2 * x_3) = (x_2 * x_3) + x_1$ for any $*$ in Ω_2'.

(8). For each ordered pair $(\cdot, *) \in \Omega_2' \times \Omega_2'$, there is a word w such that

$$(x_1 \cdot x_2) * x_3 = w(x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1, (x_3x_2)x_1, x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2),$$

where each juxtaposition represents an operation in Ω_2'.

Remark 1.13. $w(0,0,0,0,0,0,0,0) = 0$ since $w(0,0,0,0,0,0,0,0) \in \Omega_0$.

Jordan algebras satisfy (7) since + is commutative, but not (8). The object B in Example 1.12 satisfies (8) since $*$ is associative; but (7) fails for B.

Definition 1.14. A category C is called a *category of interest* if it satisfies (1)–(8).

It is easy to see that the following theorem holds.

Theorem 1.15. *If B is an object in a category of interest and $A < B$, then $Z(B, A) < B$.*

In this paper, we will always assume that we are working in categories of interest. These include many, but not all, of the familiar algebraic categories. For example, groups, groups with operators, varieties of groups, rings, associative algebras, commutative associative algebras, modules over a ring, alternative algebras and Lie algebras can all be interpreted as categories of interest. We have already noted that Jordan algebras do not constitute a category of interest because of the failure of axiom (8).

Using the method developed by Barr and Beck in [4], it has been shown that triple cohomology coincides with cohomology theories of Eilenberg–MacLane (for groups), Hochschild (for associative algebras), Harrison (for commutative algebras), and Shukla (for associative algebras). The theorems in this paper and its sequel give a simultaneous treatment of results recorded in [7], [9], [10] and [14].

§2. R-structures and R-modules

Let C be a fixed category of interest and assume all objects and morphisms belong to C unless otherwise specified. We say that an extension $0 \to A \xrightarrow{i} E \xrightarrow{p} R \to 0$ is *singular* if A is singular and that it is *split* if there is a morphism $s : R \to E$ such that $ps = \mathrm{id}_R$.

Definition 2.1. A split extension of R by A is called an *R-structure*. A singular R-structure is called an *R-module.*

An R-structure induces actions of R on A corresponding to the operations in C. If we assume $A < E$ with $i : A \to E$ the inclusion, then for $r \in R$, $a \in A$ and $* \in \Omega_2'$, we have

$$(2.2a)\qquad r + a - r = s(r) + a - s(r),$$

$$(2.2b)\qquad r * a = s(r) * a.$$

We will call (2.2a) and (2.2b) *derived actions* of R on A.

In familiar categories like groups and commutative rings, R-modules are defined in terms of such actions. We need a simple way of checking whether a particular set of actions is a set of derived actions.

Definition 2.3. Given a set of actions of R on A – one for each operation in Ω_2 – let $\overline{R \times A}$ be a universal algebra whose underlying set is $R \times A$ and whose operations are

$$(r, a) + (r', a') = (r + r', (-r + a + r') + a'),$$

$$(r, a) * (r', a') = (r * r', r * a' + a * r' + a * a').$$

Theorem 2.4. *A set of actions of R on A is a set of derived actions iff* $\overline{R \times A}$ *is an object in* $\mathcal{C}$.

Proof. It is easy to see that if

$$0 \to A \xrightarrow{i} E \underset{s}{\overset{p}{\rightleftarrows}} R \to 0$$

is an R-structure, then φ: $\overline{R \times A} \to E$ given by $\varphi((r, a)) = s(r)+i(a)$ is an isomorphism. Thus, $\overline{R \times A}$ is an object in $\mathcal{C}$.

Conversely, if $\overline{R \times A}$ is in $\mathcal{C}$, then

$$0 \to A \to \overline{R \times A} \rightleftarrows R \to 0,$$

with all maps defined in the obvious way, is an R-structure which induces the given actions of R on A.

From now on we use the terminology: A is an R-structure if there is a split extension of R by A. When A is singular, we call it an R-module.

Proposition 2.5. *If A is an R-structure and* $f: S \to R$ *is a morphism in* $\mathcal{C}$, *then f induces a set of derived actions of S on A.*

Proof. The actions induced by f are:

$$s + a - s = f(s) + a - f(s),$$

$$s * a = f(s) * a$$

for all $a \in A$ and $s \in S$. Let $\overline{S \times A}$ be defined using these actions. Then $\overline{S \times A} \simeq Q$ in the pullback diagram:

$$\begin{array}{ccc} Q & \longrightarrow & \overline{R \times A} \\ \downarrow & & \downarrow \text{projection} \\ S & \xrightarrow[f]{} & R \end{array}$$

Definition 2.6. If A is an R-module, a map $\xi : R \to A$ is called a *derivation* if

$$\xi(r + r') = (-r' + \xi(r) + r') + \xi(r'),$$

and for all $* \in \Omega_2'$,

$$\xi(r * r') = r * \xi(r') + \xi(r) * r'.$$

The definition of R-module given here corresponds to that given in [8], but in his thesis, Beck gives definitions of R-module and extension which are meaningful in general. Since our categories of interest are special cases of the categories he considers, we will remark on the relation between our definitions and his.

Let K be any category and R an object in K. (K, R) is the category which has morphisms $E \to R$ in K as objects and commutative triangles

$$\begin{array}{ccc} E & \to & E' \\ & \searrow \swarrow & \\ & R & \end{array}$$

as morphisms. For Beck, an R-module is an abelian group object in (K, R) (see [5]).

Theorem 2.7. *For $\mathcal{C}$, a category of interest, the notion of R-module given by Definition 2.1 is equivalent to that of Beck [5].*

Proof. If

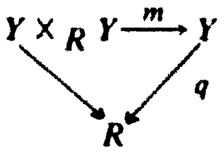
$$0 \to A \to Y \underset{s}{\overset{q}{\rightleftarrows}} R \to 0$$

is an R-module via Definition 2.1, then $Y \xrightarrow{q} R$ is an abelian group object in $(\mathcal{C}, R)$. This is easy to check.

If $Y \xrightarrow{q} R$ is an abelian group object in $(\mathcal{C}, R)$, there is a unit morphism

$$\begin{array}{ccc} R & \xrightarrow{s} & Y \\ {\scriptstyle \mathrm{id}_R} \searrow & & \swarrow {\scriptstyle q} \\ & R & \end{array}$$

since id_R is the terminal object of (C, R), and a multiplication morphism

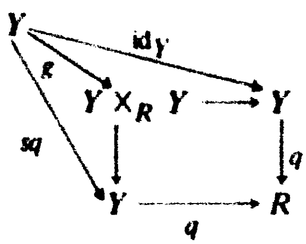


where $Y \times_R Y$ is the kernel pair of $Y \xrightarrow{q} R$. It is immediate that

$$0 \to K \to Y \underset{s}{\overset{q}{\rightleftarrows}} R \to 0$$

is an R-structure. We must check that K is singular.

We can identify Y with $\overline{R \times K}$ and let g be the unique morphism making the diagram



commutative. In this case $g((r, k)) = ((r, k), (r, 0))$. Since $Y \xrightarrow{q} R$ is an abelian group object in (C, R), $mg = \mathrm{id}_Y$. Therefore

$$mg((r, k)) = m((r, k), (r, 0)) = (r, k).$$

By a similar argument,

$$m((r, 0), (r, k)) = (r, k).$$

This can be used to show that $(0, k_1 + k_2) = (0, k_2 + k_1)$ and $(0, k_1 * k_2) = (0, 0)$.

For Beck, a derivation from R to an R-module Y is a morphism from $R \xrightarrow{\mathrm{id}} R$ to $Y \xrightarrow{q} R$ in (C, R). This also is equivalent to our definition of derivation; for if A is an R-module via Definition 2.1, it is easy to see that $f: R \to \overline{R \times A}$ is a morphism in (C, R) iff $f(r) = (r, \xi(r))$, where $\xi : R \to A$ is a derivation via Definition 2.6.

The reader may check that in categories of interest, Beck's description of extensions as principal objects coincides with our Definition 2.1. A detailed proof of this is laborious, but the ideas needed are available in [5].

§ 3. Technical observations

Let C be a category of interest. The objects of C have underlying groups, pointed sets and sets. In fact, as a consequence of axiom (2), the underlying group structure is defined on the underlying pointed set in such a way that the identity of the group is the base point.

If A and B are objects of C and $\alpha, \beta: U(A) \to U(B)$ are morphisms of their underlying pointed sets, we can define $\alpha + \beta$ (in S_*) by

$$(3.1) \qquad (\alpha + \beta)(a) = \alpha(a) + \beta(a)$$

since $\alpha(a)$ and $\beta(a)$ belong to the group underlying B. If $\gamma, \omega: UA' \to UA$ are morphisms in S_*, then for any $a' \in A'$,

$$(3.2) \qquad (\alpha + \beta)\gamma(a') = \alpha\gamma(a') + \beta\gamma(a').$$

This follows from (3.1).

If α is a morphism on the underlying pointed set of A' but not necessarily on the underlying group, then all we can say is

$$\alpha(\gamma + \omega)(a') = \alpha(\gamma a' + \omega a').$$

However, if $\hat{\alpha}: A \to B$ is a morphism in C and $a' \in UA'$, then

$$(U\hat{\alpha})(\gamma + \omega)a' = (U\hat{\alpha})(\gamma a' + \omega a') = \hat{\alpha}(\gamma a' + \omega a') = (U\hat{\alpha})\gamma a' + (U\hat{\alpha})\omega a'.$$

Thus,

$$(3.3) \qquad (U\hat{\alpha})(\gamma + \omega) = (U\hat{\alpha})\gamma + (U\hat{\alpha})\omega.$$

Since $+$ is not necessarily commutative we must keep in mind that $-(\alpha + \beta) = -\beta - \alpha$. Finally, if $\alpha: UA \to UB$ is as above, we write

$$\ker \alpha = \{a \in A \mid \alpha a = 0\},$$

$$\operatorname{im} \alpha = \{b \in B \mid b = \alpha a \text{ for some } a \in UA\}.$$

Now recall that $U: C \to S_*$ has a left adjoint $F: S_* \to C$ and that this adjoint pair gives rise to the triple $\mathbf{T}'$ on S_* and a cotriple $\mathbf{G} = (G, \epsilon, \delta)$ on C.

To compute cohomology groups, given an object R in C, one takes a resolution

$$(3.4) \qquad 0 \leftarrow R \xleftarrow{e^0} X_0 \underset{t^0}{\overset{e^0,\, e^1}{\leftleftarrows\!\rightarrow}} X_1 \underset{t^0,\, t^1}{\overset{e^0,\, e^1,\, e^2}{\equiv}} X_2 \;\cdots$$

as described in [4]. That is, the X_i are G-projective and the complex satisfies an acyclicity property. Any such complex can be used for computing triple cohomology, but for most of this paper the standard resolution

$$0 \leftarrow R \xleftarrow{e^0} GR \underset{\delta^0}{\overset{e^0,\, e^1}{\leftleftarrows\!\!\rightarrow}} G^2R \ldots$$

is entirely satisfactory.

Given a resolution such as (3.4), let

$$(3.5) \qquad e_n = \begin{cases} (Ue^n) - (Ue^{n-1}) + \ldots + (-1)^n(Ue^0) & \text{if } n \text{ is odd,} \\ (Ue^0) - (Ue^1) + \ldots + (-1)^n(Ue^n) & \text{if } n \text{ is even.} \end{cases}$$

These sums are defined in the sense of (3.1). The e_n are morphisms in S_* rather than in C. It is easy to check that $e_0e_1 = 0$, $e_1e_2 = 0$ and $e_2e_3 = 0$.

In addition, we note:

Remark 3.6. If $e_1(x) = 0$, then $x \in \operatorname{im} e_2$.

This is a consequence of a general fact about simplicial complexes in which each X_i is a group and all morphisms are group homomorphisms. Such a complex is said to satisfy the *full box condition* if given $x^0, \ldots, x^{n+1}$ in X_n such that $e^ix^j = e^{j-1}x^i$ for $i < j$, there exists x in X_{n+1} such that $e^ix = x^i$ for $0 \leqslant i \leqslant n+1$. It follows from a theorem in [13] and a routine computation that an acyclic group complex always satisfies this condition.

Let us use this to verify Remark 3.6. Suppose we have a resolution (3.4) of R. If $e_1(x) = 0$, then $e^0(x) = e^1(x)$. Let $x^0 = x^1 = x^2 = x \in X_1$. Then $e^ix^j = e^{j-1}x^i$ for $0 \leqslant i < j \leqslant 2$, and therefore there is $y \in X_2$ such that $e^0y = e^1y = e^2y = x$. Thus $e_2y = e^0y - e^1y + e^2y = x$.

Finally, we make note of another useful fact whose proof is straightforward.

Remark 3.7. If n is an object in S_* and $\alpha: F(n) \to R$ is a surjection, then there is a simplicial resolution

$$0 \leftarrow R \xleftarrow{\alpha} F(n) \leftrightarrows X_1 \ldots$$

in which each X_n is in fact free.

§4. The class E*A*

Let A be an object in C. If $A < T$ it follows from Theorem 1.7 that a set of actions of T on A is induced by the operations in T. These actions induce corresponding actions of $T/Z(T, A)$ on A by

(4.1) $$(t + Z(T, A)) + a - (t + Z(T, A)) = t + a - t,$$
$$(t + Z(T, A)) * a = t * a \qquad \text{for } * \in \Omega_2'.$$

Clearly, these are well-defined.

Definition 4.2. Let $\mathbf{E}A$ be the collection of equivalence classes of sequences (in $\mathcal{C}$) of the form:

$$0 \to ZA \to A \to T/Z(T, A) \to T/(A + Z(T, A)) \to 0,$$

where $A < T$ and equivalence is by isomorphisms which leave A fixed and preserve the actions of $T/Z(T, A)$ on A.

The second condition is not, as one might suspect, simply a consequence of A remaining fixed. This can be seen in the following example.

Example 4.3. Let $\mathcal{C}$ be the category of groups. Take G to be the free group on X and Y, and H the normal subgroup of G generated by $5X$ and $Y + X - Y - 2X$. Let $T = G/H$, and let A be the subgroup of T generated by $\bar{X} = X + H$. Then A is a normal subgroup of T. It is easy to see that

$$T/Z(T, A) = \{Z(T, A), \bar{Y} + Z(T, A), 2\bar{Y} + Z(T, A), 3\bar{Y} + Z(T, A)\}.$$

Moreover, $\Psi : T/Z(T, A) \to T/Z(T, A)$, defined by $\Psi(\bar{Y} + Z(T, A)) = 3\bar{Y} + Z(T, A)$, is an isomorphism which leaves the image of A fixed and does not fix the action of $T/Z(T, A)$ on A.

As in [1] we can construct a natural representative for each class in $\mathbf{E}A$. Let

$$0 \to ZA \to A \xrightarrow{\lambda} E \xrightarrow{\pi} M \to 0$$

represent a particular class in $\mathbf{E}A$. Then $E = T/Z(T, A)$ and $M = T/(A + Z(T, A))$ for some T such that $A < T$. Let $K \rightrightarrows T$ be the kernel pair of $T \to T/A$. Then

$$K = \{(t, t') \mid t, t' \in T \text{ and } t + A = t' + A\},$$

the two morphisms being the restrictions to K of the coordinate projections. Let

$$\Delta_Z = \{(z, z) \mid z \in Z(T, A)\}.$$

It is easy to check that $\Delta_Z < K$. Let

$$P = K/\Delta_Z.$$

The projections $K \rightrightarrows T$ induce morphisms $d^0, d^1 \colon P \rightrightarrows E$ which are given by

$$(4.4)\qquad \begin{aligned} d^0((t,t') + \Delta_Z) &= t + Z(T,A),\\ d^1((t,t') + \Delta_Z) &= t' + Z(T,A). \end{aligned}$$

We can write P in the form

$$P = \{(t+a,t) + \Delta_Z \mid t \in T, a \in A\}.$$

The map $\varphi \colon P \to \overline{E \times A}$ defined by

$$\varphi((t+a,t) + \Delta_Z) = (t + Z(T,A), a)$$

is easily checked to be an isomorphism between P and $\overline{E \times A}$. This shows that the set of actions (4.1) of $T/Z(T,A)$ on A is a set of derived actions and that P is independent of the choice of the representative.

As maps from $\overline{E \times A}$ to E, d^0 and d^1 are

$$d^0((e,a)) = e + \lambda a,$$

$$d^1((e,a)) = e.$$

From now on we will not distinguish between P and $\overline{E \times A}$.

In the following we enumerate several useful facts.

Proposition 4.5. *If* $0 \to ZA \to A \xrightarrow{\lambda} E \xrightarrow{\pi} M \to 0$ *represents a class in* EA, *then*

(i). $A < \overline{E \times A}$.

(ii). $Z(\overline{E \times A}, A) = \ker d^0$.

(iii). *Elements of* $\ker d^0$ *and* $\ker d^1$ *commute under* $+$.

(iv). $\ker d^0 * \ker d^1 = 0$ *for all* $* \in \Omega_2'$.

(v). $\ker d^0 \cap \ker d^1 \simeq ZA$.

Proof. (i). It is easy to see that $0 \times A < \overline{E \times A}$.

(ii). If $(e,a) \in Z(P,A)$ and if we write $e = t + Z(T,A)$, then we can show that $(t+a) \in Z(T,A)$ and therefore

$$e = t + Z(T,A) = -a + Z(T,A) = -\lambda(a).$$

That is, $(e,a) = (-\lambda(a), a) \in \ker d^0$. The converse is immediate.

(iii) and (iv) are easily checked after noting that $(e, a) \in \ker d^0$ implies $(e, a) = (-\lambda(a), a)$ and $(e', a') \in \ker d^1$ implies $(e', a') = (0, a')$.

(v). $d^0((e, a)) = d^1((e, a)) = 0$ iff $e + \lambda(a) = e = 0$ iff $e = 0$, and $\lambda(a) = 0$ iff $(e, a) \in 0 \times \ker \lambda = 0 \times ZA$.

Proposition 4.6. *The sequence*

$$0 \to ZA \to A \to P/Z(P, A) \to P/(A + Z(P, A)) \to 0$$

is equivalent to

$$0 \to ZA \to A \to E \to M \to 0.$$

Proof. Let $\varphi: P/Z(P, A) \to E$ be given by

$$\varphi((e, a) + Z(P, A)) = e + \lambda a.$$

We leave it to the reader to the reader to check that φ is a well-defined isomorphism that leaves the image of A fixed. We indicate how to check that φ preserves the action of $P/Z(P, A)$ on A. Let $a' \in A$; then

$$\begin{aligned}((e, a) + Z(P, A)) * a' &= (e, a) * (0, a') = (0, e * a' + a * a') \\ &= e * a' + \lambda(a) * a' = (e + \lambda(a)) * a' \\ &= \varphi((e, a) + Z(P, A)) * a'.\end{aligned}$$

To each class in $\mathbf{E}A$ we associate a truncated simplicial object by letting B be the kernel triple of $d^0, d^1 : P \rightrightarrows E$. That is,

$$B = \{(p_0, p_1, p_2) \mid d^0 p_0 = d^0 p_1,\ d^1 p_1 = d^1 p_2,\ d^1 p_0 = d^0 p_2\}.$$

Define $d^i: B \to P$ for $i = 0, 1, 2$ by $d^i((p_0, p_1, p_2)) = p_i$. If $p_i = (e_i, a_i)$, then

$$\begin{aligned} e_0 + \lambda a_0 &= e_1 + \lambda a_1, \\ e_1 &= e_2, \\ e_0 &= e_2 + \lambda a_2. \end{aligned} \tag{4.7}$$

From this one easily checks that

$$\ker d^1 \cap \ker d^2 \simeq ZA. \tag{4.8}$$

The only degeneracy we use is $s^0: E \to P$ defined by

$$s^0(e) = (e, 0).$$

We have remarked that A is an E-structure. More is true.

Proposition 4.9. *ZA is an E-module and an M-module. Moreover, ZA is a B-module and a P-module and all face operators preserve this structure.*

Proof. The actions of E on A leave ZA invariant. Take $z \in ZA$ and $e = t + Z(T, A) \in E$. First we check that for any $* \in \Omega_2'$, $e * z \in ZA$. Let $a \in A$ and $*' \in \Omega_2'$. Then

$$\begin{aligned}(e * z) *' a &= (t * z) *' a \\ &= w(t(za), t(az), (za)t, (az)t, z(ta), z(at), (ta)z, (at)z) \\ &= w(0, 0, 0, 0, 0, 0, 0, 0) = 0\end{aligned}$$

and

$$(e * z) + a = (t * z) + a = a + (t * z) = a + (e * z).$$

The latter follows from axiom (7).

Similarly we can check that $e + z - e \in ZA$, and thus, ZA is an E-module.

To see that ZA is an M-module, define actions of M on ZA by

$$\begin{aligned} m * z &= e * z, \\ m + z - m &= e + z - e \end{aligned}$$

for some e such that $\pi(e) = m$. We check that these operations are well-defined. Suppose $\pi(e) = \pi(e') = m$. Then

$$e - e' = \lambda(a) = a + Z(T, A)$$

for some $a \in A$. Moreover, if $e = t + Z(T, A)$ and $e' = t' + Z(T, A)$, then

$$(t - t') + Z(T, A) = a + Z(T, A).$$

That is,

$$t - t' - a = z' \in Z(T, A),$$

and

$$t = z' + a + t'.$$

Hence,

$$\begin{aligned}(e+z-e)-(e'+z-e') &= t+z-t+t'-z-t' \\ &= t+z-t'-a-z'+t'-z-t' \\ &= t+z+(-t'-a+t')+(-t'-z'+t')-z-t' \\ &= t-t'-a-z',\end{aligned}$$

and similarly, $e * z - e' * z = 0$.

This is a set of derived actions since $\overline{M \times ZA}$ defined via these actions is the cokernel of $\overline{\lambda A \times ZA} \hookrightarrow \overline{E \times ZA}$.

The last assertion of this theorem follows from Proposition 2.5, once we notice that for all $p \in P$, $\pi d^0 p = \pi d^1 p$, and thus, for all $b \in B$, $\pi d^0 d^0 b = \pi d^0 d^1 b = \pi d^1 d^1 b = \pi d^1 d^2 b = \pi d^0 d^2 b = \pi d^1 d^0 b$.

Finally, we prove the following important fact, which corresponds to [1, Proposition 1.3].

Proposition 4.10. *There is a derivation $\partial : B \to ZA$ given by the formula*

$$\partial x = (-s^0 d^0 + 1)dx = (-s^0 d^1 + 1)dx.$$

Proof. Using the relations recorded in (4.7), one checks that for any x in B,

$$(-s^0 d^0 + 1)dx = (-s^0 d^1 + 1)dx.$$

Next, we check that $\partial x \in ZA$:

$$d^0 \partial x = (-d^0 + d^0)dx = 0,$$

$$d^1 \partial x = (-d^1 + d^1)dx = 0.$$

Therefore $\partial x \in 0 \times ZA = \ker d^0 \cap \ker d^1$ for any $x \in B$.

A long computation follows to show that ∂ is a derivation. First consider

$$\begin{aligned}\partial(b_1 + b_2) &= (-s^0 d^0 + 1)d(b_1 + b_2) \\ &= -s^0 d^0 d^2 b_2 - s^0 d^0 d^2 b_1 + d^0 b_1 + d^0 b_2 \\ &\quad - d^1 b_2 - d^1 b_1 + d^2 b_1 + d^2 b_2.\end{aligned}$$

Note that

$$d^0(d^0 b_2 - d^1 b_2) = 0,$$

$$d^1(-d^1 b_1 + d^2 b_1) = 0,$$

so that Proposition 4.5 (ii) applies. Therefore

$$\begin{aligned}\partial(b_1 + b_2) &= -s^0d^0d^2b_2 - s^0d^0d^2b_1 + db_1 + db_2\\ &= -s^0d^0d^2b_2 - s^0d^0d^2b_1 + db_1 + s^0d^0d^2b_2 - s^0d^0d^2b_2 + db_2\\ &= -s^0d^0d^2b_2 + \partial b_1 + s^0d^0d^2b_2 + \partial b_2\\ &= -b_2 + \partial b_1 + b_2 + \partial b_2.\end{aligned}$$

Note the use of Theorem 4.9 to justify the last equality.

To check that $b_1 * \partial b_2 + \partial b_1 * b_2 = \partial(b_1 * b_2)$, we use the equations

$$\begin{aligned}(d^2b_1 - d^1b_1) * (d^2b_2 - s^0d^2b_2) = (s^0d^0d^2b_1 - d^2b_1) * (d^0b_2 - s^0d^0d^2b_2)\\ = 0,\end{aligned}$$

which are valid by virtue of Proposition 4.5 (iv). Then,

$$\begin{aligned}b_1 * \partial b_2 + \partial b_1 * b_2 = d^1b_1 * \partial b_2 + \partial b_1 * d^0b_2\\ + (d^2b_1 - d^1b_1) * (d^2b_2 - s^0d^0d^2b_2)\\ + (s^0d^0d^2b_1 - d^2b_1) * (d^0b_2 - s^0d^0d^2b_2).\end{aligned}$$

We leave it to the reader to expand the above expression fully. Using the fact that sums of "products" commute, a consequence of Axiom 5 for categories of interest, he can rearrange terms so as to arrive at the expression

$$\begin{aligned}-s^0d^0d^2(b_1 * b_2) + d(b_1 * b_2) = (-s^0d^0 + 1)d(b_1 * b_2)\\ = \partial(b_1 * b_2).\end{aligned}$$

§5. Obstructions

Let R be an object in C and $0 \to A \to T \to R \to 0$ an extension with which we associate a class in $\mathbf{E}A$ represented by

$$0 \to ZA \to A \xrightarrow{\lambda} T/Z(T, A) \xrightarrow{\pi} T/(A + Z(T, A)) \to 0.$$

There is then a surjective morphism $\rho : R \to M$ such that the diagram

$$\begin{array}{ccccccccc} & & 0 & \to & A & \longrightarrow & T & \longrightarrow & R & \to & 0\\ & & & & \| & & \downarrow & & \downarrow{\scriptstyle \rho} & & \\ 0 & \to & ZA & \to & A & \to & T/Z(T, A) & \longrightarrow & T/(A + Z(T, A)) & \to & 0\end{array}$$

commutes. ρ is said to be induced by the extension. Via ρ, ZA acquires the structure of an R-module.

In obstruction theory, one attacks the problem of finding all extensions which induce a given surjection $\rho: R \to M$ where

$$0 \to ZA \to A \to E \to M \to 0$$

represents some class in EA.

Let

$$0 \leftarrow R \xleftarrow{e^0} X_0 \underset{t^0}{\overset{e^0, e^1}{\leftleftarrows\!\rightarrow}} X_1 \underset{t^0, t^1}{\overset{e^0, e^1, e^2}{\Lleftarrow\!\rightrightarrows}} X_2 \ldots$$

be a resolution of R by G-free objects of $\mathcal{C}$. Morphisms $\rho_0: X_0 \to E$, $\rho_1: X_1 \to P$ and $\rho_2: X_2 \to B$ can be constructed as in [1] so that the following diagram is commutative:

$$\begin{array}{ccccccccc} X_2 & \Rrightarrow & X_1 & \rightrightarrows & X_0 & \to & R & \to & 0 \\ \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_0} & & \downarrow{\scriptstyle \rho} & & \\ B & \Rrightarrow & P & \rightrightarrows & E & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Using Proposition 2.5, we see that ZA is an X_n-module with derived actions induced by $\rho(e^0)^{n+1}$ for each n.

In general, if X is an N-module and $\gamma: M \to N$ induces an M-module structure on X, then for any derivation $\omega: N \to X$, $\omega\gamma: M \to X$ is also a derivation. With this in mind we see that $\partial\rho_2$ is a derivation and that for any $e^i: X_n \to X_{n-1}$, $(e^i)^*: \mathrm{Der}(X_{n-1}, ZA) \to \mathrm{Der}(X_n, ZA)$ can be defined by composition.

Recall that $\mathrm{Der}(X, ZA)$ is an abelian group. Hence,

$$e^{\#}: \mathrm{Der}(X_2, ZA) \to \mathrm{Der}(X_3, ZA)$$

and

$$e^{\#}: \mathrm{Der}(X_1, ZA) \to \mathrm{Der}(X_2, ZA)$$

can be defined by

$$e^{\#} = e^{3*} - e^{2*} + e^{1*} - e^{0*}$$

and

$$e^{\#} = e^{0*} - e^{1*} + e^{2*},$$

respectively.

The following useful observation can be proved by straightforward computation.

Remark 5.1. For $\tau \in \mathrm{Der}(X_2, ZA)$, $\sigma \in \mathrm{Der}(X_1, ZA)$, and $x \in X_3$,

$$e^*(\tau)(x) = -x + \tau(e^3 - e^2 + e^1 - e^0)(x) + x$$

and

$$e^*(\sigma) = \sigma(e^0 - e^1 + e^2).$$

This remark is useful, for example, in showing that $\partial\rho_2$ is a cocycle in $\mathrm{Der}(X_2, ZX)$:

$$\begin{aligned} e^*(\partial\rho_2)(x) &= -x + \partial\rho_2(e^3 - e^2 + e^1 - e^0)(x) + x \\ &= -x + (-s^0d^0 + 1)(d^0 - d^1 + d^2)\rho_2\,(e^3 - e^2 + e^1 - e^0)(x) + x \\ &= -x + (-s^0d^0 + 1)\rho_1(e^0 - e^1 + e^2)(e^3 - e^2 + e^1 - e^0)(x) + x \\ &= 0. \end{aligned}$$

Proposition 5.2. *The cohomology class of* $\partial\rho_2$ *in* $\mathrm{Der}(X_2, ZA)$ *does not depend on the choices of* ρ_0, ρ_1 *and* ρ_2.

Proof. This proof is the same as the proof of [1, Proposition 2.1] except that a little more care must be taken in carrying out the computations.

Since $\partial\rho_2 = (-s^0d^0 + 1)\rho_1 e$, the choice of ρ_2 is irrelevant. Let σ_0, σ_1 be new choices replacing ρ_0 and ρ_1. Construct $h^0\colon X_0 \to P$ and $h^0, h^1\colon X_1 \to B$ in the category C, so that for $h^0\colon X_0 \to P$,

$$d^0h^0 = \rho_0$$

$$d^1h^0 = \sigma_0$$

and for $h^0, h^1\colon X_1 \to B$,

$$\begin{aligned} &d^0h^1 = h^0e^0,\ d^0h^0 = \rho_1, \\ &d^1h^0 = d^1h^1, \\ &d^2h^0 = h^0e^1,\ d^2h^1 = \sigma_1. \end{aligned}$$

These constructions are carried out by using universal mapping properties of pullbacks and kernel triples and are therefore valid in C.

Next, we will show that

$$\alpha = \partial(-h^0 + h^1) \in \mathrm{Der}(X_1, ZA)$$

and

$$e^*(\alpha) = -\partial\rho_2 + \partial\rho_2.$$

First note that h^0 and h^1 preserve actions on ZA. Therefore, ∂h^0 and ∂h^1 are in $\mathrm{Der}(X_1, ZA)$. Furthermore, for $x \in X_1$,

$$\begin{aligned}
\alpha(x) &= \partial(-h^0 + h^1)(x) \\
&= -h^1x + \partial(-h^0x) + h^1x + \partial h^1x \\
&= -h^1x + h^0x - \partial h^0x + h^0x + h^1x + \partial h^1x \\
&= \pi d^0 d^0(-h^1x + h^0x) - (\partial h^0x) - \pi d^0 d^0(-h^1x + h^0x) + \partial h^1x \\
&= -\partial h^0x + \partial h^1x.
\end{aligned}$$

Since $-\partial h^0 + \partial h^1$ is in $\mathrm{Der}(X_1, ZA)$, we are done.

Then

$$\begin{aligned}
e^*(\alpha) &= \partial(-h^0 + h^1)(e^0 - e^1 + e^2) \\
&= (-s^0d^0 + 1)(-\rho_1 + h^0e^0 - h^0e^1 + \sigma_1)(e^0 - e^1 + e^2) \\
&= (-s^0d^0 + 1)(-\rho_1(e^0 - e^1 + e^2) + \sigma_1(e^0 - e^1 + e^2)) \\
&= (-s^0d^0 + 1)(-d\rho_2 + d\sigma_2) \\
&= -s^0d^0d\rho_2 + (s^0d^0d\rho_2 - s^0d^0d\sigma_2) + (s^0d^0d\rho_2 - d\rho_2) + d\sigma_2.
\end{aligned}$$

But $\pi d^1(s^0d^0\rho_2 - s^0d^0d\sigma_2) = 0$ and therefore

$$(s^0d^0d\rho_2 - s^0d^0d\sigma_2) \in \lambda A \times 0.$$

Also, $s^0d^0d\rho_2 - d\rho_2$ is in $0 \times ZA$ by Remark 4.5(v). Elements of $\lambda A \times 0$ and $0 \times ZA$ are easily seen to commute under addition. Thus,

$$\begin{aligned}
e^*(\alpha) &= -s^0d^0d\rho_2 + (s^0d^0d\rho_2 - d\rho_2) + (s^0d^0d\rho_2 - s^0d^0d\sigma_2) + d\sigma_2 \\
&= -\partial\rho_2 + \partial\sigma_2.
\end{aligned}$$

Let $[\rho]$ denote the cohomology class of $\partial\rho_2$.

Definition 5.3. $[\rho]$ is called the *obstruction* of ρ. ρ is said to be *unobstructed* if $[\rho] = 0$.

Theorem 5.4. *A surjection $\rho : R \to M$ arises from an extension iff ρ is unobstructed.*

Proof. The proof that if ρ arises from an extension then $[\rho] = 0$ is exactly the same as the corresponding proof in [1].

The proof of the converse is also essentially the same, with some modifications in the computations. If $\rho, \rho_0, \rho_1, \rho_2$ are given and there is derivation $\tau : X_1 \to ZA$ such that $e^*\tau = \partial\rho_2$, then let $\widetilde{\rho}_1 : X_1 \to P$ be $\rho_1 - \tau$. It is easily seen that $\widetilde{\rho}_1$ is a morphism in $\mathcal{C}$.

Proceeding exactly as in the proof of [1, Theorem 2.2], we see that ρ_2 can be chosen so that $\partial\rho_2 = 0$. First note that $d^0\widetilde{\rho}_1 = \rho_0 e^0$ and $d^1\widetilde{\rho}_1 = \rho_0 e^1$; then choose $\widetilde{\rho}_2$ over $\widetilde{\rho}_1$. Then we have

$$\begin{aligned}
\partial\rho_2 &= (-s^0d^0 + 1)(d^0 - d^1 + d^2)\widetilde{\rho}_2 \\
&= (-s^0d^0 + 1)(\widetilde{\rho}_1 e^0 - \widetilde{\rho}_1 e^1 + \widetilde{\rho}_1 e^2) \\
&= (-s^0d^0 + 1)(\rho_1 - \tau)e \\
&= +s^0d^0\tau e - s^0d^0\rho_1 e + \rho_1 e - \tau e \\
&= \partial\rho_2 - \tau e \text{ (by Proposition 4.5)} \\
&= \partial\rho_2 - e^*\tau = 0.
\end{aligned}$$

We are now ready to construct an extension that induces ρ. Let

$$\begin{array}{ccc}
Q & \xrightarrow{q_1} & P \\
{\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle d^1} \\
X_0 & \xrightarrow[\rho_0]{} & E
\end{array} \tag{5.5}$$

be a pullback diagram. q_2 is onto since d^1 is. Since $\mathcal{C}$ is a pointed category, $\ker q_2 \simeq \ker d^1 \simeq A$. We will identify $\ker q_2$ with $a : A \to Q$.

Since (5.5) is a pullback diagram, there exist unique u^0 and $u^1 : X_1 \to Q$ such that

$$\begin{aligned}
&q_1 u^0 = s^0 d^0 \rho_1, q_2 u^0 = e^0, \\
&q_1 u^1 = \rho_1, q_2 u^1 = e^1.
\end{aligned} \tag{5.6}$$

Consider the diagram

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & A & \longrightarrow & A \\
 & & {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle \bar{a}} \\
X_1 & \overset{u^0,\, u^1}{\rightrightarrows} & Q & \xrightarrow{q} & T \\
\| & & {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle \varphi} \\
X_1 & \underset{e^0,\, e^1}{\rightrightarrows} & X_0 & \xrightarrow[e]{} & R \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array} \tag{5.7}$$

where $q = \text{coeq}(u^0, u^1)$ and $\bar{a}$ and φ are induced maps. The right hand column of (5.7) is exact by commutativity of colimits. We will show that,

$$0 \to A \xrightarrow{\bar{a}} T \xrightarrow{\varphi} R \to 0$$

is an extension and that it induces $\rho : R \to M$.

First we show that $\bar{a}$ is monic. This is true if $\text{im}\, a \cap \ker q = 0$. But $\text{im}\, a = \ker q_2$ and $\ker q$ is the ideal generated by $\text{im}\,(u^1 - u^0)$. Let $u = u^1 - u^0$. For $\bar{a}$ to be monic, we need the following proposition.

Proposition 5.8. *The image of u is an ideal and* $\text{im}\, u \cap \ker q_2 = 0$.

This corresponds to [1, Proposition 1.2.3]. The proof is the same, but the computations are more elaborate.

Proof. To see that $\text{im}\, u < Q$ we must show that for each $x \in X_1$, $y \in Q$ and for each operation $* \in \Omega_2''$, there exist x_* and x_g such that

(1). $u(x_*) = u(x) * y$.

(2). $u(x_g) = y + u(x) - y$.

Let $x' = t^0 q_2(y)$ and set $x_* = x * x'$ and $x_g = x' + x - x'$. It suffices to check that

$$q_i u(x * x') = q_i(u(x) * y)$$

and

$$q_i(u(x' + x - x')) = q_i(y + u(x) - y)$$

for $i = 1, 2$. This is immediate for $i = 2$. Moreover, since $\partial \rho_2 = 0$,

$$(-s^0 d^0 + 1)\rho_1 |_{\text{im}\, e\, =\, \ker e} = 0.$$

In particular, $(-s^0 d^0 + 1)\rho_1 t^0 = 0$, and therefore

$$s^0 d^0 \rho_1 t^0 = \rho_1 t^0. \tag{5.9}$$

Thus,

$$\begin{aligned}
q_1 u(x * x') &= (q_1 u^1 - q_1 u^0)(x * x') \\
&= (\rho_1 - s^0 d^0 \rho_1)(x * x') \quad \text{(by (5.6))} \\
&= \rho_1 x * \rho_1 x' - \rho_1 x * s^0 d^0 \rho_1 x' + \rho_1 x * s^0 d^0 \rho_1 x' - s^0 d^0 \rho_1 x * s^0 d^0 \rho_1 x' \\
&= \rho_1 x * (1 - s^0 d^0)\rho_1 x' + (1 - s^0 d^0)\rho_1 x * s^0 d^0 \rho_1 x' \\
&= \rho_1 x * (\rho_1 t^0 (q_2 y) - s^0 d^0 \rho_1 t^0 (q_2 y)) + (1 - s^0 d^0)\rho_1 x * s^0 d^0 \rho_1 t^0 q_2(y) \\
&= (1 - s^0 d^0)\rho_1 x * s^0 d^0 \rho_1 t^0 q_2(y) \quad \text{(by (5.9))}
\end{aligned}$$

$$
\begin{aligned}
&= (1 - s^0d^0)\rho_1 x * s^0\rho_0 q_2(y)\\
&= (1 - s^0d^0)\rho_1 x * s^0d^1 q_1(y)\\
&= (1 - s^0d^0)\rho_1 x * q_1(y) - (1 - s^0d^0)\rho_1 x * (1 - s^0d^1)q_1(y)\\
&= (1 - s^0d^0)\rho_1 x * q_1(y) \qquad \text{(by Proposition 4.5(iv))}\\
&= (q_1u^1x - q_1u^0x) * q_1(y)\\
&= q_1u(x) * q_1(y)\\
&= q_1(u(x) * y),
\end{aligned}
$$

and

$$
\begin{aligned}
q_1u(x' + x - x') &= (\rho_1 - s^0d^0\rho_1)(x' + x - x')\\
&= \rho_1t^0q_2y + \rho_1 x - \rho_1t^0q_2y + s^0d^0\rho_1t^0q_2y\\
&\quad - s^0d^0\rho_1 x - s^0d^0\rho_1t^0q_2y\\
&= s^0\rho_0q_2y + \rho_1x - s^0\rho_0e^0x - s^0\rho_0q_2y\\
&= q_1y + (-q_1y + s^0d^1q_1y) + (\rho_1x - s^0d^0\rho_1x)\\
&\quad + (-s^0d^1q_1y + q_1y) - q_1y\\
&= q_1y + (\rho_1x - s^0d^0\rho_1x) - q_1y \qquad \text{(by Proposition 4.5(ii))}\\
&= q_1y + q_1ux - q_1y\\
&= q_1(y + ux - y).
\end{aligned}
$$

So im u is an ideal in Q. If $ux \in \ker q_2$ then $0 = q_2ux = ex$. That is, $x \in \ker e = \operatorname{im} e$. Therefore

$$0 = (-s^0d^0 + 1)\rho_1(x) = q_1ux,$$

and so $ux = 0$. That is,

$$\operatorname{im} u \cap \ker q_2 = 0.$$

To finish the proof of Theorem 5.4, we will show that the extension just constructed induces ρ. Since $d^0q_1u^0 = d^0s^0d^0\rho_1 = d^0\rho_1 = d^0q_1u^1$, there exists a unique $\tau: T \to E$ such that $\tau q = d^0q_1$. Moreover,

$$\pi\tau q = \pi d^0q_1 = \pi d^1q_1 = \pi\rho_0q_2 = \rho e q_2 = \rho\varphi q.$$

Since q is epic, $\pi\tau = \rho\varphi$.

τ is seen to be onto by chasing the following diagram:

$$\begin{array}{ccccccccc} 0 & \to & A & \to & T & \to & R & \to & 0 \\ & & \downarrow {\scriptstyle 1_A} & & \downarrow {\scriptstyle \tau} & & \downarrow {\scriptstyle \rho(\text{onto})} & & \\ & & A & \to & E & \to & M & \to & 0 \end{array}$$

To see that $\ker \tau = Z(T, A)$, observe that

(i) $q(\ker \tau q) = \ker \tau$,

(ii) $q(Z(Q, A)) = Z(T, A)$,

(iii) $Z(Q, A) = \ker \tau q$.

This can be shown as follows:

(i). $x \in \ker \tau q$ implies $q(x) \in \ker \tau$. Conversely, if $x \in \ker \tau$, then, since q is onto, there is some $y \in Q$ such that $q(y) = x$. Therefore $\tau q(y) = \tau(x) = 0$. So $x = q(y)$, where $y \in \ker \tau q$.

(ii). Take $x \in Z(Q, A)$. For any $y \in A$,

$$x * a(y) = a(y) * x = 0$$

and

$$a(y) + x = x + a(y).$$

Hence,

$$qa(y) * q(x) = \bar{a}(y) * q(x) = 0,$$

$$q(x) * qa(y) = q(x) * \bar{a}(y) = 0,$$

$$\bar{a}(y) + q(x) = q(a(y) + x) = q(x + a(y)) = q(x) + \bar{a}(y).$$

So $q(x) \in Z(T, A)$.

If $x \in Z(T, A)$, then for any $y \in A$,

$$\bar{a}(y) * x = x * \bar{a}(y) = 0$$

and

$$\bar{a}(y) + x = x + \bar{a}(y).$$

Since $x \in T$, there exists $z \in Q$ such that $x = q(z)$. Hence,

$$q(a(y) * z) = qa(y) * q(z) = \bar{a}(y) * x = x * \bar{a}(y) = q(z * a(y)) = 0,$$

and

$$q(a(y) + z - a(y) - z) = \bar{a}(y) + x - \bar{a}(y) - x = 0.$$

Since q is one-one on $a(A)$, this means

$$a(y) * z = z * a(y) = 0$$

and

$$a(y) + z = z + a(y),$$

for any $y \in A$. Therefore $z \in Z(Q, A)$ and $x \in q(Z(Q, A))$.

(iii).

$$\ker \tau q = \{(x, p \in X_0 \times P | d^0(p) = 0\},$$

$$Q = \{(x, p) \in X_0 \times P | \rho_0(x) = d^1(p)\}.$$

The image of A in Q is

$$\{(0, p) | d^1(p) = 0\},$$

since the image is the kernel of q_2. Hence,

$$\begin{aligned}
Z(Q, A) &= \{(x, p) \in Q | d^1(p') = 0 \Rightarrow [(x, p) * (0, p') = (0, p * p') = 0,\\
&\qquad (0, p') * (x, p) = (0, p' * p) = 0, \quad (x, p) + (0, p\ ) = (0, p') + (x, p)]\}\\
&= \{(x, p) \in Q | d^1(p') = 0 \Rightarrow [p * p' = p' * p = 0, \quad p + p' = p' + p]\}\\
&= \{(x, p) \in Q | p \in Z(P, A)\}\\
&= \{(x, p) \in Q | p \in \ker d^0\} \qquad \text{(by Proposition 4.5(i))}\\
&= \ker \tau q.
\end{aligned}$$

Using (i), (ii) and (iii),

$$\ker \tau = q(\ker \tau q) = q(Z(Q, A)) = Z(T, A).$$

§6. The action of H^1

In [1, §3], the following theorem is proved for commutative algebras:

Theorem 6.1. *Let $\rho : R \to M$ be unobstructed. Let $\Sigma = \Sigma_\rho$ denote the equivalence classes of extensions*

$$0 \to A \to T \to R \to 0$$

which induce ρ. Then the group $H^1(R, ZA)$ acts in Σ_ρ as a principal homogeneous representation.

The theorem is also true if objects and morphisms are assumed to belong to a category of interest C, and the cohomology is computed by means of the triple arising in axiom (1). The proof is nearly identical to the one given by Barr.

Using a result from Beck's thesis [5], $\Lambda = H^1(R, ZA)$ consists of equivalence classes of extensions

$$0 \to ZA \to U \to R \to 0$$

in which ZA is assumed to have the R-module structure arising from ρ. The only changes in the proof are in the section which corresponds to [1, Proposition 3.2]. Consider, for example, part b) of that proposition, in which it is shown that $(\Sigma_1 - \Sigma_2) + \Sigma_2 = \Sigma_1$. If $0 \to A \to T \to R \to 0$ represents $(\Sigma_1 - \Sigma_2) + \Sigma_2$, we have, just as in [1], that any element of T can be represented by a 3-tuple (t_1, t_2, t_2') for which $\tau_1 t_1 = \tau_2 t_2$ and $\varphi_1 t_1 = \varphi_2 t_2 = \varphi_2 t_2'$. In defining the morphism $\sigma : T \to T'$, we must be cautious about order and signs. First observe that $\varphi_2(t_2) = \varphi_2(t_2')$ implies $(-t_2 + t_2') \in A$. Let

$$\sigma(t_1, t_2, t_2') = t_1 - t_2 + t_2'.$$

Then σ is well-defined, since $(t_1, t_2, t_2') = (s_1, s_2, s_2')$ iff $(-t_1 + s_1, -t_2 + s_2, -t_2' + s_2') = (-z, 0, z)$ for some $z \in ZA$. That is, $s_1 = t_1 - z$, $s_2 = t_2$ and $s_2' = t_2' + z$. Thus,

$$\begin{aligned} s_1 - s_2 + s_2' &= t_1 - z - t_2 + t_2' + z \\ &= t_1 - t_2 + t_2' \end{aligned}$$

(since $(-t_2 + t_2') \in A$). In checking that σ is a morphism in C we use the fact that $\tau_1 x = \tau_2 y$ implies that x and y have the same actions on A:

$$\begin{aligned} \sigma(t_1, t_2, t_2') + \sigma(s_1, s_2, s_2') &= (t_1 - t_2 + t_2') + (s_1 - s_2 + s_2') \\ &= t_1 - t_2 + t_2' - t_1 + t_1 + s_1 - s_2 + s_2' \\ &= [t_1 + (-t_2 + t_2') - t_1] + (t_1 + s_1) + (-s_2 + s_2') \\ &= [t_2 + (-t_2 + t_2') - t_2] + (t_1 + s_1) + (-s_2 + s_2') \\ &= (t_2' - t_2) + (t_1 + s_1) + (-s_2 + s_2') \\ &= (t_1 + s_1) - (t_1 + s_1) + (t_2' - t_2) + (t_1 + s_1) + (-s_2 + s_2') \end{aligned}$$

$$
\begin{aligned}
&= (t_1+s_1) - (t_2+s_2) + (t_2'-t_2) + (t_2+s_2) + (-s_2+s_2')\\
&= (t_1+s_1) - (t_2+s_2) + (t_2'+s_2')\\
&= \sigma(t_1+s_1, t_2+s_2, t_2'+s_2').
\end{aligned}
$$

The proof that σ preserves operations $* \in \Omega_2'$ is easier since sums of products commute.

In part c) it can be shown, as in [1], that $(\Lambda + \Sigma) - \Sigma = \Lambda$. If $0 \to ZA \to U' \overset{\psi'}{\to} R \to 0$ represents $(\Lambda + \Sigma) - \Sigma$, then a typical element of U' is represented by a 3-tuple (t, u, t') in which $\varphi(t) = \psi(u) = \varphi(t')$ and $\tau(t) = \tau(t')$. Since $\varphi(t - t') = 0$ and $\tau(t - t') = 0$,

$$(t - t') \in A \cap Z(T, A) = ZA.$$

Moreover, $(t, u, t') = (s, v, s')$ iff $s - t = s' - t' = a$, for some $a \in A$, and $u = v$.

Define $\sigma : U' \to U$ by $\sigma(t, u, t') = (t - t') + u$. σ is well-defined, for if $(t, u, t') = (s, v, s')$, then

$$
\begin{aligned}
\sigma(s, v, s') &= (s - s') + v\\
&= ((a + t) - (a + t')) + u\\
&= (a + (t - t') - a) + u\\
&= (t - t') + u \qquad \text{(since } t - t' \in ZA)\\
&= (t, u, t').
\end{aligned}
$$

Moreover, σ is a morphism in C since

$$
\begin{aligned}
&\sigma(t, u, t')) + \sigma(s, v, s') - \sigma((t, u, t') + (s, v, s'))\\
&= (t - t' + u) + (s - s' + v) - (t + s - s' - t' + u + v)\\
&= (t - t') + u + (s - s') + v - v - u + t' + s' - s - t\\
&= [(t - t') + u + (s - s') - u - (t - t')] + [t + (s' - s) - t]\\
&= [\varphi(t - t') + (u + (s - s') - u) - \varphi(t - t')] + [\varphi(t) + (s' - s) - \varphi(t)]\\
&= (u + (s - s') - u) + (\psi(u) + (s' - s) - \psi(u))\\
&= (u + (s - s') - u) + (u + (s' - s) - u)\\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sigma((t, u, t')) * \sigma((s, v, s'))\\
&= (t - t' + u) * (s - s' + v)
\end{aligned}
$$

$$= (t - t') * (s - s') + (t - t') * v + u * (s - s') + u * v$$

$$= (t - t') * \varphi(s - s') + (t - t') * \psi(v) + \psi(u) * (s - s') + u * v$$

$$= (t - t') * [\varphi(s) - \varphi(s') + \varphi(s')] + \varphi(t') * (s - s') + u * v$$

$$= (t - t') * s + t' * (s - s') + u * v$$

$$= t * s - t' * s' + u * v$$

$$= \sigma((t, u, t') * (s, v, s')).$$

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