Paracompactness in Fuzzy Topological Spaces*

MAO-KANG LUO

Department of Mathematics, Sichuan University, China

Communicated by L. Zadeh

Received May 17, 1985

In this paper, we introduce two kinds of fuzzy paracompactnesses which are defined on general fuzzy subsets. Each of them is a good extension of crisp paracompactness. They are all hereditary with respect to closed subsets and take $N$-compactness and some other compactnesses in fuzzy topological spaces as special cases of them. Furthermore, in a class of spaces (so-called "weakly induced spaces") which connects crisp spaces with fuzzy spaces, both a fuzzy regular Lindelöf property and a fuzzy pseudo-metric property imply these two kinds of paracompactnesses.

1. INTRODUCTION

Paracompactness describes the relation between a locally finite property and an entire property of spaces, occupies an important position in general topology, and is a problem in fuzzy topology which attracts the attention of all of us. But up until now, fuzzy paracompactness has not had a better or more ideal definition. There were some works about this problem [1, 2], but the definitions of them used the rather special "B set" or depended upon the concepts of covers; all have many limitations. In fact, neither takes one of the $N$-compactnesses [3] or some other compactness in fuzzy topological space as a special case and is not a good extension [4] of crisp paracompactness. One of the purposes of setting up fuzzy paracompactnesses in this paper is to overcome these limitations. Moreover, there exist close relations among the metric property, the Lindelöf property, and paracompactness, and trying to set up or discover these relations within the scope of fuzzy topology is also a purpose of this paper.

The complex nature of the neighborhood structure in fuzzy topological space makes the relation between the local property and the entire property of fuzzy topological space more difficult to handle. On the other hand, the essential difference between fuzzy and crisp topological spaces

* The work of this paper was supported by the Conference of Scientific Funds of The Academy of Sciences of China.
lies in the differences and connections among each horizontal level of space. It is also an important point for deep study of fuzzy topology. In view of these situations, after analyzing the above-mentioned purposes, from the difference between two kinds of neighborhood structures, this paper has set up two kinds of fuzzy paracompactnesses to describe the properties of every horizontal level of space. They have the advantages stated in the abstract of this paper. We have studied the basic properties of these two kinds of paracompactnesses in greater depth, e.g., the problem of equivalent description etc. and used full counterexamples to explain the relations among the conditions of the theorems and the differences between these conditions and crisp situations carefully. By using this paper as a foundation, we can study the problems of fuzzy paracompactness more fully.

We let $q$ denote the quasi-coincident relation [5], $Q(A)$ denote the $Q$-neighborhood system of fuzzy set $A$ [5], $\chi$ denote the characteristic function, and $|C|$ denote the cardinality of a crisp set $C$. The concepts of base and subbase for a fuzzy topological space and other concepts which have not been defined in this paper are taken from [5]. We let $fts$ denote fuzzy topological space for convenience. For every real number $\alpha$ and every fuzzy set $A$ on $X$, we let $\alpha A$ denote a function such that $(\alpha A)(x) = \alpha A(x)$ for every $x \in X$; we let $A > \alpha$ denote the relation $A(x) > \alpha$ for every $x \in X$. If it will not cause any confusion, we will call the fuzzy set the set and the fuzzy point, the point directly, and we will not often differentiate a crisp set from its characteristic function.

The author is indebted to Professor Liu Ying-Ming for his careful instruction.

2. Definitions and Basic Properties

2.1. Definition. Let $\mathcal{A}$, $\mathcal{B}$ be two families of sets in $fts (X, \mathcal{F})$. $\mathcal{A}$ is called a refinement of $\mathcal{B}$ if for any $A \in \mathcal{A}$, there exist a $B \in \mathcal{B}$ such that $A \subseteq B$.

2.2. Definition. Let $\mathcal{A}$ be a family of sets and $B$ be a set in $fts (X, \mathcal{F})$. We say that $\mathcal{A}$ is locally finite (resp. *-locally finite) in $B$ if for each point $e$ in $B$, there exists a $U \in Q(e)$ such that $U$ is quasi-coincident (resp. intersects) with at most a finite number of sets of $\mathcal{A}$; we often omit the word "in $B$" when $B = X$.

2.3. Definition. Let $A$ be a set in $fts (X, \mathcal{F})$ and let $\alpha \in (0, 1]$, $\beta \in [0, 1)$; we define
2.4. **Definition.** A family of sets $\mathcal{A}$ is called a $Q$-cover [6] of a set $B$ if for each $x \in \text{supp}(B)$, there exist an $A \in \mathcal{A}$ such that $A$ and $B$ are quasi-coincident at $x$. Let $\alpha \in (0, 1]$. $\mathcal{A}$ is called an $\alpha$-$Q$-cover of $B$ if $\mathcal{A}$ is a $Q$-cover of $B\langle \alpha \rangle$.

2.5. **Definition.** Let $\alpha \in (0, 1]$, $A$ be a set in fts $(X, \mathcal{F})$. We say that $A$ is $\alpha$-paracompact (resp. $\alpha^*$-paracompact) if for each $\alpha$-open $Q$-cover of $A$ there exists an open refinement of it which is both locally finite (resp. $*$-locally finite) in $A$ and an $\alpha$-$Q$-cover of $A$. $A$ is called $S$-paracompact (resp. $S^*$-paracompact) if for every $\alpha \in (0, 1]$, $A$ is $\alpha$-paracompact (resp. $\alpha^*$-paracompact).

We say that $(X, \mathcal{F})$ is $\alpha$-paracompact (resp. $\alpha^*$-paracompact, $S$-paracompact, $S^*$-paracompact) if set $X$ is $\alpha$-paracompact (resp. $\alpha^*$-paracompact, $S$-paracompact, $S^*$-paracompact).

2.6. **Remark.** It is obvious that

*$\Rightarrow$ locally finite,

so we get the relations

$\alpha^*$-paracompact $\Rightarrow$ $\alpha$-paracompact,

$S^*$-paracompact $\Rightarrow$ $S$-paracompact;

but it is easy to find that the inverses of these relations are not true.


*Proof.* It is obvious that strong fuzzy compact spaces are $S^*$-paracompact. On the other hand, from [3] we know that for each fts there exists the relation

ultra-fuzzy compact $\Rightarrow$ $N$-compact $\Rightarrow$ strong fuzzy compact,

so the theorem is true.  

2.9. EXAMPLE. There exists a fuzzy compact [7] fts which is not 1-paracompact. Let \( X = [0, +\infty) \), for each \( x \in X \), let
\[
U_x(y) = \begin{cases} 
e^{-y}, & 0 \leq y < x, \\ 0, & y \leq x, \end{cases}
\]
\[
U_\infty(y) = e^{-y}, \quad y \in X,
\]
\[ \mathcal{T} = \{ U_x, X \} \cup \{ U_x : x \in X \}, \]
then it is easy to know that \((X, \mathcal{T})\) is a fuzzy compact fts. Let \( \mathcal{U} = \{ U_x : x \in X \} \), \( \mathcal{U} \) is an open \( Q \)-cover of \( X \). Let \( \mathcal{V} \) be both an open refinement of \( \mathcal{U} \) and an open \( Q \)-cover of \( X \), then from the structure of \( \mathcal{T} \) we know \( \mathcal{V} \subset \mathcal{U} \). Furthermore we have \( \sup \{ x : U_x \in \mathcal{V} \} = +\infty \), so \(|\mathcal{V}| \geq \omega \). Hence \( \mathcal{V} \) is not locally finite at the fuzzy point \((0)_1\), and \((X, \mathcal{T})\) is not 1-paracompact.

2.10. DEFINITION [5]. An fts \((X, \mathcal{T})\) is called a \( T_2 \)-fts if for each pair of fuzzy points \( x_1, y_1, x \neq y \), there exist \( U \in Q(x_1) \) and \( V \in Q(y_1) \) such that \( U \cap V = \emptyset \).

2.11. THEOREM. If \( T_2 \)-fts \((X, \mathcal{T})\) is fuzzy compact, then \((X, \mathcal{T})\) is \( S^* \)-paracompact.

\[ \text{Proof.} \] From [3] we know that every fuzzy compact \( T_2 \)-fts is strong fuzzy compact, hence we know from 2.7 that the theorem is true.

2.12. THEOREM. If a family of sets \( \{A_i\}_{i \in T} \) of fts \((X, \mathcal{T})\) is locally finite in a set \( A \), then
\[
\bigcup_{i \in T} A_i \cap A = \bigcup_{i \in T} (A_i \cap A).
\]

\[ \text{Proof.} \] It need only to show that \( \bigcup_{i \in T} A_i \cap A \subset \bigcup_{i \in T} (A_i \cap A) \). Let \( B = \bigcup_{i \in T} A_i \) and \( e \in B \cap A \); then from [5] we know that \( e \) is an adherence point of \( B \). Since \( \{ A_i \}_{i \in T} \) is locally finite in \( A \), so there exists a \( V \in Q(e) \) which is quasi-coincident with only a finite number of members \( A_{i_1}, \ldots, A_{i_n} \) of \( \{A_i\}_{i \in T} \). If \( e \in \bigcup_{i=1}^n \bar{A}_{i} \), then there exists a \( V_i \in Q(e) \) which is not quasi-coincident with \( A_{i_i} \) \((i = 1, \ldots, n)\). Let \( V_0 = (\bigcap_{i=1}^n V_i) \cap V \), we have \( V_0 \in Q(e) \) and \( V_0 \) is not quasi-coincident with any \( A_{i_i} \), so \( V_0 \) is not quasi-coincident with \( B \). This is in contradiction with the fact that \( e \) is an adherence point of \( B \), hence \( e \in \bigcup_{i=1}^n (\bar{A}_{i_i} \cap A) \subset \bigcup_{i \in T} (A_i \cap A) \).
2.13. **Corollary.** If a family of sets \( \{ A_i \}_{i \in T} \) of fits \((X, \mathcal{F})\) is locally finite in a set \( A \) and \( \bigcup_{i \in T} A_i \subset A \), then
\[
\bigcup_{i \in T} A_i = \bigcup_{i \in T} \overline{A}_i.
\]

2.14. **Theorem.** If a family of sets \( \{ A_i \}_{i \in T} \) of fits \((X, \mathcal{F})\) is locally finite in a set \( A \), then the family of sets \( \{ \overline{A}_i \}_{i \in T} \) is also locally finite in \( A \).

2.15. **Definition.** An fits \((X, \mathcal{F})\) is called to be regular if for each point \( e \) in \((X, \mathcal{F})\) and each \( U \in \mathcal{Q}(e) \), there exists a \( V \in \mathcal{Q}(e) \) such that \( V \subset U \).

2.16. **Remark.** [8] has proved that 2.15 is an equivalence form of the definition of regularity in [9].

2.17. **Theorem.** Let \( A, B \) be sets in a regular fits \((X, \mathcal{F})\) and each open \( \mathcal{Q}\)-cover of \( A \) have a refinement which is both locally finite in \( B \) and a \( \mathcal{Q}\)-cover of \( A \). Then for each open \( \mathcal{Q}\)-cover \( \{ U_i \}_{i \in T} \), when \( \bigcup_{i \in T} U_i \subset B \), there exists a closed \( \mathcal{Q}\)-cover \( \{ F_i \}_{i \in T} \) of \( A \) which is locally finite in \( B \) such that for each \( t \in T \) we have \( F_i \subset U_i \).

**Proof.** From the regularity, \( A \) has an open \( \mathcal{Q}\)-cover \( \mathcal{W} \) such that \( \{ W : W \in \mathcal{W} \} \) is a refinement of \( \{ U_i \}_{i \in T} \). Take a \( \mathcal{Q}\)-cover \( \{ A_r \}_{r \in S} \) of \( A \) such that \( \{ A_r \}_{r \in S} \) is both a refinement of \( \mathcal{W} \) and locally finite in \( B \). From the way we get \( \mathcal{W} \) we know that there exists a mapping \( f: S \to T \) such that \( A_r \subset U_{f(r)} \). Let \( F_i = \bigcup \{ \overline{A}_r : f(r) = i \} \); then from 2.13 we know that \( F_i \) is closed set, and it is easy to know from 2.14 that the closed \( \mathcal{Q}\)-cover \( \{ F_i \}_{i \in T} \) of \( A \) is locally finite in \( B \). Furthermore, for each \( t \in T \) we have \( F_i \subset U_i \).

If we let the \( A_i \)'s in the proof of 2.16 be open, then next theorem is obvious.

2.18. **Theorem.** Let \( A \) be an \( \alpha\)-paracompact set in a regular fits \((X, \mathcal{F})\), then for each \( \alpha\)-open \( \mathcal{Q}\)-cover \( \{ U_i \}_{i \in T} \) of \( A \), when \( \bigcup_{i \in T} U_i \subset A \), there exists an \( \alpha\)-open \( \mathcal{Q}\)-cover \( \{ V_i \}_{i \in T} \) of \( A \) which is locally finite in \( A \) and for each \( t \in T \) we have \( V_i \subset U_i \).

2.19. **Definition.** Let \( \alpha \in [0, 1) \). We say that fits \((X, \mathcal{F})\) is \( \alpha\)-crisp if for each \( U \in \mathcal{F} \) we have \( U_{(\alpha)} \in \mathcal{F} \).

The proof of next theorem is direct.

2.20. **Theorem.** If a family of sets \( \{ A_i \}_{i \in T} \) in a \( \alpha\)-crisp fits \((X, \mathcal{F})\) is \( *\)-locally finite in a set \( A \), then so is the family of sets \( \{ \overline{A}_i \}_{i \in T} \).
The example below explains that why 0-crispness is a condition which cannot be eliminated from 2.20.

2.21. Example. Let \( X = [0, 1) \). For each \( \alpha \in X \) and each \( n \in \mathbb{N} \), we let \( U_\alpha = \alpha \chi_{(0,1)} \) and \( A_n = (1 - 1/n)_1 \), then \( \mathcal{F} = \{ U_\alpha : \alpha \in X \} \cup \{ X \} \) is a fuzzy topology on \( X \) and \( \{ A_n : n \in \mathbb{N} \} \) is a \( * \)-locally finite family of sets in \( (X, \mathcal{F}) \). But the family of sets \( \{ A_n : n \in \mathbb{N} \} \) is not \( * \)-locally finite in \( X \), because we have \( A_n = (U_{1 - 1/n})' > 0 \).

Being similar to 2.17 and 2.18, we have two theorems below:

2.22. Theorem. Let \( A, B \) be two sets in 0-crisp regular fts \( (X, \mathcal{F}) \). Suppose that each open \( \mathcal{Q} \)-cover of \( A \) has a refinement which is both \( * \)-locally finite in \( B \) and a \( \mathcal{Q} \)-cover of \( A \), then for each open \( \mathcal{Q} \)-cover \( \{ U_t : t \in \mathbb{T} \} \) of \( A \), when \( \bigcup_{t \in \mathbb{T}} U_t \subset B \), there exists a closed \( \mathcal{Q} \)-cover \( \{ F_t : t \in \mathbb{T} \} \) of \( A \) which is \( * \)-locally finite in \( B \) and for each \( t \in \mathbb{T} \) we have \( F_t \subset U_t \).

2.23. Theorem. Let \( A \) be an \( \alpha * \)-paracompact set in a 0-crisp regular fts \( (X, \mathcal{F}) \). Then for each \( \alpha \)-open \( \mathcal{Q} \)-cover \( \{ U_t : t \in \mathbb{T} \} \) of \( A \), when \( \bigcup_{t \in \mathbb{T}} U_t \subset A \), there exists an \( \alpha \)-open \( \mathcal{Q} \)-cover \( \{ V_t : t \in \mathbb{T} \} \) of \( A \) which is \( * \)-locally finite in \( A \) such that for each \( t \in \mathbb{T} \) we have \( V_t \subset U_t \).

2.24. Theorem. If \( A \) an \( \alpha \)-paracompact (resp. \( \alpha * \)-paracompact) set in fts \( (X, \mathcal{F}) \), then for each closed set \( B \) in \( (X, \mathcal{F}) \), each \( \alpha \)-open \( \mathcal{Q} \)-cover of set \( B \cap A \) has an open refinement which is both an \( \alpha \)-\( \mathcal{Q} \)-cover of \( B \cap A \) and locally finite (resp. \( * \)-locally finite) in \( A \).

Proof. We only prove the case of \( \alpha \)-paracompactness. Let \( \mathcal{U} \) be an \( \alpha \)-open \( \mathcal{Q} \)-cover of \( C = B \cap A \); then \( \mathcal{U} \cup \{ B' \} \) is an open \( \mathcal{Q} \)-cover of \( A <_\alpha \), and it has an open refinement \( \varphi \) which is both locally finite in \( A \) and a \( \mathcal{Q} \)-cover of \( A < \alpha \). Let \( \varphi_0 = \{ V \in \varphi : \exists U \in \mathcal{U}, V \subset U \} \), then \( \varphi_0 \) is an open refinement of \( \mathcal{U} \) which is locally finite in \( A \). We say with certainty that \( \varphi_0 \) is a \( \mathcal{Q} \)-cover of \( C < \alpha \). Suppose that it does not hold, then there exists a \( x \in \text{supp}(C < \alpha) \) such that \( (\bigcup \varphi_0)(x) < 1 - \alpha \). But \( \varphi \) is a \( \mathcal{Q} \)-cover of \( A < \alpha \), so from \( C \subset_\varphi A < \alpha \) we know that there exists a \( V \in \varphi \) such that \( V(x) > 1 - \alpha \). Since \( \varphi \) is a refinement of \( \mathcal{U} \cup \{ B' \} \) we know \( V \subset B' \); from \( C(x) \geq \alpha \) we know \( B(x) \geq \alpha \). Hence
\[
1 - \alpha < V(x) \leq B'(x) \leq 1 - \alpha;
\]
this is a contradiction.

2.25. Corollary. Every closed set of an \( \alpha \)-paracompact (resp. \( \alpha * \)-paracompact) fts is \( \alpha \)-paracompact (resp. \( \alpha * \)-paracompact).
2.26. Definition [10]. Let $\alpha \in (0, 1]$. A set $A$ in fts $(X, \mathcal{I})$ is called a $Q_\alpha$-compact set if each $\alpha$-open $Q$-cover of $A$ has a finite subfamily which is an $\alpha$-$Q$-cover of $A$. $A$ is called a strong $Q$-compact set if $A$ is $Q_\alpha$-compact for each $\alpha \in (0, 1]$.

2.27. Theorem. Both $N$-compact sets and strong $Q$-compact sets are $S^*$-paracompact.

Proof. It is obvious that strong $Q$-compact sets are $S^*$-paracompact. On the other hand, from [10] we know that $N$-compact sets are strong $Q$-compact, so the theorem is true. $\blacksquare$

3. Paracompactness in Weakly Induced Fuzzy Topological Spaces

3.1. Definition. For each fts $(X, \mathcal{I})$, the family of crisp sets

$$[\mathcal{I}] = \{ A \subseteq X : \chi_A \in \mathcal{I} \}$$

is called the original topology of $\mathcal{I}$ and the crisp topological spaces $(X, [\mathcal{I}])$ are called original topological spaces of $(X, \mathcal{I})$.

3.2. Remark. The fact that $[\mathcal{I}]$ is a crisp topology on $X$ is certain.

3.3. Definition [11]. We say that fts $(X, \mathcal{I})$ is a weak inducement of crisp topological space $(X, \mathcal{I}_0)$ if $[\mathcal{I}] = \mathcal{I}_0$, and every $U \in \mathcal{I}$ is lower semi-continuous when we regard it as a mapping between $(X, \mathcal{I}_0)$ and $[0, 1]$. We say $(X, \mathcal{I})$ is weakly induced if there exists a crisp topological space $(X, \mathcal{I}_0)$ such that $(X, \mathcal{I})$ is a weak inducement of $(X, \mathcal{I}_0)$.

3.4. Remark. Clearly, every induced fts is weakly induced, but the inverse is not true. Hence the concept of weak inducement is a real extension of the concept of inducement. Since the concept of induced fts is an extension of the concept of crisp topological space, hence so is the concept of weak inducement.

From the properties of lower semi-continuous functions which are well known by us, we have

3.5. Theorem. $(X, \mathcal{I})$ is a weakly induced fts if and only if $(X, \mathcal{I})$ is $\alpha$-crisp for every $\alpha \in [0, 1]$.

3.6. Theorem. For every weakly induced fts $(X, \mathcal{I})$ the following conditions are equivalent:

(i) $(X, \mathcal{I})$ is $S^*$-paracompact.

(ii) There exist a $\alpha \in (0, 1)$ such that $(X, \mathcal{I})$ is $\alpha^*$-paracompact.
(iii) \((X, \mathcal{T})\) is \(S\)-paracompact.

(iv) There exist a \(x \in (0, 1)\) such that \((X, \mathcal{T})\) is \(x\)-paracompact.

(v) \((X, [\mathcal{T}])\) is paracompact.

Proof. (i) \(\Rightarrow\) (iii) \(\Rightarrow\) (iv) Obvious.

(iv) \(\Rightarrow\) (v) Let \(\mathcal{U} \subset [\mathcal{T}]\) be an open cover of \(X\); then \(\{\chi_U: U \in \mathcal{U}\}\) is an open \(Q\)-cover of \(1_x\) and it has a locally finite open refinement \(\mathcal{V}\) which is a \(Q\)-cover of \(1_x\) also. Let \(\mathcal{W} = \{V_{(1-\lambda)}: V \in \mathcal{V}\}\); then \(\mathcal{W}\) is both a refinement of \(\mathcal{U}\) and a cover of \(X\). From the properties of weak inducement we know \(\mathcal{W} \subset \mathcal{F}\), so we need only to prove that \(\mathcal{W}\) is locally finite. For each \(x \in X\), take \(O_1 \in Q_\mathcal{T}(x_{1-\lambda})\) such that \(O_1\) is quasi-coincident with only a finite number of members \(V_1, \ldots, V_n\) of \(\mathcal{V}\). Let \(O = (O_1)_{(1-\lambda)}\); then \(x \in O \in [\mathcal{F}]\). For each \(V \in \mathcal{V}\), if \(O \cap V_{(1-\lambda)} \neq \emptyset\), we have a crisp point \(y \in O \cap V_{(1-\lambda)}\), hence \(O_1(y) > \alpha\), \(V(y) > 1 - \alpha\), \(O_1(y) + V(y) > 1\), and \(O\) and \(V\) are quasi-coincident, so \(V \in \{V_1, \ldots, V_n\}\). Hence the neighborhood \(O\) of \(x\) intersects with only a finite number of members \((V_1)_{(1-\lambda)}, \ldots, (V_n)_{(1-\lambda)}\) of \(\mathcal{W}\).

(v) \(\Rightarrow\) (i) For each \(x \in (0, 1]\), let \(\mathcal{U} \subset \mathcal{F}\) be a \(Q\)-cover of \(1_x\). Then from the properties of weak inducement we know that the family of crisp sets \(\{U_{(1-\lambda)}: U \in \mathcal{U}\}\) is an open cover of \((X, [\mathcal{T}])\) and it has a locally finite open refinement \(\mathcal{V} \subset [\mathcal{T}]\) which is a cover of \(X\). For each \(V \in \mathcal{V}\) take a \(U_V \in \mathcal{U}\) such that \(V \subset (U_V)_{(1-\lambda)}\) and let \(\mathcal{W} = \{\chi_V \cap U_V: V \in \mathcal{V}\}\). Then \(\mathcal{W} \subset \mathcal{F}\) is both a refinement of \(\mathcal{U}\) and a \(Q\)-cover of \(1_x\). For each \(x \in X\) and each \(\lambda \in (0, 1]\), take a \(O \in N_{(1-\lambda)}(x)\) such that \(O\) intersects with only finite number of members of \(\mathcal{V}\). Then \(\chi_O \in Q_\mathcal{T}(x_{(1-\lambda)})\) intersects with only a finite number of members of \(\mathcal{W}\). So we know from the arbitrariness of \(x\) that \((X, \mathcal{T})\) is \(S^\star\)-paracompact.

(ii) \(\Leftrightarrow\) (i) Clearly (ii) \(\Rightarrow\) (iv); furthermore we have proved (iv) \(\Rightarrow\) (i), so we can know (ii) \(\Leftrightarrow\) (i) from (i) \(\Rightarrow\) (ii).

3.7. Corollary. Both \(S\)-paracompactness and \(S^\star\)-paracompactness are good extensions [4] of crisp paracompactness.

3.8. Example. The open interval \((0, 1)\) in 3.6 cannot be substituted by the half open interval \((0, 1]\). Let \(X = [0, 1]\) and

\[\mathcal{P} = \{1_x: x \in (0, 1]\} \cup \{\chi_{[0x]}: x \in X\},\]

then we can use \(\mathcal{P}\) as a subbase to generate a weakly induced fuzzy topology \(\mathcal{F}\) on \(X\).

Let \(\mathcal{U} \subset \mathcal{F}\) be a \(Q\)-cover of \(1_1 = X\). Generally, we can suppose that every member of \(\mathcal{U}\) is not empty and cannot consist of a \(Q\)-cover of \(X\) by itself.
Then we can see from the structure of $\mathcal{P}$ that for each $U \in \mathcal{U}$ there exists a $x_U \in X$ such that
\[(U \setminus [0, x_U)) > 0, \quad U[[X_U, 1]] = \{0\}.
\]
Since $\mathcal{U}$ is a $Q$-cover of $X$, so $\sup \{x_U: U \in \mathcal{U}\} = 1$. Take a countable family $\{U_i\}_{i \in N} \subset \mathcal{U}$ such that $\sup \{X_U: i \in N\} = 1$ and let $V_i = 1_{1/2} \cap U_i$; then $\mathcal{V} = \{V_i\}_{i \in N}$ is both an open refinement of $\mathcal{U}$ and a $Q$-cover of $X$. For each $x \in X$ and each $\lambda \in (0, 1]$, take $k \in N$ such that $1/k < \lambda$; then $1_{1 - 1/k} \in Q(x_i)$ and is not quasi-coincident with $V_i$ when $i \geq k$. Hence $\mathcal{V}$ is locally finite in $X$ and $(X, \mathcal{T})$ is 1-paracompact.

But it is easy to show that $\mathcal{C} = (\mathcal{x} \times > u \{ x \}): x \in X,$ so $\mathcal{X} = \{ [0, x): x \in X\}$ is an open cover of $(X, [\mathcal{T}])$. Clearly, $\mathcal{X}$ has not any locally finite open refinement which is a cover of $X$, so $(X, [\mathcal{T}])$ is not paracompact.

3.9. Remark. We can know from 3.6 and 3.8 that there exists a 1-paracompact fts which is not $x$-paracompact for every $x \in (0, 1)$.

3.10. Lemma. For every weakly induced fts $(X, \mathcal{T})$, if there exist an $x \in (0, 1)$ such that every $x$-open $Q$-cover of $X$ has a locally finite closed refinement which is a $x$-$Q$-cover of $X$, then $(X, \mathcal{T})$ is $x$-paracompact.

Proof. Let $\mathcal{U}$ be an open $Q$-cover of $1_x$. Take a locally finite refinement $\mathcal{A} = \{A_t\}_{t \in T}$ of $\mathcal{U}$ such that $\mathcal{A}$ is a $Q$-cover of $1_x$. We let $\beta$ denote $\min \{x, 1 - x\}$, then $\beta \in (0, 1)$. Take $U_x \in Q(x_{\beta})$ such that $U_x$ is quasi-coincident with only a finite number of members of $\mathcal{A}$ and let $\tilde{U}_x = (U_x)_{(1 - x)} \cap U_x$; then $\mathcal{U}_x = \{\tilde{U}_x: x \in X\}$ is an open $Q$-cover of $1_x$ and it has a locally finite closed refinement $\mathcal{F}$ which is $Q$-cover of $1_x$ also. For each $t \in T$ let $W_t = \bigcap \{F: F \in \mathcal{F}, F \supseteq (A_t)_{(1 - x)}\},$

then $W_t \in \mathcal{T}$ and, for each $F \in \mathcal{F}$, we have $W_t qF \Leftrightarrow (A_t)_{(1 - x)} qF.$

Take $U_t \in \mathcal{U}$ for each $t \in T$ such that $A_t \subset U_t$ and let $V_t = (W_t)_{(1 - x)} \cap U_t$. Then $\mathcal{V} = \{V_t\}_{t \in T}$ is an open refinement of $\mathcal{U}$. For each $x \in X$, take a $t \in T$
such that \( A_i(x) > 1 - \alpha \). Since \( W_i = (A_i)_{t \in T} \), we have \( W_i(x) = 1 > 1 - \beta \), so

\[
V_i(x) = U_i(x) \supseteq A_i(x) > 1 - \alpha;
\]

hence \( \mathcal{V} \) is a \( \mathbb{Q} \)-cover of \( 1_x \) also.

At last we prove that \( \mathcal{V} \) is locally finite. For each \( x \in X \) and each \( \lambda \in (0, 1] \), let \( \gamma = \min\{\lambda, \beta\} \), take \( U \in \mathcal{Q}(x_\gamma) \) such that \( U \) is quasi-coincident with only a finite number of members \( F_1, \ldots, F_n \) of \( \mathcal{F} \). For each \( i \in \{1, \ldots, n\} \), take a \( x' \in X \) such that \( F_i \subseteq \overline{U}_{x'} \); then from the way we get \( \mathcal{U}_i \) we know the set \( T_0 = \{ t \in T : A_i \cup \bigcup_{j=1}^n \overline{U}_{x'} \} \) is finite. It can be said with certainty that for each \( t \in T \) and each \( i \in \{1, \ldots, n\} \) we have

\[
(A_i)_{(1 - \alpha)} q F_i \Rightarrow t \in T_0.
\]

In fact, if \( (A_i)_{(1 - \alpha)} q F_i \), then there exists a \( y \in X \) such that \( A_i(y) > 1 - \alpha \), \( \overline{U}_{x'}(y) > 0 \), \( \overline{U}_{x'}(y) + A_i(y) > 1 - \beta + 1 - \alpha > \alpha + 1 - \alpha = 1 \), and \( A_i q \bigcup_{j=1}^n \overline{U}_{x'} \), \( t \in T_0 \). So take \( V = U_{(1 - \gamma)} \cap U \), then \( V \in \mathcal{Q}(x_\gamma) \). If \( V q V_t \), then there exists a \( y \in X \) such that \( V(y) > 1 - \gamma \geq 1 - \beta \), \( V_i(y) > 0 \). Take \( F \in \mathcal{F} \) such that \( F(y) > 1 - \alpha \leq \beta \), then \( V q F \), \( F \in \{F_1, \ldots, F_n\} \). On the other hand, since \( V_i(y) > 0 \), so \( W_i(y) > 1 - \beta \), \( W_i q F \). From relation \((*)\) we have \( (A_i)_{(1 - \alpha)} q F \) and from relation \((***)\) we know that \( t \in T_0 \). So \( V \) is quasi-coincident only with the members of the finite subfamily \( \{V_i : t \in T_0\} \) of \( \mathcal{V} \) and \( \mathcal{V} \) is locally finite.

3.11. DEFINITION. Let \( \mathcal{A} \) be a family of sets and \( B \) be a set in \( \text{ft}(X, \mathcal{F}) \). We say that \( \mathcal{A} \) is \( \sigma \)-locally finite (respectively: \( \sigma^* \)-locally finite) in \( B \) if \( \mathcal{A} \) can be represented as a countable union of subfamilies and each of these subfamily is locally finite (respectively: \( \sigma^* \)-locally finite) in \( B \); we often omit the word "in \( B \)" when \( B = X \).

3.12. LEMMA. Let \( \alpha \in (0, 1] \). If \( (X, \mathcal{F}) \) is a \((1 - \alpha)\)-crisp \( \text{ft}(X, \mathcal{F}) \), then each \( \alpha \)-locally finite \( \alpha \)-open \( \mathbb{Q} \)-cover of \( X \) has a locally finite refinement which is a \( \alpha \)-\( \mathbb{Q} \)-cover of \( X \) also.

Proof. Let \( \mathcal{V} = \bigcup_{i \in N} V_i \) be an open \( \mathbb{Q} \)-cover of \( 1_x \); here every \( V_i = \{V_i\}_{t \in T_i} \) is locally finite in \( X \) and \( T_i \cap T_j = \emptyset \) when \( i \neq j \). For each \( i \in N \) and each \( t \in T \) we let

\[
A_i = V_t \cap \left( \bigcup_{k < i, t' \in T_k} (V_{t'})_{(1 - \alpha)} \right),
\]

\[
T = \bigcup_{i \in N} T_i,
\]

\[
\mathcal{A} = \{A_i\}_{t \in T},
\]

then \( \mathcal{A} \) is a refinement of \( \mathcal{V} \).
For each $x \in X$, let $i_0 = \min \{i \in \mathbb{N} : \bigcup \mathcal{V}_x \in \mathcal{Q}(x) \}$ and take $t_0 \in T_{i_0}$ such that $V_{i_0}(x) \subseteq 1 - \alpha$ and we have $A_{i_0}(x) = V_{i_0}(x) > 1 - \alpha$ and $\mathcal{A}$ is a $Q$-cover of $1_x$. On the other hand, for each $i \in (0, 1]$ and $i = i_1, \ldots, i_0$, take $U_i \in \mathcal{Q}(x_i)$ such that $U_i$ is quasi-coincident with only a finite number of members of $\mathcal{V}_i$. Let

$$U = U_1 \cap \cdots \cap U_{i_0} \cap (V_{i_0}(1 - \alpha));$$

then $U \in \mathcal{Q}(x_i)$ and, for each $i \in \mathbb{N}$, $i > i_0$, and each $t \in T_i$, we have

$$A' = V_i \cup \bigcup_{k < i \in T_k} \bigcup_{i \in T_k} (V_r)(1 - \alpha) \supseteq \mathcal{V}_{i_0}(1 - \alpha) \supseteq U.$$

$U$ and $A'$ are not quasi-coincident. Since for each $i \in \{1, \ldots, i_0\}$, $U$ is quasi-coincident with only a finite number of members of $\{A_i \in T_i\}$, so $U$ is quasi-coincident with only a finite number of members of $\mathcal{A}$ and $\mathcal{A}$ is locally finite.

3.13. EXAMPLE. $(1 - \alpha)$-crispness is the condition which cannot be eliminated from 3.12. Take $X = [0, + \infty)$, let $\mathcal{T}_0$ denote usual crisp topology on $X$. For each $\beta \in [0, 1]$ let

$$U_\beta = (\beta \chi_{\{0\}}) \cup \left(\frac{1}{2} \chi_{\{1/n : n \in \mathbb{N}\}} \cup \chi_{X \setminus \{0\} \cup \{1/n : n \in \mathbb{N}\}}\right),$$

$$\mathcal{P} = \{U_\beta, U'_\beta : \beta \in [0, 1]\} \cup \{\chi_U : U \in \mathcal{T}_0\};$$

then we can use $\mathcal{P}$ as a subbase to generate a fuzzy topology on $X$. Since $\mathcal{T}_0$ has a countable base, we can see easily that $\mathcal{T}$ has a countable subbase $\mathcal{P} \subset \mathcal{P}$ and hence $\mathcal{T}$ has a countable base. Hence every open $Q$-cover of $1_x$ has a $\sigma$-locally finite open refinement for every $x \in (0, 1]$. Furthermore, we know that $(X, \mathcal{T})$ is a $T_2$-fts from $\mathcal{T}_0 \subset \mathcal{T}$. But the open $Q$-cover

$$\{U_{2/3} \cup \{\chi_{(1/n - 1/2n(n + 1), 1/n + 1/2n(n + 1))} : n \in \mathbb{N}\} \}$$

of $1_{1/2}$ does not have any locally finite refinement which is a $Q$-cover of $1_{1/2}$.

3.14. THEOREM. For every regular weakly induced fts $(X, \mathcal{T})$ the following conditions are equivalent:

(i) $(X, \mathcal{T})$ is $S$-paracompact.

(ii) For every $x \in (0, 1]$, every $x$-open $Q$-cover of $X$ has a $\sigma$-locally finite open refinement which is an $x$-$Q$-cover of $X$ also.

(iii) There exists an $x \in (0, 1)$ such that every $x$-open $Q$-cover of $X$ has a $\sigma$-locally finite open refinement which is an $x$-$Q$-cover of $X$ also.

(iv) For every $x \in (0, 1]$, every $x$-open $Q$-cover of $X$ has a locally finite refinement which is an $x$-$Q$-cover of $X$ also.
(v) There exists an \( \alpha \in (0, 1) \) such that every \( \alpha \)-open \( Q \)-cover of \( X \) has a locally finite refinement which is an \( \alpha \)-\( Q \)-cover of \( X \) also.

(vi) For every \( \alpha \in (0, 1] \), every \( \alpha \)-open \( Q \)-cover of \( X \) has a locally finite closed refinement which is an \( \alpha \)-\( Q \)-cover of \( X \) also.

(vii) There exists an \( \alpha \in (0, 1) \) such that every \( \alpha \)-open \( Q \)-cover of \( X \) has a locally finite closed refinement which is an \( \alpha \)-\( Q \)-cover of \( X \) also.

Proof. (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) Obvious.

(iii) \( \Rightarrow \) (v) From 3.12.

(v) \( \Rightarrow \) (vii) From 2.17.

(vii) \( \Rightarrow \) (i) From 3.10 and 3.6.

(i) \( \Rightarrow \) (iv) Obvious.

(iv) \( \Rightarrow \) (vi) From 2.17.

(vi) \( \Rightarrow \) (i) Clearly, we have (vi) \( \Rightarrow \) (vii). On the other hand, we have proved (vii) \( \Rightarrow \) (i). So (vi) \( \Rightarrow \) (i).

3.15. Theorem. For every regular weakly induced fts \( (X, \mathcal{F}) \) the following conditions are equivalent:

(i) \( (X, \mathcal{F}) \) is \( S^* \)-paracompact.

(ii) For every \( \alpha \in (0, 1] \), every \( \alpha \)-open \( Q \)-cover of \( X \) has a \( \sigma^* \)-locally finite open refinement which is an \( \alpha \)-\( Q \)-cover of \( X \) also.

(iii) There exists an \( \alpha \in (0, 1) \) such that every \( \alpha \)-open \( Q \)-cover of \( X \) has a \( \sigma^* \)-locally finite open refinement which is an \( \alpha \)-\( Q \)-cover of \( X \) also.

(iv) For every \( \alpha \in (0, 1] \), every \( \alpha \)-open \( Q \)-cover of \( X \) has a *-locally finite refinement which is an \( \alpha \)-\( Q \)-cover of \( X \) also.

(v) There exists an \( \alpha \in (0, 1) \) such that every \( \alpha \)-open \( Q \)-cover of \( X \) has a *-locally finite refinement which is an \( \alpha \)-\( Q \)-cover of \( X \) also.

(vi) For every \( \alpha \in (0, 1] \), every \( \alpha \)-open \( Q \)-cover of \( X \) has a *-locally finite closed refinement which is an \( \alpha \)-\( Q \)-cover of \( X \) also.

(vii) There exists an \( \alpha \in (0, 1) \) such that every \( \alpha \)-open \( Q \)-cover of \( X \) has a *-locally finite closed refinement which is an \( \alpha \)-\( Q \)-cover of \( X \) also.

Proof. (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) Obvious.

(iii) \( \Rightarrow \) (i) Clearly we have: (iii) \( \Rightarrow \) 3.14(iii) \( \Rightarrow \) 3.14(i) \( \Rightarrow \) (i).

(iv) \( \Rightarrow \) (vi) From 2.22.

(vi) \( \Rightarrow \) (iv) Clearly we have: (vi) \( \Rightarrow \) 3.14(vi) \( \Rightarrow \) 3.14(i) \( \Rightarrow \) (i) \( \Rightarrow \) (iv).

The following lemma can be proved easily:

3.16. Lemma. Let \( \mathcal{P} \) be a subbase of an fts \( (X, \mathcal{F}) \). If for each point \( e \) in
(\(X, \mathcal{T}\)) and each \(U \in \mathcal{P} \cap Q(e)\) there exists a \(V \in Q(e)\) such that \(V \subset U\), then \((X, \mathcal{T})\) is regular.

3.17. Example. There exist an fts \((X, \mathcal{T})\) such that:

(i) \((X, \mathcal{T})\) is regular;

(ii) \((X, \mathcal{T})\) satisfies condition (vi) of 3.15;

(iii) \((X, \mathcal{T})\) is not \(x\)-paracompact for every \(x \in (0, 1]\).

Take \(X = \omega_1\). For each \(x \in (0, 1]\), each \(\beta \in (0, 1]\), and each \(x \in X\), let

\[ U_x = x\chi_{\{0\}}, \quad U_{\beta, x} = (\beta \chi_{\{0\}}) \cup \chi_{\{x\}}, \quad V_{\beta, x} = (\beta \chi_{\{0\}}) \cup \chi_{X \setminus \{0, x\}}, \]

\[ \mathcal{P}_1 = \{ U_x, U_{x}^* : x \in (0, 1) \}, \quad \mathcal{P}_2 = \{ U_{\beta, x}, V_{\beta, x} : \beta \in (0, 1], x \in X \}, \]

\[ \mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2, \]

then we use \(\mathcal{P}\) as a subbase to generate a fuzzy topology \(\mathcal{T}\) on \(X\).

Proof of (i). Take \(x \in X\), \(\lambda \in (0, 1]\), and \(U \in \mathcal{P} \cap Q(x_i)\) arbitrarily. If \(U \in \mathcal{P}_1\), then \(U = U\lambda\) holds naturally. If \(U \in \mathcal{P}_2\) and \(U = U_{\lambda, x}\), then \(x = 0\) or \(x = y\). If \(x = 0\), take \(\alpha \in (0, 1]\) such that \(1 - \lambda < \alpha < \beta\); then we have \(U_x \in Q(x_i)\) and \(U_x = U_x \subset U\). If \(x = y\), take \(\alpha \in (0, 1]\) such that \(0 < 1 - \alpha < \beta\); then we have \(V_{\lambda, x} = U_{1 - \alpha, x} \in Q(x_i)\) and \(V_{\lambda, x} = U_{1 - \alpha, x} \subset U\). If \(U \in \mathcal{P}_2\) and \(U = V_{\lambda, x}\), then \(x = 0\) or \(x \neq 0\). If \(x = 0\), take \(\alpha \in (0, 1]\) such that \(1 - \lambda < \alpha < \beta\); then we have \(U_x \in Q(x_i)\) and \(U_x = U_x \subset U\). If \(x \neq 0\), take \(\alpha \in (0, 1]\) such that \(0 < \alpha < \beta\); then we have \(U_{\alpha, x} \in Q(x_i)\) and \(U_{\alpha, x} = U_{\alpha, x} \subset U\). So we have proved that \((X, \mathcal{T})\) satisfies the condition in 3.16; hence \((X, \mathcal{T})\) is regular indeed.

Proof of (ii). Let \(\mathcal{U}\) be an open \(Q\)-vector of \(1_x\). Since \(V_{1, x} \in \mathcal{T}\) for each \(x \in X \setminus \{0\}\), so \(V_{1, x}\) is a closed set. Take \(U_0 \in \mathcal{U}\) such that \(U_0 \in Q((0, x_i))\) and \(\beta \in (0, 1]\) such that \(1 - \alpha < \beta < U_0(0)\); then \(U_\beta \in Q((0, x_i))\) is a closed set too. So \(\mathcal{T} = \{ U_\beta \} \cup \{ V_{1, x} : x \in X \setminus \{0\} \}\) is a closed refinement of \(\mathcal{U}\) which is a \(Q\)-cover of \(1_x\) also. Clearly, \(\mathcal{T}\) is \(\ast\)-locally finite.

Proof of (iii). \(\mathcal{U} = \{ U_{1, x} : x \in X \}\) is an open \(Q\)-cover of \(1_x\) for each \(x \in (0, 1]\). From the structure of \(\mathcal{P}\) we can know that \(U(0) > 0\) holds for every \(U \in \mathcal{T}\) indeed, so if \(\mathcal{V}\) is both an open refinement of \(\mathcal{U}\) and a \(Q\)-cover of \(1_x\), then there must exist a crisp set \(\{ x_\alpha : x \in X \setminus \{0\} \} \subset (0, 1]\) such that \(\{ U_{x, x_\alpha} : x \in X \setminus \{0\} \}\) \(\subset \mathcal{V}\). Since \(|X \setminus \{0\}| = \omega_1\), there exists a \(k \in N\) and an uncountable crisp subset \(C\) of \(X \setminus \{0\}\) such that \(\{ x_\alpha : x \in C \}\subset (1/k, 1]\). Then for each \(U \in Q((0), 1/k)\) and each \(x \in C\) we always have

\[ U(0) \cap U_{x, x_\alpha}(0) > 1 - \frac{1}{k} + \frac{1}{k} = 1, \]
which means that every $Q$-neighborhood of point $(0)_{1/k}$ is quasi-coincident with every member of the uncountable subfamily $\{U_{a,x}: x \in C\}$ of $V$, $V$ is not locally finite.

4. PARACOMPACTNESS AND LINDELÖF PROPERTIES

4.1. DEFINITION. Let $\alpha \in (0, 1]$. We say that a set $A$ in fts $(X, \mathcal{V})$ is $\alpha$-Lindelöf if every $\alpha$-open $Q$-cover of $A$ has a countable subfamily which is a $Q$-cover of $A$ also. We say that $A$ is $S$-Lindelöf if $A$ is $\alpha$-Lindelöf for every $\alpha \in (0, 1]$.

We say that $(X, \mathcal{V})$ is $\alpha$-Lindelöf (resp. $S$-Lindelöf) if set $X$ is $\alpha$-Lindelöf (resp. $S$-Lindelöf).

4.2. THEOREM. Ultra-fuzzy compact spaces, $N$-compact spaces, and strong fuzzy compact spaces are $S$-Lindelöf.

Being similar to 2.24, we have

4.3. THEOREM. If $A$ is an $\alpha$-Lindelöf set in an fts $(X, \mathcal{V})$, then for every closed set $B$ in $(X, \mathcal{V})$, set $B \cap A$ is $\alpha$-Lindelöf too.

4.4. COROLLARY. Every closed set in an $\alpha$-Lindelöf is $\alpha$-Lindelöf.

4.5. THEOREM. If $A$ is an $S$-Lindelöf set in an fts $(X, \mathcal{V})$, then for every family of sets $\mathcal{C}$ which is locally finite in $A$, $\mathcal{C}$ is countable when $C \cap A \neq \emptyset$ for every $C \in \mathcal{C}$.

Proof. For each $n \in N$ and each $x \in \text{supp}(A_{<1/n})$, take $U_{n,x} \in \mathcal{Q}(x_{1/n})$ such that $U_{n,x}$ is quasi-coincident with only a finite number of members of $\mathcal{C}$. From the $S$-Lindelöf property of $A$, for each $n \in N$, there exists a countable subfamily $\mathcal{U}_n$ of $\{U_{n,x}: x \in \text{supp}(A_{<1/n})\}$ which is a $Q$-cover of $A_{<1/n}$. So $\mathcal{U} = \bigcup_{n \in N} \mathcal{U}_n$ is countable; hence $\mathcal{U}$ is quasi-coincident with at most a countable number of members of $\mathcal{C}$. On the other hand, since $\text{supp}(A) \subset \bigcup \mathcal{U}$ and $C \cap A \neq \emptyset$ for every $C \in \mathcal{C}$, we know that $C \cap (\bigcup \mathcal{U}) \neq \emptyset$ for every $C \in \mathcal{C}$, so $\mathcal{C}$ is countable.

4.6. EXAMPLE. A locally finite family in a $1$-Lindelöf fts which consists of nonempty sets need not be countable. Let $X$ be the set of all the real numbers. For each $x \in X$ and each $\epsilon > 0$, let $U_{x,\epsilon} = \chi_{(x)} \cup (\frac{1}{2} \chi_{(x-\epsilon,x+\epsilon)})$ and use $\{U_{x,\epsilon}: x \in X, \epsilon > 0\}$ as a subbase to generate a fuzzy topology $\mathcal{T}$ on $X$. Then it is obvious that $(X, \mathcal{T})$ is $1$-Lindelöf. Take $\mathcal{C} = \{\frac{1}{2} \chi_{\{x\}}: x \in X\}$; then $\mathcal{C}$ is locally finite but is uncountable.
4.7. **Theorem.** If a weakly induced fts \((X, \mathcal{F})\) is regular, then so is its original topological space \((X, [\mathcal{F}])\).

**Proof.** For each \(x \in X\), let \(N_{[\mathcal{F}]}(x)\) denote the neighborhood system of \(x\) in \((X, [\mathcal{F}])\); then \(x, U \in Q_{\mathcal{F}}(x)\) for every \(U \in N_{[\mathcal{F}]}(x)\). Take \(V \in Q_{\mathcal{F}}(x)\) such that \(V \subseteq U\), and take \(\alpha\) such that \(0 < \alpha < V(x)\), then \(V(\alpha) \subseteq (V)_{(\alpha)} \subseteq U\). Since \((X, \mathcal{F})\) is weakly induced, from 3.5 it follows that there exists an \(x \in supp(V(\alpha)) \subseteq [\mathcal{F}]\) and \(sup((V)_{(\alpha)}) \subseteq U\) is a closed set in \((X, [\mathcal{F}])\). So if we let \(W = supp(V(\alpha))\), then the following relation holds in \((X, [\mathcal{F}])\):

\[ x \in W \subseteq W \subseteq U. \]

4.8. **Example.** A weakly induced fts \((X, \mathcal{F})\) of a crisp \(T_\alpha\)-topological space \((X, Y_0)\) need not be regular. Let \((X, Y_0)\) be the usual real number space and \(\mathcal{B} = \{x, U, \frac{3}{2}X : U \in Y_0\}\); then we can use \(\mathcal{B}\) as a base to generate a fuzzy topology \(\mathcal{F}\) on \(X\), and we can see that \((X, \mathcal{F})\) is a weak inducement of \((X, Y_0)\). But for each \(x, X\), and the \(Q\)-neighborhood \(\frac{3}{2}x, 3\) of \(x, x/2\) there does not exist any \(V \in Q(x)\) such that \(V(x) \subseteq \frac{3}{2}x\).

4.9. **Theorem.** If a regular \(S\)-Lindelöf fts \((X, \mathcal{F})\) is \((1 - \alpha)\)-crisp, then \((X, \mathcal{F})\) is \(\alpha\)-paracompact.

**Proof.** Let \(\mathcal{U}\) be an open \(Q\)-cover of \(1_\alpha\). From the \(S\)-Lindelöf property we know that \(\mathcal{U}\) has a countable subfamily \(\{U_i\}_{i \in N}\) which is a \(Q\)-cover of \(1_\alpha\). So there exists a mapping \(f: X \to N\) such that

\[ f(x) = \min\{i \in N: U_i(x) > 1 - \alpha\}. \]

For each \(n \in N\) and each \(x \in X\), from the \((1 - \alpha)\)-crispness and regularity, we can take \(V_{n, x} \in Q(x/ln)\) such that \(V_{n, x} \subseteq (U_{f(x)})_{(1 - \alpha)}\). Take a countable subfamily \(\{V_{n, x, j}: j \in N\}\) of \(\{V_{n, x}: x \in X\}\) such that it is a \(Q\)-cover of \(1_{1/n}\), let \(V_{n, j}\) denote \(V_{n, x, j}\), and for each \(i \in N\) let

\[ W_i = U_i \cap \left( \bigcup \left\{ V_{n, j}: n < i, j < i, f(x^{n,j}) < i \right\} \right); \]

then \(W = \{W_i\}_{i \in N}\) is an open refinement of \(\mathcal{U}\). For each \(x \in X\) we have \(U_{f(x)}(x) > 1 - \alpha\), from the definition we have \(V_{n, j} \subseteq (U_{f(x)})_{(1 - \alpha)}\), and when \(f(x^{n,j}) < f(x)\) we have

\[ V_{n, j}(x) \subseteq (U_{f(x)})_{(1 - \alpha)}(x) = 1, \]

so

\[ W_{f(x)}(x) = U_{f(x)}(x) > 1 - \alpha, \]

where \(W\) is a \(Q\)-cover of \(1_\alpha\). On the other hand, for each \(x \in X\) and each \(\lambda \in (0, 1]\), take \(n \in N\) such that \(1/n \leq \lambda\); then there exists a \(j \in N\) such that
\[ V_{n,j}(x) > 1 - 1/n \geq 1 - \lambda, \text{ so } V_{n,j} \in \mathcal{Q}(x). \] Hence when \( i = \max \{ n, j, f(x^{n,j}) \} \) we have

\[ V_{n,j} \subset V_{n,j} \subset \bigcup \{ \overline{V}_{m,k} : m < i, k < i, f(x^{m,k}) < i \} \subset W_i, \]

where \( V_{n,j} \) and \( W_i \) are not quasi-coincident and \( \mathcal{W} \) is locally finite. \( \qed \)

4.10. **Corollary.** Every weakly induced regular S-Lindelöf fits is S-paracompact.

4.11. **Example.** There exists a regular \( T_1 \)-fits \((X, \mathcal{T})\) which has a countable base (so it is S-Lindelöf), but it is not 1-paracompact.

We take \( X = [0, +\infty) \), let \( \mathcal{F}_0 \) denote the usual crisp topology on \( X \), and let

\[ U_0 = \frac{1}{2} \chi_{\{0\}}, \quad U_1 = \left( \frac{1}{2} \chi_{\{0\}} \right) \cup \chi_{X \setminus \{0\} \cup \{1/n : n \in \mathbb{N}\}}, \]

\[ \mathcal{P} = \{ U_0, U_0', U_1 \} \cup \{ \chi_U : U \in \mathcal{F}_0 \}; \]

then we can use \( \mathcal{P} \) as a subbase to generate a fuzzy topology \( \mathcal{F} \) on \( X \). Thereupon we have

(i) \((X, \mathcal{F})\) is \( T_2 \)-fits. It is obvious.

(ii) \((X, \mathcal{F})\) is regular. In fact, for each fuzzy point \( e \), from the way we take \( \mathcal{P} \) we can know that for each \( U \in \mathcal{P} \cap \mathcal{Q}(e) \), there exists a \( V \in \mathcal{Q}(e) \) such that \( \overline{V} \subset U \). Hence we know from 3.16 that \((X, \mathcal{F})\) is regular.

(iii) \((X, \mathcal{F})\) has a countable base. Since \( \mathcal{F}_0 \) has a countable base, we can see that \( \mathcal{F} \) has a countable subbase \( \mathcal{P}_1 \subset \mathcal{P} \) and so \( \mathcal{F} \) has a countable base.

(iv) The open \( \mathcal{Q}\)-cover \( \mathcal{U} = \{ U_1 \} \cup \{ \chi_{(1/n - 1/2(n+1), 1/n + 1/2(n+1))} : n \in \mathbb{N} \} \) of \( X \) does not have any open refinement \( \mathcal{V} \) which both is a \( \mathcal{Q}\)-cover of \( X \) and locally finite on point \( (0)_{1/2} \). In fact, for each \( n \in \mathbb{N} \), let \( V_n \in \mathcal{V} \) such that \( V_n(1/n) > 0 \), then from that \( \mathcal{V} \) is a refinement of \( \mathcal{U} \) we can know \( V_n \subset \chi_{(1/n - 1/2(n+1), 1/n + 1/2(n+1))} \), so \( V_m \cap V_n = \emptyset \) when \( m \neq n \), so the correspondence \( 1/n \mapsto V_n \) is one to one. On the other hand, we have \( U(0) > \frac{1}{2} \) for each \( U \in \mathcal{Q}((0)_{1/2}) \), so from the structure of \( \mathcal{P} \) we know that there exists a \( U_0 \in \mathcal{F}_0 \) such that \( U \supset \chi_{U_0} \). Hence \( U \) is quasi-coincident with an infinite number of \( V_n \).

4.12. **Example.** 0-crisp regular 1-Lindelöf fits need not be 1-paracompact. Let \( X = [0, +\infty) \) and

\[ \mathcal{P}_1 = \{ U \in I_X : \exists x \in X, \{ y \in [0, x] : U(y) \neq \frac{1}{2} \} \in \omega, \]

\[ U([x, +\infty)) \in \{ \{0\}, \{1\} \} \}, \]

\[ \mathcal{P}_2 = \{ U(0) : U \in \mathcal{P}_1 \}, \]

\[ \mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2; \]

then we can use \( \mathcal{P} \) as a subbase to generate a 0-crisp fuzzy topology \( \mathcal{F} \).
Take a fuzzy point $x_1$ from $(X, \mathcal{T})$ arbitrarily; for each $U \in \mathcal{P}_2 \cap Q(x_1)$, let $W(\frac{1}{2}X_{(0,x_1)}) \cup X_{(x_1)}$ and $V = U \cap W$. Then $V \in Q(x_1) \cap \mathcal{P}_1$, $V$ is an open-closed set, and we have $V = V \subset U$; and for each $U \in \mathcal{P}_1 \cap Q(x_1)$, $U$ is closed, and certainly we have $\bar{U} \subset U$. So from 3.16 we know that $(X, \mathcal{T})$ is regular.

Now we will prove that $(X, \mathcal{T})$ is 1-Lindelöf. For each $U \in \mathcal{T}$, let

$$X_U = \inf \{ x \in X: U[(x, +\infty)] = \{0\} \},$$

$$F_U = \{ x \in [0, x_U): U(x) = 0 \};$$

here the infimum of the empty set is defined as $+\infty$. Let

$$\mathcal{B} = \left\{ \bigcap \mathcal{A}: \mathcal{A} \subset \mathcal{P}, |\mathcal{A}| < \omega \right\},$$

then $\mathcal{B}$ is the base of $\mathcal{T}$. For each $U \in \mathcal{T}$, suppose that $U = \bigcup \mathcal{A}$, $\mathcal{A} \subset \mathcal{B}$, then from the properties of real number space we know that there exists a countable subfamily $\mathcal{A}_1$ of $\mathcal{A}$ such that

$$\sup \{ x_U: V \in \mathcal{A}_1 \} = x_U. \quad (*)$$

Since $|F_U| < \omega$ for each $V \in \mathcal{B}$, it also does for each $V \in \mathcal{A}_1$. From the equality $(*)$ and $|\mathcal{A}_1| \leq \omega$ we have

$$|F_U| \leq |F_{U \cup \mathcal{A}_1}| \leq \left| \bigcup \{ F_U: V \in \mathcal{A}_1 \} \right| \leq \omega. \quad (**)$$

Let $\mathcal{U}$ be an open $Q$-cover of $X$; then $\sup \{ x_U: U \in \mathcal{U} \} = +\infty$. Take a countable subfamily $\{ U_i \}_{i \in N}$ of $\mathcal{U}$ such that $\sup \{ x_U: i \in N \} = +\infty$; then from the inequality $(**)$ we have $|\bigcup \{ F_U: i \in N \}| \leq \omega$ and from the definition of $F_U$ we can easily find that $\mathcal{U}$ has a countable subfamily which is a $Q$-cover of $X$ also, so that $(X, \mathcal{T})$ is 1-Lindelöf.

At last, we will prove that $(X, \mathcal{T})$ is not 1-paracompact. Let $\mathcal{V} = \{ \frac{1}{2}X_{(0,x)}: x \in X \}$, then $\mathcal{V}$ is an open $Q$-cover of $X$. Suppose that an open refinement $\mathcal{W}$ of $\mathcal{V}$ is a $Q$-cover of $X$, then from the 1-Lindelöf property we have proved we can suppose that $\mathcal{W}$ is countable. For each $U \in \mathcal{T}$, let $C_U = \{ x \in [0, x_U): U(x) < \frac{1}{2} \}$; then we can see that $|C_U| \leq \omega$ for every $U \in \mathcal{B}$. Similar to the proof of $(**)$ we can prove that $|C_U| \leq \omega$ for every $U \in \mathcal{T}$. Hence $C = \bigcup \{ C_W: W \in \mathcal{W} \}$ is countable. Take $x^0 \in X \setminus C$; then for each $W \in \mathcal{W}$, when $x_W > x^0$, we have $W(x^0) = \frac{1}{2}$. So from $\sup \{ x_W: W \in \mathcal{W} \} = \infty$ and the fact that $\mathcal{W}$ is both an open refinement of $\mathcal{V}$ and a $Q$-cover of $X$, we know that there are an infinite number of members of $\mathcal{W}$ which take the value $\frac{1}{2}$ at point $x$ and each $Q$-neighborhood of $(x^0)_{\frac{1}{2}}$ is quasi-coincident with an infinite number of members of $\mathcal{W}$.
4.13. Theorem. For every weakly induced fts \((X, \mathcal{F})\) the following conditions are equivalent:

(i) \((X, \mathcal{F})\) is S-Lindelöf.

(ii) There exists an \(\alpha \in (0, 1]\) such that \((X, \mathcal{F})\) is \(\alpha\)-Lindelöf.

(iii) \((X, [\mathcal{F}])\) is Lindelöf.

Proof. (i) \(\Rightarrow\) (ii) Obvious.

(ii) \(\Rightarrow\) (iii) Let \(\mathcal{U} \subseteq [\mathcal{F}]\) be a cover of \(X\), then \(\{\chi_U : U \in \mathcal{U}\} \subseteq \mathcal{F}\) is a \(Q\)-cover of \(1_x\); it has a countable subfamily \(\{\chi_{U_i} : i \in N\}\) which is a \(Q\)-cover of \(1_x\) also. Then \(\{U_i\}_{i \in N}\) is a countable subcover of \(\mathcal{U}\).

(iii) \(\Rightarrow\) (i) For each \(\alpha \in (0, 1]\), let \(\mathcal{U} \subseteq \mathcal{F}\) be a \(Q\)-cover of \(1_x\); then \(\{U_{(1-\alpha)} : U \in \mathcal{U}\} \subseteq [\mathcal{F}]\) is a cover of \(X\) and it has a countable subcover \(\{(U_i)_{(1-\alpha)} : i \in N\}\), here \(\{U_i\}_{i \in N} \subseteq \mathcal{U}\). Hence \(\{U_i\}_{i \in N}\) is a \(Q\)-cover of \(1_x\) too.

From 4.13, 4.10, and 3.6 we have

4.14. Theorem. If there exists an \(\alpha \in (0, 1]\) such that a weakly induced regular fts \((X, \mathcal{F})\) is \(\alpha\)-Lindelöf, then \((X, \mathcal{F})\) is \(S^*\)-paracompact.

5. Paracompact in Metric Spaces

5.1. Definition [12] Let \(L\) be a completely distributive lattice with order reversing involution \(a \mapsto a'\):

(a) A mapping \(p : L^X \times L^X \rightarrow [0, \infty]\) is called a fuzzy \(p \cdot q \cdot\) metric on \(X\) if \(p\) satisfies following conditions:

- \([\text{M1}]\) \[ p(\phi, \lambda) = \infty, \quad \lambda \in L^X \setminus \{\phi\}, \]

- \[ p(\lambda, \lambda) = 0, \quad \lambda \in L^X, \]

- \[ p(\lambda, \phi) = 0, \quad \lambda \in L^X; \]

- \([\text{M2}]\) \[ p(\lambda, \mu) \leq p(\lambda, \beta) + p(\beta, \mu), \quad \lambda, \beta, \mu \in L^X; \]

- \([\text{M3}]\) (i) if \(\lambda \subseteq \mu\), then \(p(\lambda, \beta) \geq p(\mu, \beta)\) for every \(\beta \in L^X\),

(ii) \[ p(\beta, \bigcup_{\alpha} \lambda_\alpha) = \bigvee_{\alpha} p(\beta, \lambda_\alpha); \]

- \([\text{M4}]\) let \(r\) be a positive real number, \(\mu \in L^X\), \(A\) is a set of indexes, and \(\lambda_\alpha \in L^X\) for every \(\alpha \in A\). If

- \(\alpha \in A, \quad \beta \in L^X, \quad p(\lambda_\alpha, \beta) < r \Rightarrow \beta \subseteq \mu,\)

then

\[ p\left(\bigcup_{\alpha} \lambda_\alpha, \gamma\right) < r \Rightarrow \gamma \subseteq \mu. \]
(b) Let \( p \) be a fuzzy \( p \cdot q \)-metric on \( X \). For each \( r > 0 \) and each \( \lambda \in L^X \), let

\[
D_r(\lambda) = \bigcup \{ \mu \in L^X : p(\lambda, \mu) < r \}.
\]

\( \{ D_r : r > 0 \} \) is called the associated neighborhood maps of \( p \) and \( (L^X, p, D_r) \) is called a fuzzy \( p \cdot q \)-metric space.

We often omit the word "fuzzy" in the discussion about metric.

5.2. THEOREM [13]. Let \( (L^X, p, D_r) \) be a \( p \cdot q \)-metric space, then the associated neighborhood map \( \{ D_r : r > 0 \} \) satisfies:

\[
\begin{align*}
\langle A1 \rangle & \quad D_r(\phi) = \varnothing; \\
\langle A2 \rangle & \quad \lambda \subset D_r(\lambda); \\
\langle A3 \rangle & \quad D_r \left( \bigcup_{\alpha} \lambda_{\alpha} \right) = \bigcup_{\alpha} D_r(\lambda_{\alpha}); \\
\langle A4 \rangle & \quad D_r \circ D_s \leq D_{r+s}; \\
\langle A5 \rangle & \quad D_r = \bigvee_{s < r} D_s.
\end{align*}
\]

Proof. See also [13 or 12].

5.3. THEOREM [13]. Let \( \{ D_r : D_r : L^X \to L^X, \ r > 0 \} \) be a family of mappings which satisfies conditions \( \langle A1 \rangle - \langle A5 \rangle \). For each \( \lambda, \mu \in L^X \), let

\[
p(\lambda, \mu) = \bigwedge \{ r : \mu \subset D_r(\lambda) \},
\]

then \( p \) is a \( p \cdot q \)-metric on \( X \) and \( \{ D_r : r > 0 \} \) is exactly the associated neighborhood map of \( p \).

Proof. See also [13 or 12].

5.4. Remark. From 5.2 and 5.3 we know that a \( p \cdot q \)-metric and its associated neighborhood maps are decided by each other, so we can consider only one of them. For example, a \( p \cdot q \)-metric space can be considered as \( (L^X, D_r) \), where the \( D_r \)'s satisfy \( \langle A1 \rangle - \langle A5 \rangle \).

5.5. DEFINITION [12]. A \( p \cdot q \)-metric space \( (L^X, D_r) \) is called a pseudometric space if \( \{ D_r : r > 0 \} \) better satisfies the condition:

\[
\langle A6 \rangle \quad D_r = D_r^{-1},
\]
where $D_r^{-1}$ is defined as

$$D_r^{-1}(\lambda) = \bigcap \{ \mu: D_r(\mu') < \lambda' \}.$$

5.6. **Definition** [12]. For a $p \cdot q$-metric space $(L^X, D_r)$, a topology on $X$ which is generated from the base $\{D_r(\lambda): \lambda \in L^X, r > 0\}$ is called the $p \cdot q$-metric topology of $(L^X, D_r)$. Fts $(X, \mathcal{F})$ is called $p \cdot q$-metrizable (resp. pseudometrizable) if there exist a $p \cdot q$-metric (resp. pseudometric) on $X$ such that the topology it generates is exactly $\mathcal{F}$.

5.7. **Remark.** As [12] has proved, in a $p \cdot q$-metric space $(L^X, D_r)$, the family $\{D_r(\lambda): \lambda \in L^X, r > 0\}$ is a base for a topology indeed.

5.8. **Lemma** [13]. Let $(L^X, D_r)$ be a $p \cdot q$-metric space and $x_\lambda$ is a fuzzy point, then

$$\mathcal{B}(x_\lambda) = \{ D_r(x_\lambda): r > 0, \alpha \in (1 - \lambda, 1] \}$$

is a base of the $Q$-neighborhood system of $x_\lambda$.

**Proof.** From $\langle A2 \rangle$ and 5.6 we have $\mathcal{B}(x_\lambda) \subseteq Q(x_\lambda)$. For each $U \in Q(x_\lambda)$, from [12] we know $U = \bigcup \{ A \in I^X: \exists r > 0, D_r(A) \subseteq U \}$, so we have an $A \in I^X$ and an $r > 0$ such that $A(x) > 1 - \lambda$ and $D_r(A) \subseteq U$. Take $\alpha = A(x)$, then from $\langle A3 \rangle$ we have

$$D_r(x_\lambda) \subseteq D_r(A) \subseteq U.$$

5.9. **Lemma.** In every pseudometric space $(L^X, D_r)$ we have

$$A_q D_r(B) \Rightarrow D_{r/2}(A) \subseteq D_{q'r/2}(B).$$

**Proof.** Since

$$D_{r/2}(D_{r/2}(B)) = D_{r/2} \circ D_{r/2}(B) \subseteq D_r(B),$$

so from $A \subseteq D_r(B)'$ we have

$$D_{r/2}(A) \subseteq D_{r/2}(D_r(B)) = D_{r/2}^{-1}(D_r(B)) = \bigcap \{ C': D_{r/2}(C) \subseteq D_r(B) \} \subseteq D_r(B).$$

5.10. **Definition.** Let $\mathcal{A}$ be a family of sets and $B$ be a set in fts $(X, \mathcal{F})$. We say that $\mathcal{A}$ is discrete (resp. *-discrete) in $B$ if for each point $e$ in $B$, there exists a $U \in Q(e)$ such that $U$ is quasi-coincident (resp. intersects) with at most one member of $\mathcal{A}$; we say that $\mathcal{A}$ is $\sigma$-discrete (resp. $\sigma^*$-discrete) in $B$ if $\mathcal{A}$ can be represented as a countable union of subfamilies and each of these subfamilies is discrete (resp. *-discrete) in $B$; we often omit the word “in $B$” when $B = X$. 


5.11. THEOREM. If a weakly induced fts \((X, \mathcal{F})\) is pseudometrizable, then for each \(\alpha \in (0, 1]\), each \(\alpha\)-open \(Q\)-cover of \(X\) has an open refinement which is both \(\sigma\)-discrete and locally finite in \(X\), and it is an \(\alpha\)-Q-cover of \(X\) also.

Proof. Let \(\{U_i\}_{i \in T}\) be an open \(Q\)-cover of \(\alpha\). Taking a well-ordering relation \(<\) on the index set \(T\), we can take a mapping

\[ f : X \to T, \quad f(x) = \min \{ t \in T : x \notin qU_t \}. \]

Let \(\mathcal{A} = \{x : x \in X, \, \lambda \in (0, 1]\}\), then for each \(x \in \mathcal{A}\) we have \((U_{f(x)})(1-\alpha) \in Q(x_{1-\lambda})\), so from 5.8 we know that there exist an \(r > 0\) and a \(\mu \in (\lambda, 1]\) such that \(D_r(x_{\lambda}) \subset D_r(x_{\mu}) \subset (U_{f(x)})(1-\alpha)\); we can take a mapping

\[ g : \mathcal{A} \to N, \quad g(x) = \min \{ n \in N : D_{2/2^n}(x_{\lambda}) \subset (U_{f(x)})(1-\alpha) \}. \]

Let \(\beta = \frac{1}{2} \sqrt{(1-\alpha)}\), then \(\beta \in \left[\frac{1}{2}, 1\right)\). We also denote \(D(A, r) D(A)\) for convenience. For each \(i \in N\) and each \(t \in T\), let

\[ W_{i,t} = \bigcup \{ D(x_{\lambda}, 1/2^t)_{(\beta)} : x_{\lambda} \in \mathcal{A}, f(x) = t, g(x_{\lambda}) = i \}; \]

then for the same reason used to prove the existence of \(g\), we can take the mapping

\[ h_{i,t} : \{ x_{\lambda} \in \mathcal{A} : x_{\lambda} \in W_{i,t} \} \to N, \quad h_{i,t}(x_{\lambda}) = \min \{ n \in N : D(x_{\lambda}, 1/2^n) \subset W_{i,t} \}. \]

For \(i, j \in N\), we inductively define:

\[ C_{i,j} = \{ x_{\lambda} \in \mathcal{A} : x_{\lambda} \in W_{i,t}, f(x) = t, g(x_{\lambda}) = i, h_{i,t}(x_{\lambda}) = j, \forall s \in T, \forall k < i \cup j, \forall l < i \cup j, x_{\lambda} \not\in qV_{k,s,l} \}, \]

\[ V_{i,t} = U_t \cap \bigcup \{ D(x_{\lambda}, 1/2^t+j)_{(\beta)} : x_{\lambda} \in C_{i,j} \}, \quad i, j \in N, \quad t \in T, \]

\[ \bigcup_{i,j} \mathcal{V}_{i,j} = \bigcup_{i,j \in N} \mathcal{V}_{i,j}. \]

Then we have following results:

(1) \(\mathcal{V}\) is an open refinement of \(\mathcal{U}\).

(2) \(\mathcal{V}\) is an \(\alpha\)-open \(Q\)-coer of \(X\). In fact, for every \(x \in X\), let \(t = f(x)\), since \(x_{\lambda} \notin qU_t\), so \(x \in (U_t)(1-\alpha)\), \((U_t)(1-\alpha) \in Q(x_{1-\lambda})\), and there exists an \(i \in N\) and a \(\lambda \in (\beta, 1]\) such that \(D(x_{\lambda}, 2/2^i) \subset (U_t)(1-\alpha)\). Then let \(i = g(x_{\lambda})\) and we have \(D(X_{\lambda}, 2/2^i) \subset (U_t)(1-\alpha)\), \(x_{\lambda} \in D(x_{\lambda}, 2/2^i(\beta) \subset W_{i,t}\). Let \(j = h_{i,t}(x_{\lambda})\); now
if there exists \( s \in T \), \( k < i \lor j \), \( 1 < i \lor j \) such that \( x_s q V_{k,s,l} \), then we have nothing to prove. If not, then \( x_s \in C_{i,j} \), \( D(x_s, 1/2(i+j))(x) \geq \beta \geq 1 - \alpha \), \( V_{l,i,j}(x) \geq U_i(x) \land D(x_j, 1/2^i+j))(x) > 1 - \alpha \), \( x_s q V_{l,i,j} \).

(3) For every \( i \in N \) and every pair of \( t, t' \in T \), and \( t \neq t' \), we have \( W_{i,j} \cap W_{l,i} = \emptyset \). Suppose this is not so, and there is no harm in supposing \( i < t' \), then there exist \( x_j \in \mathcal{A} \), \( y_j \in \mathcal{A} \), and \( z \in X \) such that \( g(x_j) = g(y_j) = i \), \( f(x) = t \), \( f(y) = t' \), \( z_1 \in D(x_j, 1/2^i)(\beta) \land D(y_j, 1/2^i)(\beta) \). Since \( \beta \geq 1/2 \),

\[
D(X_1, 1/2^i)(z) + D(y_j, 1/2^i)(z) > \beta + \beta \geq \frac{1}{2} + \frac{1}{2} = 1.
\]

\( D(x_j, 1/2^i) \) and \( D(y_j, 1/2^i) \) are quasi-coincident at the point \( z \). But from \( f(y) > f(x) = t \) we have \((U_i)(1-\alpha)(y) = 0\), from \( D(x_j, 2/2') \in (U_i)(1-\alpha) \), and from 5.9 we know that \( D(x_j, 1/2^i) \) is not quasi-coincident with \( D(y_j, 1/2^i) \); this is a contradiction.

(4) For each \( x_j \in \mathcal{A} \) such that \( D(x_j, 1/2^m) \in (V_{k,s,l})(1-\alpha) \) \((k, l, m = 1, 2, ..., )\), we have that if \( i, j \in N \), \( i \lor j \leq k + l + m \), \( t \in T \), then \( D(x_j, 1/2^k+i+m) \in (\mathcal{A}_{i,j}) \). In fact, for each \( y_j \in C_{i,j} \), we have \( y_j q V_{k,s,l} \) and \( y \in (V_{k,s,l})(1-\alpha) \). Then from \( D(x_j, 1/2^m) \in (\mathcal{A}_{i,j}) \) we know \( y_j q D(x_j, 1/2^m) \) and from 5.9 we know \( D(x_j, 1/2^k+i+m) \in (\mathcal{A}_{i,j}) \). So from the definition of \( V_{i,j} \) we know \( D(x_j, 1/2^k+i+m) \in (\mathcal{A}_{i,j}) \).

(5) \( \mathcal{V} \) is \( \sigma \)-discrete. Let \( i, j \in N \), \( x \in X \), and \( \lambda \in (0, 1] \), take a \( \mu \in (1 - \lambda, 1) \); then from (3) we know that \( x_\mu \) intersects with at most one member of the family of sets

\[
\left\{ \bigcup \{ D(z_\delta, 1/2^i): z_\delta \in C_{i,j}\}: t \in T \right\}.
\]

So from 5.9 we know that \( D(x_\mu, 1/2^{i+j}) \in (\mathcal{Q}(x_\lambda)) \) is quasi-coincident with at most one member of the family of sets

\[
\left\{ \bigcup \{ D(z_\delta, 1/2^{i+j}): z_\delta \in C_{i,j}\}: t \in T \right\}.
\]

Hence from the definition of \( V_{i,j} \) we know that the \( \mathcal{Q} \)-neighborhood \( D(x_\mu, 1/2^{i+j}) \) of \( x_\lambda \) is quasi-coincident with at most one member of \( \mathcal{V}_{i,j} \).

(6) \( \mathcal{V} \) is locally finite. Let \( x \in X \), \( \lambda \in (0, 1] \), from (2) we know that there exist \( k, l \in N \), \( s \in T \) such that \( V_{k,s,i}(x) > -\alpha \), \( (V_{k,s,i})(1-\alpha) \in (\mathcal{Q}(x_\lambda)) \). From 5.8 we know that there exists a \( \mu \in (1 - \alpha, 1) \) and an \( m \in N \) such that \( D(x_\mu, 1/2^m) \in (V_{k,s,i})(1-\alpha) \), so from (4) and (5) we know that only at that time \( i \lor j \leq k + l + m \), \( D(x_\mu, 1/2^k+\alpha+m) \) has the possibility to be quasi-coincident with one and at most one member of \( \mathcal{V}_{i,j} \). Hence the \( \mathcal{Q} \)-neighborhood \( D(x_\mu, 1/2^{k+l+m}) \) of \( x_\lambda \) is quasi-coincident with at most

\[
|\{(i, j) \in N \times N: i \lor j \leq k + l + m\}| = (k + l + m - 1)^2
\]

members of \( \mathcal{V} \) and \( \mathcal{V} \) is locally finite.
From 5.11 and 3.6 we have

5.12. THEOREM. If weakly induced fifs $(X, \mathcal{F})$ is pseudometrizable, then $(X, \mathcal{F})$ is $S^*$-paracompact.

REFERENCES