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Paracompactness in Fuzzy Topological Spaces*

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In this paper, we introduce two kinds of fuzzy paracompactnesses which are defined on general fuzzy subsets. Each of them is a good extension of crisp paracompactness. They are all hereditary with respect to closed subsets and take N-compactness and some other compactnesses in fuzzy topological spaces as special cases of them. Furthermore, in a class of spaces (so-called "weakly induced spaces") which connects crisp spaces with fuzzy spaces, both a fuzzy regular Lindelöff property and a fuzzy pseudo-metric property imply these two kinds of paracompactnesses.

1. INTRODUCTION

Paracompactness describes the relation between a locally finite property and an entire property of spaces, occupies an important position in general topology, and is a problem in fuzzy topology which attracts the attention of all of us. But up until now, fuzzy paracompactness has not had a better or more ideal definition. There were some works about this problem [1, 2], but the definitions of them used the rather special "B set" or depended upon the concepts of covers; all have many limitations. In fact, neither takes one of the N-compactnesses [3] or some other compactness in fuzzy topological space as a special case and is not a good extension [4] of crisp paracompactness. One of the purposes of setting up fuzzy paracompactnesses in this paper is to overcome these limitations. Moreover, there exist close relations among the metric property, the Lindelöff property, and paracompactness, and trying to set up or discover these relations within the scope of fuzzy topology is also a purpose of this paper.

The complex nature of the neighborhood structure in fuzzy topological space makes the relation between the local property and the entire property of fuzzy topological space more difficult to handle. On the other hand, the essential difference between fuzzy and crisp topological spaces

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lies in the differences and connections among each horizontal level of space. It is also an important point for deep study of fuzzy topology. In view of these situations, after analyzing the above-mentioned purposes, from the difference between two kinds of neighborhood structures, this paper has set up two kinds of fuzzy paracompactnesses to describe the properties of every horizontal level of space. They have the advantages stated in the abstract of this paper. We have studied the basic properties of these two kinds of paracompactnesses in greater depth, e.g., the problem of equivalent description etc. and used full counterexamples to explain the relations among the conditions of the theorems and the differences between these conditions and crisp situations carefully. By using this paper as a foundation, we can study the problems of fuzzy paracompactness more fully.

We let q denote the quasi-coincident relation [5], $Q(A)$ denote the Q-neighborhood system of fuzzy set A [5], χ denote the characteristic function, and $|C|$ denote the cardinality of a crisp set C. The concepts of base and subbase for a fuzzy topological space and other concepts which have not been defined in this paper are taken from [5]. We let fts denote fuzzy topological space for convenience. For every real number α and every fuzzy set A on X, we let αA denote a function such that $(\alpha A)(x) = \alpha A(x)$ for every $x \in X$; we let $A > \alpha$ denote the relation $A(x) > \alpha$ for every $x \in X$. If it will not cause any confusion, we will call the fuzzy set the set and the fuzzy point, the point directly, and we will not often differentiate a crisp set from its characteristic function.

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2. Definitions and Basic Properties

2.1. Definition. Let $\mathscr{A}$, $\mathscr{B}$ be two families of sets in fts $(X, \mathscr{T})$. $\mathscr{A}$ is called a refinement of $\mathscr{B}$ if for any $A \in \mathscr{A}$, there exist a $B \in \mathscr{B}$ such that $A \subset B$.

2.2. Definition. Let $\mathscr{A}$ be a family of sets and B be a set in fts $(X, \mathscr{T})$. We say that $\mathscr{A}$ is locally finite (resp. $*$-locally finite) in B if for each point e in B, there exists a $U \in Q(e)$ such that U is quasi-coincident (resp. intersects) with at most a finite number of sets of $\mathscr{A}$; we often omit the word "in B" when $B = X$.

2.3. Definition. Let A be a set in fts $(X, \mathscr{T})$ and let $\alpha \in (0, 1]$, $\beta \in [0, 1)$; we define

$$A_{[\alpha]} = \chi_{\{x \in X:\, A(x) \geqslant \alpha\}},$$

$$A_{(\beta)} = \chi_{\{x \in X:\, A(x) > \beta\}},$$

$$A_{\langle\alpha\rangle} = \alpha A_{[\alpha]},$$

$$1_\alpha = \alpha\chi_X.$$

2.4. Definition. A family of sets $\mathscr{A}$ is called a Q-cover [6] of a set B if for each $x \in \operatorname{supp}(B)$, there exist an $A \in \mathscr{A}$ such that A and B are quasi-coincident at x. Let $\alpha \in (0, 1]$. $\mathscr{A}$ is called an α-Q-cover of B if $\mathscr{A}$ is a Q-cover of $B_{\langle\alpha\rangle}$.

2.5. Definition. Let $\alpha \in (0, 1]$, A be a set in fts $(X, \mathscr{T})$. We say that A is α-paracompact (resp. α^*-paracompact) if for each α-open Q-cover of A there exists an open refinement of it which is both locally finite (resp. $*$-locally finite) in A and an α-Q-cover of A. A is called S-paracompact (resp. S^*-paracompact) if for every $\alpha \in (0, 1]$, A is α-paracompact (resp. α^*-paracompact).

We say that $(X, \mathscr{T})$ is α-paracompact (resp. α^*-paracompact, S-paracompact, S^*-paracompact) if set X is α-paracompact (resp. α^*-paracompact, S-paracompact, S^*-paracompact).

2.6. *Remark.* It is obvious that

$$*\text{-locally finite} \Rightarrow \text{locally finite},$$

so we get the relations

$$\alpha^*\text{-paracompact} \Rightarrow \alpha\text{-paracompact},$$

$$S^*\text{-paracompact} \Rightarrow S\text{-paracompact};$$

but it is easy to find that the inverses of these relations are not true.

2.7. Theorem. *Ultra-fuzzy compact* [4] *spaces, N-compact spaces, and strong fuzzy compact* [4] *spaces are S^*-paracompact.*

Proof. It is obvious that strong fuzzy compact spaces are S^*-paracompact. On the other hand, from [3] we know that for each fts there exists the relation

$$\text{ultra-fuzzy compact} \Rightarrow N\text{-compact} \Rightarrow \text{strong fuzzy compact},$$

so the theorem is true. ∎

2.8. Theorem. *Q-compact* [6] *fts is 1^*-paracompact.*

2.9. EXAMPLE. There exists a fuzzy compact [7] fts which is not 1-paracompact. Let $X=[0,+\infty)$, for each $x\in X$, let

$$U_x(y)=\begin{cases} e^{-y}, & 0\leqslant y<x,\\ 0, & y\leqslant x,\end{cases}$$

$$U_\infty(y)=e^{-y}, \qquad y\in X,$$

$$\mathscr{T}=\{U_\infty, X\}\cup\{U_x: x\in X\},$$

then it is easy to know that $(X,\mathscr{T})$ is a fuzzy compact fts. Let $\mathscr{U}=\{U_x: x\in X\}$, $\mathscr{U}$ is an open Q-cover of X. Let $\mathscr{V}$ be both an open refinement of $\mathscr{U}$ and an open Q-cover of X, then from the structure of $\mathscr{T}$ we know $\mathscr{V}\subset\mathscr{U}$. Furthermore we have $\sup\{x: U_x\in\mathscr{V}\}=+\infty$, so $|\mathscr{V}|\geqslant\omega$. Hence $\mathscr{V}$ is not locally finite at the fuzzy point $(0)_1$, and $(X,\mathscr{T})$ is not 1-paracompact.

2.10. DEFINITION [5]. An fts $(X,\mathscr{T})$ is called a T_2-fts if for each pair of fuzzy points x_λ, y_μ, $x\neq y$, there exist $U\in Q(x_\lambda)$ and $V\in Q(y_\mu)$ such that $U\cap V=\varnothing$.

2.11. THEOREM. *If T_2-fts $(X,\mathscr{T})$ is fuzzy compact, then $(X,\mathscr{T})$ is S^*-paracompact.*

Proof. From [3] we know that every fuzzy compact T_2-fts is strong fuzzy compact, hence we know from 2.7 that the theorem is true. ∎

2.12. THEOREM. *If a family of sets $\{A_t\}_{t\in T}$ of fts $(X,\mathscr{T})$ is locally finite in a set A, then*

$$\overline{\bigcup_{t\in T} A_t}\cap A=\bigcup_{t\in T}(\bar{A}_t\cap A).$$

Proof. It need only to show that $\overline{\bigcup_{t\in T}A_t}\cap A\subset\bigcup_{t\in T}(\bar{A}_t\cap A)$. Let $B=\bigcup_{t\in T}A_t$ and $e\in\bar{B}\cap A$; then from [5] we know that e is an adherence point of B. Since $\{A_t\}_{t\in T}$ is locally finite in A, so there exists a $V\in Q(e)$ which is quasi-coincident with only a finite number of members $A_{t_1},\dots,A_{t_n}$ of $\{A_t\}_{t\in T}$. If $e\,\bar{\in}\,\bigcup_{i=1}^n\bar{A}_{t_i}$, then there exists a $V_i\in Q(e)$ which is not quasi-coincident with A_{t_i} $(i=1,\dots,n)$. Let $V_0=(\bigcap_{i=1}^n V_i)\cap V$, we have $V_0\in Q(e)$ and V_0 is not quasi-coincident with any A_t, so V_0 is not quasi-coincident with B. This is in contradiction with the fact that e is an adherence point of B, hence $e\in\bigcup_{i=1}^n(\bar{A}_{t_i}\cap A)\subset\bigcup_{t\in T}(\bar{A}_t\cap A)$. ∎

2.13. COROLLARY. *If a family of sets* $\{A_t\}_{t\in T}$ *of fts* $(X, \mathscr{T})$ *is locally finite in a set* A *and* $\overline{\bigcup_{t\in T} A_t} \subset A$, *then*

$$\overline{\bigcup_{t\in T} A_t} = \bigcup_{t\in T} \bar{A}_t.$$

2.14. THEOREM. *If a family of sets* $\{A_t\}_{t\in T}$ *of fts* $(X, \mathscr{T})$ *is locally finite in a set* A, *then the family of sets* $\{\bar{A}_t\}_{t\in T}$ *is also locally finite in* A.

2.15. DEFINITION. An fts $(X, \mathscr{T})$ is called to be regular if for each point e in $(X, \mathscr{T})$ and each $U \in Q(e)$, there exists a $V \in Q(e)$ such that $\bar{V} \subset U$.

2.16. *Remark.* [8] has proved that 2.15 is an equivalence form of the definition of regularity in [9].

2.17. THEOREM. *Let* A, B *be sets in a regular fts* $(X, \mathscr{T})$ *and each open* Q*-cover of* A *have a refinement which is both locally finite in* B *and a* Q*-cover of* A. *Then for each open* Q*-cover* $\{U_t\}_{t\in T}$, *when* $\overline{\bigcup_{t\in T} U_t} \subset B$, *there exists a closed* Q*-cover* $\{F_t\}_{t\in T}$ *of* A *which is locally finite in* B *such that for each* $t \in T$ *we have* $F_t \subset U_t$.

Proof. From the regularity, A has an open Q-cover $\mathscr{W}$ such that $\{\bar{W}: W \in \mathscr{W}\}$ is a refinement of $\{U_t\}_{t\in T}$. Take a Q-cover $\{A_r\}_{r\in S}$ of A such that $\{A_r\}_{r\in S}$ is both a refinement of $\mathscr{W}$ and locally finite in B. From the way we get $\mathscr{W}$ we know that there exists a mapping $f: S \to T$ such that $\bar{A}_r \subset U_{f(r)}$. Let $F_t = \bigcup \{\bar{A}_r : f(r) = t\}$; then from 2.13 we know that F_t is closed set, and it is easy to know from 2.14 that the closed Q-cover $\{F_t\}_{t\in T}$ of A is locally finite in B. Furthermore, for each $t \in T$ we have $F_t \subset U_t$. ∎

If we let the A_r's in the proof of 2.16 be open, then next theorem is obvious.

2.18. THEOREM. *Let* A *be an* α*-paracompact set in a regular fts* $(X, \mathscr{T})$, *then for each* α*-open* Q*-cover* $\{U_t\}_{t\in T}$ *of* A, *when* $\overline{\bigcup_{t\in T} U_t} \subset A$, *there exists an* α*-open* Q*-cover* $\{V_t\}_{t\in T}$ *of* A *which is locally finite in* A *and for each* $t \in T$ *we have* $\bar{V}_t \subset U_t$.

2.19. DEFINITION. *Let* $\alpha \in [0, 1)$. We say that fts $(X, \mathscr{T})$ is α-crisp if for each $U \in \mathscr{T}$ we have $U_{(\alpha)} \in \mathscr{T}$.

The proof of next theorem is direct.

2.20. THEOREM. *If a family of sets* $\{A_t\}_{t\in T}$ *in a* 0*-crisp fts* $(X, \mathscr{T})$ *is* $*$*-locally finite in a set* A, *then so is the family of sets* $\{\bar{A}_t\}_{t\in T}$.

The example below explains that why 0-crispness is a condition which cannot be eliminated from 2.20.

2.21. EXAMPLE. Let X $[0,1)$. For each $\alpha \in X$ and each $n \in N$, we let $U_\alpha = \alpha\chi_{[0,\alpha)}$ and $A_n = (1-1/n)_1$, then $\mathscr{T} = \{U_\alpha : \alpha \in X\} \cup \{X\}$ is a fuzzy topology on X and $\{A_n : n \in N\}$ is a $*$-locally finite family of sets in $(X, \mathscr{T})$. But the family of sets $\{\bar{A}_n : n \in N\}$ is not $*$-locally finite in X, because we have $\bar{A}_n = (U_{1-1/n})' > 0$.

Being similar to 2.17 and 2.18, we have two theorems below:

2.22. THEOREM. *Let A, B be two sets in 0-crisp regular fts $(X, \mathscr{T})$. Suppose that each open Q-cover of A has a refinement which is both $*$-locally finite in B and a Q-cover of A, then for each open Q-cover $\{U_t\}_{t\in T}$ of A, when $\overline{\bigcup_{t\in T} U_t} \subset B$, there exists a closed Q-cover $\{F_t\}_{t\in T}$ of A which is $*$-locally finite in B and for each $t \in T$ we have $F_t \subset U_t$.*

2.23. THEOREM. *Let A be an α^*-paracompact set in a 0-crisp regular fts $(X, \mathscr{T})$. Then for each α-open Q-cover $\{U_t\}_{t\in T}$ of A, when $\overline{\bigcup_{t\in T} U_t} \subset A$, there exists an α-open Q-cover $\{V_t\}_{t\in T}$ of A which is $*$-locally finite in A such that for each $t \in T$ we have $\bar{V}_t \subset U_t$.*

2.24. THEOREM. *If A an α-paracompact (resp. α^*-paracompact) set in fts $(X, \mathscr{T})$, then for each closed set B in $(X, \mathscr{T})$, each a-open Q-cover of set $B \cap A$ has an open refinement which is both an α-Q-cover of $B \cap A$ and locally finite (resp. $*$-locally finite) in A.*

Proof. We only prove the case of α-paracompactness. Let $\mathscr{U}$ be an α-open Q-cover of $C = B \cap A$; then $\mathscr{U} \cup \{B'\}$ is an open Q-cover of $A_{\langle\alpha\rangle}$ and it has an open refinement $\mathscr{V}$ which is both locally finite in A and a Q-cover of $A_{\langle\alpha\rangle}$. Let $\mathscr{V}_0 = \{V \in \mathscr{V} : \exists U \in \mathscr{U},\ V \subset U\}$, then $\mathscr{V}_0$ is an open refinement of $\mathscr{U}$ which is locally finite in A. We say with certainty that $\mathscr{V}_0$ is a Q-cover of $C_{\langle\alpha\rangle}$. Suppose that it does not hold, then there exists a $x \in \operatorname{supp}(C_{\langle\alpha\rangle})$ such that $(\bigcup \mathscr{V}_0)(x) \leqslant 1-\alpha$. But $\mathscr{V}$ is a Q-cover of $A_{\langle\alpha\rangle}$, so from $C_{\langle\alpha\rangle} \subset A_{\langle\alpha\rangle}$ we know that there exists a $V \in \mathscr{V}$ such that $V(x) > 1-\alpha$. Since $\mathscr{V}$ is a refinement of $\mathscr{U} \cup \{B'\}$ we know $V \subset B'$; from $C(x) \geqslant \alpha$ we know $B(x) \geqslant \alpha$. Hence

$$1-\alpha < V(x) \leqslant B'(x) \leqslant 1-\alpha;$$

this is a contradiction. ∎

2.25. COROLLARY. *Every closed set of an α-paracompact (resp. α^*-paracompact) fts is α-paracompact (resp. α^*-paracompact).*

2.26. DEFINITION [10]. Let $\alpha \in (0, 1]$. A set A in fts $(X, \mathscr{T})$ is called a Q_α-compact set if each α-open Q-cover of A has a finite subfamily which is an α-Q-cover of A. A is called a strong Q-compact set if A is Q_α-compact for each $\alpha \in (0, 1]$.

2.27. THEOREM. *Both N-compact sets and strong Q-compact sets are S^*-paracompact.*

Proof. It is obvious that strong Q-compact sets are S^*-paracompact. On the other hand, from [10] we know that N-compact sets are strong Q-compact, so the theorem is true. ∎

3. PARACOMPACTNESS IN WEAKLY INDUCED FUZZY TOPOLOGICAL SPACES

3.1. DEFINITION. For each fts $(X, \mathscr{T})$, the family of crisp sets

$$[\mathscr{T}] = \{A \subset X: \chi_A \in \mathscr{T}\}$$

is called the original topology of $\mathscr{T}$ and the crisp topological spaces $(X, [\mathscr{T}])$ are called original topological spaces of $(X, \mathscr{T})$.

3.2. *Remark.* The fact that $[\mathscr{T}]$ is a crisp topology on X is certain.

3.3. DEFINITION [11]. We say that fts $(X, \mathscr{T})$ is a weak inducement of crisp topological space $(X, \mathscr{T}_0)$ if $[\mathscr{T}] = \mathscr{T}_0$, and every $U \in \mathscr{T}$ is lower semi-continuous when we regard it as a mapping between $(X, \mathscr{T}_0)$ and $[0, 1]$. We say $(X, \mathscr{T})$ is weakly induced if there exists a crisp topological space $(X, \mathscr{T}_0)$ such that $(X, \mathscr{T})$ is a weak inducement of $(X, \mathscr{T}_0)$.

3.4. *Remark.* Clearly, every induced fts is weakly induced, but the inverse is not true. Hence the concept of weak inducement is a real extension of the concept of inducement. Since the concept of induced fts is an extension of the concept of crisp topological space, hence so is the concept of weak inducement.

From the properties of lower semi-continuous functions which are well known by us, we have

3.5. THEOREM. *$(X, \mathscr{T})$ is a weakly induced fts if and only if $(X, \mathscr{T})$ is α-crisp for every $\alpha \in [0, 1)$.*

3.6. THEOREM. *For every weakly induced fts $(X, \mathscr{T})$ the following conditions are equivalent*:

(i) *$(X, \mathscr{T})$ is S^*-paracompact.*

(ii) *There exist a $\alpha \in (0, 1)$ such that $(X, \mathscr{T})$ is α^*-paracompact.*

(iii) $(X, \mathscr{T})$ *is S-paracompact.*

(iv) *There exist a* $\alpha \in (0, 1)$ *such that* $(X, \mathscr{T})$ *is* α*-paracompact.*

(v) $(X, [\mathscr{T}])$ *is paracompact.*

Proof. (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) Obvious.

(iv) $\Rightarrow$ (v) Let $\mathscr{U} \subset [\mathscr{T}]$ be an open cover of X; then $\{\chi_U: U \in \mathscr{U}\}$ is an open Q-cover of 1_α and it has a locally finite open refinement $\mathscr{V}$ which is a Q-cover of 1_α also. Let $\mathscr{W} = \{V_{(1-\alpha)}: V \in \mathscr{V}\}$; then $\mathscr{W}$ is both a refinement of $\mathscr{U}$ and a cover of X. From the properties of weak inducement we know $\mathscr{W} \subset [\mathscr{T}]$, so we need only to prove that $\mathscr{W}$ is locally finite. For each $x \in X$, take $O_1 \in Q_{\mathscr{T}}(x_{1-\alpha})$ such that O_1 is quasi-coincident with only a finite number of members $V_1, ..., V_n$ of $\mathscr{V}$. Let $O = (O_1)_{(\alpha)}$; then $x \in O \in [\mathscr{T}]$. For each $V \in \mathscr{V}$, if $O \cap V_{(1-\alpha)} \neq \varnothing$, we have a crisp point $y \in O \cap V_{(1-\alpha)}$, hence $O_1(y) > \alpha$, $V(y) > 1 - \alpha$, $O_1(y) + V(y) > 1$, and O and V are quasi-coincident, so $V \in \{V_1, ..., V_n\}$. Hence the neighborhood O of x intersects with only a finite number of members $(V_1)_{(1-\alpha)}, ..., (V_n)_{(1-\alpha)}$ of $\mathscr{W}$.

(v) $\Rightarrow$ (i) For each $\alpha \in (0, 1]$, let $\mathscr{U} \subset \mathscr{T}$ be a Q-cover of 1_α. Then from the properties of weak inducement we know that the family of crisp sets $\{U_{(1-\alpha)}: U \in \mathscr{U}\}$ is an open cover of $(X, [\mathscr{T}])$ and it has a locally finite open refinement $\mathscr{V} \subset [\mathscr{T}]$ which is a cover of X. For each $V \in \mathscr{V}$ take a $U_V \in \mathscr{U}$ such that $V \subset (U_V)_{(1-\alpha)}$ and let $\mathscr{W} = \{\chi_V \cap U_V: V \in \mathscr{V}\}$. Then $\mathscr{W} \subset \mathscr{T}$ is both a refinement of $\mathscr{U}$ and a Q-cover of 1_α. For each $x \in X$ and each $\lambda \in (0, 1]$, take a $O \in N_{[\mathscr{T}]}(x)$ such that O intersects with only finite number of members of $\mathscr{V}$. Then $\chi_O \in Q_{\mathscr{T}}(x_\lambda)$ intersects with only a finite number of members of $\mathscr{W}$. So we know from the arbitrariness of α that $(X, \mathscr{T})$ is S^*-paracompact.

(ii) $\Leftrightarrow$ (i) Clearly (ii) $\Rightarrow$ (iv); furthermore we have proved (iv) $\Rightarrow$ (i), so we can know (ii) $\Leftrightarrow$ (i) from (i) $\Rightarrow$ (ii). ∎

3.7. Corollary. *Both S-paracompactness and S*-paracompactness are good extensions* [4] *of crisp paracompactness.*

3.8. Example. The open interval $(0, 1)$ in 3.6 cannot be substituted by the half open interval $(0, 1]$. Let $X = [0, 1)$ and

$$\mathscr{P} = \{1_\alpha: \alpha \in (0, 1]\} \cup \{\chi_{[0x)}: x \in X\},$$

then we can use $\mathscr{P}$ as a subbase to generate a weakly induced fuzzy topology $\mathscr{T}$ on X.

Let $\mathscr{U} \subset \mathscr{T}$ be a Q-cover of $1_1 = X$. Generally, we can suppose that every member of $\mathscr{U}$ is not empty and cannot consist of a Q-cover of X by itself.

Then we can see from the structure of $\mathscr{P}$ that for each $U \in \mathscr{U}$ there exists a $x_U \in X$ such that

$$(U \mid [0, x_U)) > 0, \qquad U[[X_U, 1)] = \{0\}.$$

Since $\mathscr{U}$ is a Q-cover of X, so $\sup\{x_U : U \in \mathscr{U}\} = 1$. Take a countable family $\{U_i\}_{i \in N} \subset \mathscr{U}$ such that $\sup\{X_{U_i} : i \in N\} = 1$ and let $V_i = 1_{1/2} \cap U_i$; then $\mathscr{V} = \{V_i\}_{i \in N}$ is both an open refinement of $\mathscr{U}$ and a Q-cover of X. For each $x \in X$ and each $\lambda \in (0, 1]$, take $k \in N$ such that $1/k < \lambda$; then $1_{1-1/k} \in Q(x_\lambda)$ and is not quasi-coincident with V_i when $i \geqslant k$. Hence $\mathscr{V}$ is locally finite in X and $(X, \mathscr{T})$ is 1-paracompact.

But it is easy to show that

$$[\mathscr{T}] = \{\phi, X\} \cup \{[0, x) : x \in X\},$$

so $\mathscr{W} = \{[0, x) : x \in X\}$ is an open cover of $(X, [\mathscr{T}])$. Clearly, $\mathscr{W}$ has not any locally finite open refinement which is a cover of X, so $(X, [\mathscr{T}])$ is not paracompact.

3.9. *Remark.* We can know from 3.6 and 3.8 that there exists a 1-paracompact fts which is not α-paracompact for every $\alpha \in (0, 1)$.

3.10. LEMMA. *For every weakly induced fts $(X, \mathscr{T})$, if there exist an $\alpha \in (0, 1)$ such that every α-open Q-cover of X has a locally finite closed refinement which is a α-Q-cover of X, then $(X, \mathscr{T})$ is α-paracompact.*

Proof. Let $\mathscr{U}$ be an open Q-cover of 1_α. Take a locally finite refinement $\mathscr{A} = \{A_t\}_{t \in T}$ of $\mathscr{U}$ such that $\mathscr{A}$ is a Q-cover of 1_α. We let β denote $\min\{\alpha, 1 - \alpha\}$, then $\beta \in (0, 1)$. Take $U_x \in Q(x_\beta)$ such that U_x is quasi-coincident with only a finite number of members of $\mathscr{A}$ and let $\tilde{U}_x = (U_x)_{(1-\alpha)} \cap U_x$; then $\mathscr{U}_1 = \{\tilde{U}_x : x \in X\}$ is an open Q-cover of 1_α and it has a locally finite closed refinement $\mathscr{F}$ which is Q-over of 1_α also. For each $t \in T$ let

$$W_t = \bigcap \{F' : F \in \mathscr{F}, F' \supset (A_t)_{(1-\alpha)}\},$$

then $W_t \in \mathscr{T}$ and, for each $F \in \mathscr{F}$, we have

$$W_t q F \Leftrightarrow (A_t)_{(1-\alpha)}\, qF. \qquad (*)$$

Take $U_t \in \mathscr{U}$ for each $t \in T$ such that $A_t \subset U_t$ and let

$$V_t = (W_t)_{(1-\beta)} \cap U_t.$$

Then $\mathscr{V} = \{V_t\}_{t \in T}$ is an open refinement of $\mathscr{U}$. For each $x \in X$, take a $t \in T$

such that $A_t(x) > 1-\alpha$. Since $W_t \supset (A_t)_{(1-\alpha)}$, we have $W_t(x) = 1 > 1-\beta$, so

$$V_t(x) = U_t(x) \geqslant A_t(x) > 1-\alpha;$$

hence $\mathscr{V}$ is a Q-cover of 1_α also.

At last we prove that $\mathscr{V}$ is locally finite. For each $x \in X$ and each $\lambda \in (0, 1]$, let $\gamma = \min\{\lambda, \beta\}$, take $U \in Q(x_\gamma)$ such that U is quasi-coincident with only a finite number of members $F_1, ..., F_n$ of $\mathscr{F}$. For each $i \in \{1, ..., n\}$, take a $x^i \in X$ such that $F_i \subset \tilde{U}_{x^i}$; then from the way we get $\mathscr{U}_1$ we know the set $T_0 = \{t \in T: A_t q \bigcup_{i=1}^n \tilde{U}_{x^i}\}$ is finite. It can be said with certainty that for each $t \in T$ and each $i \in \{1, ..., n\}$ we have

$$(A_t)_{(1-\alpha)} qF_i \Rightarrow t \in T_0. \tag{$**$}$$

In fact, if $(A_t)_{(1-\alpha)} qF_i$, then there exists a $y \in X$ such that $A_t(y) > 1-\alpha$, $\tilde{U}_{x^i}(y) \geqslant F_i(y) > 0$, $\tilde{U}_{x^i}(y) + A_t(y) > 1-\beta+1-\alpha \geqslant \alpha+1-\alpha = 1$, and $A_t q \bigcup_{j=1}^n \tilde{U}_{x^j}$, $t \in T_0$. So take $V = U_{(1-\gamma)} \cap U$, then $V \in Q(x_\lambda)$. If VqV_t, then there exists a $y \in X$ such that $V(y) > 1-\gamma \geqslant 1-\beta$, $V_t(y) > 0$. Take $F \in \mathscr{F}$ such that $F(y) > 1-\alpha \leqslant \beta$, then VqF, $F \in \{F_1, ..., F_n\}$. On the other hand, since $V_t(y) > 0$, so $W_t(y) > 1-\beta$, $W_t qF$. From relation ($*$) we have $(A_t)_{(1-\alpha)} qF$ and from relation ($**$) we know that $t \in T_0$. So V is quasi-coincident only with the members of the finite subfamily $\{V_t : t \in T_0\}$ of $\mathscr{V}$ and $\mathscr{V}$ is locally finite. ∎

3.11. Definition. Let $\mathscr{A}$ be a family of sets and B be a set in fts $(X, \mathscr{T})$. We say that $\mathscr{A}$ is σ-locally finite (respectively: σ^*-locally finite) in B if $\mathscr{A}$ can be represented as a countable union of subfamilies and each of these subfamily is locally finite (respectively: $*$-locally finite) in B; we often omit the word "in B" when $B = X$. ∎

3.12. Lemma. *Let $\alpha \in (0, 1]$. If $(X, \mathscr{T})$ is a $(1-\alpha)$-crisp fts, then each σ-locally finite α-open Q-cover of X has a locally finite refinement which is a α-Q-cover of X also.*

Proof. Let $\mathscr{V} = \bigcup_{i \in N} \mathscr{V}_i$ be an open Q-cover of 1_α; here every $\mathscr{V}_i = \{V_t\}_{t \in T_i}$ is locally finite in X and $T_i \cap T_j = \varnothing$ when $i \neq j$. For each $i \in N$ and each $t \in T$ we let

$$A_t = V_t \cap \left(\bigcup_{k<i} \bigcup_{t' \in T_k} (V_{t'})_{(1-\alpha)} \right),$$

$$T = \bigcup_{i \in N} T_i,$$

$$\mathscr{A} = \{A_t\}_{t \in T},$$

then $\mathscr{A}$ is a refinement of $\mathscr{V}$.

For each $x \in X$, let $i_0 = \min\{i \in N\colon \bigcup \mathscr{V}_i \in Q(x_\alpha)\}$ and take $t_0 \in T_{i_0}$ such that $V_{t_0} \in Q(X_\alpha)$. Then from the fact that for each $k < i_0$ and each $t \in T_k$ we have $V_t(x) \leqslant 1-\alpha$ and we have $A_{t_0}(x) = V_{t_0}(x) > 1-\alpha$ and $\mathscr{A}$ is a Q-cover of 1_α. On the other hand, for each $\lambda \in (0, 1]$ and $i = 1,..., i_0$, take $U_i \in Q(x_\lambda)$ such that U_i is quasi-coincident with only a finite number of members of $\mathscr{V}_i$. Let

$$U = U_1 \cap \cdots \cap U_{i_0} \cap (V_{t_0})_{(1-\alpha)};$$

then $U \in Q(x_\lambda)$ and, for each $i \in N$, $i > i_0$, and each $t \in T_i$, we have

$$A'_t = V'_t \cup \bigcup_{k<i} \bigcup_{r \in T_k} (V_r)_{(1-\alpha)} \supset (V_{t_0})_{(1-\alpha)} \supset U.$$

U and A_t are not quasi-coincident. Since for each $i \in \{1,..., i_0\}$, U is quasi-coincident with only a finite number of members of $\{A_t\}_{t \in T_i}$, so U is quasi-coincident with only a finite number of members of $\mathscr{A}$ and $\mathscr{A}$ is locally finite. ∎

3.13. Example. $(1-\alpha)$-crispness is the condition which cannot be eliminated from 3.12. Take $X = [0, +\infty)$, let $\mathscr{T}_0$ denote usual crisp topology on X. For each $\beta \in [0, 1]$ let

$$U_\beta = (\beta\chi_{\{0\}}) \cup (\tfrac{1}{2}\chi_{\{1/n : n \in N\}}) \cup \chi_{X \setminus (\{0\} \cup \{1/n : n \in N\})},$$
$$\mathscr{P} = \{U_\beta, U'_\beta\colon \beta \in [0, 1]\} \cup \{\chi_U\colon U \in \mathscr{T}_0\};$$

then we can use $\mathscr{P}$ as a subbase to generate a fuzzy topology on X. Since $\mathscr{T}_0$ has a countable base, we can see easily that $\mathscr{T}$ has a countable subbase $\mathscr{P}_1 \subset \mathscr{P}$ and hence $\mathscr{T}$ has a countable base. Hence every open Q-cover of 1_α has a σ-locally finite open refinement for every $\alpha \in (0, 1]$. Furhermore, we can know that $(X, \mathscr{T})$ is a T_2-fts from $\mathscr{T}_0 \subset \mathscr{T}$. But the open Q-cover $\{U_{2/3}\} \cup \{\chi_{(1/n - 1/2n(n+1), 1/n + 1/2n(n+1))}\colon n \in N\}$ of $1_{1/2}$ does not have any locally finite refinement which is a Q-cover of $1_{1/2}$ also.

3.14. Theorem. *For every regular weakly induced fts* $(X, \mathscr{T})$ *the following conditions are equivalent*:

(i) $(X, \mathscr{T})$ *is S-paracompact.*

(ii) *For every* $\alpha \in (0, 1]$, *every* α*-open Q-cover of X has a* σ*-locally finite open refinement which is an* α*-Q-cover of X also.*

(iii) *There exists an* $\alpha \in (0, 1)$ *such that every* α*-open Q-cover of X has a* σ*-localy finite open refinement which is an* α*-Q-cover of X also.*

(iv) *For every* $\alpha \in (0, 1]$, *every* α*-open Q-cover of X has a locally finite refinement which is an* α*-Q-cover of X also.*

(v) *There exists an* $\alpha \in (0, 1)$ *such that every* α*-open* Q*-cover of* X *has a locally finite refinement which is an* α*-*Q*-cover of* X *also.*

(vi) *For every* $\alpha \in (0, 1]$, *every* α*-open* Q*-cover of* X *has a locally finite closed refinement which is an* α*-*Q*-cover of* X *also.*

(vii) *There exists an* $\alpha \in (0, 1)$ *such that every* α*-open* Q*-cover of* X *has a locally finite closed refinement which is an* α*-*Q*-cover of* X *also.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (v) From 3.12.

(v) $\Rightarrow$ (vii) From 2.17.

(vii) $\Rightarrow$ (i) From 3.10 and 3.6.

(i) $\Rightarrow$ (iv) Obvious.

(iv) $\Rightarrow$ (vi) From 2.17.

(vi) $\Rightarrow$ (i) Clarly, we have (vi) $\Rightarrow$ (vii). On the other hand, we have proved (vii) $\Rightarrow$ (i). So (vi) $\Rightarrow$ (i). ∎

3.15. THEOREM. *For every regular weakly induced fts* $(X, \mathscr{T})$ *the following conditions are equivalent*:

(i) $(X, \mathscr{T})$ *is* S^**-paracompact.*

(ii) *For every* $\alpha \in (0, 1]$, *every* α*-open* Q*-cover of* X *has a* σ^**-locally finite open refinement which is an* α*-*Q*-cover of* X *also.*

(iii) *There exists an* $\alpha \in (0, 1)$ *such that every* α*-open* Q*-cover of* X *has a* σ^**-locally finite open refinement which is an* α*-*Q*-cover of* X *also.*

(iv) *For every* $\alpha \in (0, 1]$, *every* α*-open* Q*-cover of* X *has a* $*$*-locally finite refinement which is an* α*-*Q*-cover of* X *also.*

(v) *There exists an* $\alpha \in (0, 1)$ *such that every* α*-open* Q*-cover of* X *has a* $*$*-locally finite refinement which is an* α*-*Q*-cover of* X *also.*

(vi) *For every* $\alpha \in (0, 1]$, *every* α*-open* Q*-cover of* X *has a* $*$*-locally finite closed refinement which is an* α*-*Q*-cover of* X *also.*

(vii) *There exists an* $\alpha \in (0, 1)$ *such that every* α*-open* Q*-cover of* X *has a* $*$*-locally finite closed refinement which is an* α*-*Q*-cover of* X *also.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (i) Clearly we have: (iii) $\Rightarrow$ 3.14(iii) $\Rightarrow$ 3.14(i) $\Rightarrow$ (i).

(iv) $\Rightarrow$ (vi) From 2.22.

(vi) $\Rightarrow$ (iv) Clearly we have: (vi) $\Rightarrow$ 3.14(vi) $\Rightarrow$ 3.14(i) $\Rightarrow$ (i) $\Rightarrow$ (iv). ∎

The following lemma can be proved easily:

3.16. LEMMA. *Let* $\mathscr{P}$ *be a subbase of an fts* $(X, \mathscr{T})$. *If for each point* e *in*

$(X, \mathscr{T})$ and each $U \in \mathscr{P} \cap Q(e)$ there exists a $V \in Q(e)$ such that $\bar{V} \subset U$, then $(X, \mathscr{T})$ is regular.

3.17. EXAMPLE. There exist an fts $(X, \mathscr{T})$ such that:

(i) $(X, \mathscr{T})$ is regular;

(ii) $(X, \mathscr{T})$ satisfies condition (vi) of 3.15;

(iii) $(X, \mathscr{T})$ is not α-paracompact for every $\alpha \in (0, 1]$.

Take $X = \omega_1$. For each $\alpha \in (0, 1)$, each $\beta \in (0, 1]$, and each $x \in X$, let

$$U_\alpha = \alpha\chi_{\{0\}}, \qquad U_{\beta,x} = (\beta\chi_{\{0\}}) \cup \chi_{\{x\}}, \qquad V_{\beta,x} = (\beta\chi_{\{0\}}) \cup \chi_{X\setminus\{0,x\}},$$

$$\mathscr{P}_1 = \{U_\alpha, U'_\alpha : \alpha \in (0, 1)\}, \qquad \mathscr{P}_2 = \{U_{\beta,x}, V_{\beta,x} : \beta \in (0, 1], x \in X\},$$

$$\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2,$$

then we use $\mathscr{P}$ as a subbase to generate a fuzzy topology $\mathscr{T}$ on X.

Proof of (i). Take $x \in X$, $\lambda \in (0, 1]$, and $U \in \mathscr{P} \cap Q(x_\lambda)$ arbitrarily. If $U \in \mathscr{P}_1$, then $\bar{U} = U$ hilds naturally. If $U \in \mathscr{P}_2$ and $U = U_{\beta,y}$, then $x = 0$ or $x = y$. If $x = 0$, take $\alpha \in (0, 1)$ such that $1 - \lambda < \alpha < \beta$; then we have $U_\alpha \in Q(x_\lambda)$ and $\bar{U}_\alpha = U_\alpha \subset U$. If $x = y$, take $\alpha \in (0, 1]$ such that $0 < 1 - \alpha < \beta$; then we have $V'_{\alpha,x} = U_{1-\alpha,x} \in Q(x_\lambda)$ and $\overline{V'_{\alpha,x}} = V'_{\alpha,x} = U_{1-\alpha,x} \subset U$. If $U \in \mathscr{P}_2$ and $U = V_{\beta,y}$, then $x = 0$ or $x \neq 0$. If $x = 0$, take $\alpha \in (0, 1)$ such that $1 - \lambda < \alpha < \beta$; than we have $U_\alpha \in Q(x_\lambda)$ and $\bar{U}_\alpha = U_\alpha \subset U$. If $x \neq 0$, take $\alpha \in (0, 1]$ such that $0 < \alpha < \beta$; then we have $U_{\alpha,x} \in Q(x_\lambda)$ and $\bar{U}_{\alpha,x} = U_{\alpha,x} \subset U$. So we have proved that $(X, \mathscr{T})$ satisfies the condition in 3.16; hence $(X, \mathscr{T})$ is regular indeed.

Proof of (ii). Let $\mathscr{U}$ be an open Q-vector of 1_α. Since $V_{1,x} \in \mathscr{T}$ for each $x \in X\setminus\{0\}$, so $V'_{1,x}$ is a closed set. Take $U_0 \in \mathscr{U}$ such that $U_0 \in Q((0)_\alpha)$ and $\beta \in (0, 1)$ such that $1 - \alpha < \beta < U_0(0)$; then $U_\beta \in Q((0)_\alpha)$ is a closed set too. So $\mathscr{F} = \{U_\beta\} \cup \{V'_{1,x} : x \in X\setminus\{0\}\}$ is a closed refinement of $\mathscr{U}$ which is a Q-cover of 1_α also. Clearly, $\mathscr{F}$ is $*$-locally finite.

Proof of (iii). $\mathscr{U} = \{U_{1,x} : x \in X\}$ us an open Q-cover of 1_α for each $\alpha \in (0, 1]$. From the structure of $\mathscr{P}$ we can know that $U(0) > 0$ holds for every $U \in \mathscr{T}$ indeed, so if $\mathscr{V}$ is both an open refinement of $\mathscr{U}$ and a Q-cover of 1_α, then there must exist a crisp set $\{\alpha_x : x \in X\setminus\{0\}\} \subset (0, 1]$ such that $\{U_{\alpha_x,x} : x \in X\setminus\{0\}\} \subset \mathscr{V}$. Since $|X\setminus\{0\}| = \omega_1$, there exists a $k \in N$ and an uncountable crisp subset C of $X\setminus\{0\}$ such that $\{\alpha_x : x \in C\} \subset (1/k, 1]$. Then for each $U \in Q((0)_{1/k})$ and each $x \in C$ we always have

$$U(0) + U_{\alpha_x,x}(0) > 1 - \frac{1}{k} + \frac{1}{k} = 1,$$

which means that every Q-neighborhood of point $(0)_{1/k}$ is quasi-coincident with every member of the uncountable subfamily $\{U_{\alpha_x,x}: x \in C\}$ of $\mathscr{V}$, $\mathscr{V}$ is not locally finite. ▌

4. Paracompactness and Lindelöff Properties

4.1. Definition. *Let* $\alpha \in (0, 1]$. We say that a set A in fts $(X, \mathscr{T})$ is α-Lindelöff if every α-open Q-cover of A has a countable subfamily which is a Q-cover of A also. We say that A is S-Lindelöff if A is α-Lindelöff for every $\alpha \in (0, 1]$.

We say that $(X, \mathscr{T})$ is α-Lindelöff (resp. S-Lindelöff) if set X is α-Lindelöff (resp. S-Lindelöff).

4.2. Theorem. *Ultra-fuzzy compact spaces, N-compact spaces, and strong fuzzy compact spaces are S-Lindelöff.*

Being similar to 2.24, we have

4.3. Theorem. *If A is an α-Lindelöff set in an fts $(X, \mathscr{T})$, then for every closed set B in $(X, \mathscr{T})$, set $B \cap A$ is α-Lindelöff too.*

4.4. Corollary. *Every closed set in an α-Lindelöff is α-Lindelöff.*

4.5. Theorem. *If A is an S-Lindelöff set in an fts $(X, \mathscr{T})$, then for every family of sets $\mathscr{C}$ which is locally finite in A, $\mathscr{C}$ is countable when $C \cap A \neq \varnothing$ for every $C \in \mathscr{C}$.*

Proof. For each $n \in N$ and each $x \in \operatorname{supp}(A_{\langle 1/n \rangle})$, take $U_{n,x} \in Q(x_{1/n})$ such that $U_{n,x}$ is quasi-coincident with only a finite number of members of $\mathscr{C}$. From the S-Lindelöff property of A, for each $n \in N$, there exists a countable subfamily $\mathscr{U}_n$ of $\{U_{n,x}: x \in \operatorname{supp}(A_{\langle 1/n \rangle})\}$ which is a Q-cover of $A_{\langle 1/n \rangle}$. So $\mathscr{U} = \bigcup_{n \in N} \mathscr{U}_n$ is countable; hence $\bigcup \mathscr{U}$ is quasi-coincident with at most a countable number of members of $\mathscr{C}$. On the other hand, since $\operatorname{supp}(A) \subset \bigcup \mathscr{U}$ and $C \cap A \neq \varnothing$ for every $C \in \mathscr{C}$, we know that $C \cap (\bigcup \mathscr{U})) \neq \varnothing$ for every $C \in \mathscr{C}$, so $\mathscr{C}$ is countable. ▌

4.6. Example. A locally finite family in a 1-Lindelöff fts which consists of nonempty sets need not be countable. Let X be the set of all the real numbers. For each $x \in X$ and each $\varepsilon > 0$, let $U_{x,\varepsilon} = \chi_{\{x\}} \cup (\frac{1}{2}\chi_{(x-\varepsilon, x+\varepsilon)})$ and use $\{U_{x,\varepsilon}: x \in X, \varepsilon > 0\}$ as a subbase to generate a fuzzy topology $\mathscr{T}$ on X. Then it is obvious that $(X, \mathscr{T})$ is 1-Lindelöff. Take $\mathscr{C} = \{\frac{1}{2}\chi_{\{x\}}: x \in X\}$; then $\mathscr{C}$ is locally finite but is uncountable.

4.7. THEOREM. *If a weakly induced fts $(X, \mathscr{T})$ is regular, then so is its original topological space $(X, [\mathscr{T}])$.*

Proof. For each $x \in X$, let $N_{[\mathscr{T}]}(x)$ denote the neighborhood system of x in $(X, [\mathscr{T}])$; then $\chi_U \in Q_{\mathscr{T}}(x_1)$ for every $U \in N_{[\mathscr{T}]}(x)$. Take $V \in Q_{\mathscr{T}}(x_1)$ such that $\bar{V} \subset \chi_U$ and take α such that $0 < \alpha < V(x)$, then $V_{(\alpha)} \subset (\bar{V})_{[\alpha]} \subset \chi_U$. Since $(X, \mathscr{T})$ is weakly induced, from 3.5 it follows that there exists an $x \in \text{supp}(V_{(\alpha)}) \in [\mathscr{T}]$ and $\sup((\bar{V})_{[\alpha]}) \subset U$ is a closed set in $(X, [\mathscr{T}])$. So if we let $W = \text{supp}(V_{(\alpha)})$, then the following relation holds in $(X, [\mathscr{T}])$:

$$x \in W \subset \bar{W} \subset U. \quad \blacksquare$$

4.8. EXAMPLE. A weakly induced fts $(X, \mathscr{T})$ of a crisp T_4-topological space $(X, \mathscr{T}_0)$ need not be regular. Let $(X, \mathscr{T}_0)$ be the usual real number space and $\mathscr{B} = \{\chi_U, \frac{2}{3}\chi_U : U \in \mathscr{T}_0\}$; then we can use $\mathscr{B}$ as a base to generate a fuzzy topology $\mathscr{T}$ on X, and we can see that $(X, \mathscr{T})$ is a weak inducement of $(X, \mathscr{T}_0)$. But for each x, X, and the Q-neighborhood $1_{2/3}$ of $x_{1/2}$ there does not exist any $V \in Q(x_{1/2})$ such that $\bar{V} \subset 1_{2/3}$.

4.9. THEOREM. *If a regular S-Lindelöff fts $(X, \mathscr{T})$ is $(1-\alpha)$-crisp, then $(X, \mathscr{T})$ is α-paracompact.*

Proof. Let $\mathscr{U}$ be an open Q-cover of 1_α. From the S-Lindelöff property we know that $\mathscr{U}$ has a countable subfamily $\{U_i\}_{i \in N}$ which is a Q-cover of 1_α. So there exists a mapping $f: X \to N$ such that

$$f(x) = \min\{i \in N: U_i(x) > 1 - \alpha\}.$$

For each $n \in N$ and each $x \in X$, from the $(1-\alpha)$-crispness and regularity, we can take $V_{n,x} \in Q(x_{1/n})$ such that $\bar{V}_{n,x} \subset (U_{f(x)})_{(1-\alpha)}$. Take a countable subfamily $\{V_{n,x^{n,j}}: j \in N\}$ of $\{V_{n,x}: x \in X\}$ such that it is a Q-cover of $1_{1/n}$, let $V_{n,j}$ denote $V_{n,x}n, j$, and for each $i \in N$ let

$$W_i = U_i \cap \left(\bigcup \{\bar{V}_{n,j}: n < i, j < i, f(x^{n,j}) < i\} \right)';$$

then $\mathscr{W} = \{W_i\}_{i \in N}$ is an open refinement of $\mathscr{U}$. For each $x \in X$ we have $U_{f(x)}(x) > 1 - \alpha$, from the demfiition we have $\bar{V}_{n,j} \subset (U_{f(x^{n,j})})_{(1-\alpha)}$, and when $f(x^{n,j}) < f(x)$ we have

$$\bar{V}'_{n,j}(x) \geqslant ((U_{f(x^{n,j})})_{(1-\alpha)})'(x) = 1,$$

so

$$W_{f(x)}(x) = U_{f(x)}(x) > 1 - \alpha,$$

where $\mathscr{W}$ is a Q-cover of 1_α. On the other hand, for each $x \in X$ and each $\lambda \in (0, 1]$, take $n \in N$ such that $1/n \leqslant \lambda$; then there exists a $j \in N$ such that

$V_{n,j}(x) > 1 - 1/n \geqslant 1 - \lambda$, so $V_{n,j} \in Q(x)$. Hence when $i > \max\{n, j, f(x^{n,j})\}$ we have

$$V_{n,j} \subset \bar{V}_{n,j} \subset \bigcup \{\bar{V}_{m,k} : m < i, k < i, f(x^{m,k}) < i\} \subset W_i',$$

where $V_{n,j}$ and W_i are not quasi-coincident and $\mathscr{W}$ is locally finite. ∎

4.10. Corollary. *Every weakly induced regular S-Lindelöff fts is S-paracompact.*

4.11. Example. There exists a regular T_2-fts $(X, \mathscr{T})$ which has a countable base (so it is S-Lindelöff), but it is not 1-paracompact.

We take $X = [0, +\infty)$, let $\mathscr{T}_0$ denote the usual crisp topology on X, and let

$$U_0 = \tfrac{1}{2}\chi_{\{0\}}, \qquad U_1 = (\tfrac{1}{2}\chi_{\{0\}}) \cup \chi_{X \setminus (\{0\} \cup \{1/n : n \in N\})},$$
$$\mathscr{P} = \{U_0, U_0', U_1\} \cup \{\chi_U : U \in \mathscr{T}_0\};$$

then we can use $\mathscr{P}$ as a subbase to generate a fuzzy topology $\mathscr{T}$ on X. Thereupon we have

(i) $(X, \mathscr{T})$ is T_2-fits. It is obvious.

(ii) $(X, \mathscr{T})$ is regular. In fact, for each fuzzy point e, from the way we take $\mathscr{P}$ we can know that for each $U \in \mathscr{P} \cap Q(e)$, there exists a $V \in Q(e)$ such that $\bar{V} \subset U$. Hence we know from 3.16 that $(X, \mathscr{T})$ is regular.

(iii) $(X, \mathscr{T})$ has a countable base. Since $\mathscr{T}_0$ has a countable base, we can see that $\mathscr{T}$ has a countable subbase $\mathscr{P}_1 \subset \mathscr{P}$ and so $\mathscr{T}$ has a countable base.

(iv) The open Q-cover $\mathscr{U} = \{U_1\} \cup \{\chi_{(1/n - 1/2n(n+1), 1/n + 1/2n(n+1))} : n \in N\}$ of X does not have any open refinement $\mathscr{V}$ which both is a Q-cover of X and locally finite on point $(0)_{1/2}$. In fact, for each $n \in N$, let $V_n \in \mathscr{V}$ such that $V_n(1/n) > 0$, then from that $\mathscr{V}$ is a refinement of $\mathscr{U}$ we can know $V_n \subset \chi_{(1/n - 1/2n(n+1), 1/n + 1/2n(n+1))}$, so $V_m \cap V_n = \varnothing$ when $m \neq n$, so the correspondence $1/n \mapsto V_n$ is one to one. On the other hand, we have $U(0) > \frac{1}{2}$ for each $U \in Q((0_{1/2})$, so from the structure of $\mathscr{P}$ we know that there exists a $U_0 \in \mathscr{T}_0$ such that $U \supset \chi_{U_0}$. Hence U is quasi-coincident with an infinite number of V_n.

4.12. Example. 0-crisp regular 1-Lindelöff fts need not be 1-paracompact. Let $X = [0, +\infty)$ and

$$\mathscr{P}_1 = \{U \in I^X : \exists x \in X, |\{y \in [0, x] : U(y) \neq \tfrac{1}{2}\}| < \omega,$$
$$U[(x, +\infty)] \in \{\{0\}, \{1\}\}\},$$
$$\mathscr{P}_2 = \{U_{(0)} : U \in \mathscr{P}_1\},$$
$$\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2;$$

then we can use $\mathscr{P}$ as a subbase to generate a 0-crisp fuzzy topology $\mathscr{T}$.

Take a fuzzy point x_λ from $(X, \mathscr{T})$ arbitrarily; for each $U \in \mathscr{P}_2 \cap Q(x_\lambda)$, let $W(\frac{1}{2}\chi_{[0,x)}) \cup \chi_{\{x\}}$ and $V = U \cap W$. Then $V \in Q(x_\lambda) \cap \mathscr{P}_1$, V is an open-closed set, and we have $\bar{V} = V \subset U$; and for each $U \in \mathscr{P}_1 \cap Q(x_\lambda)$, U is closed, and certainly we have $\bar{U} \subset U$. So from 3.16 we know that $(X, \mathscr{T})$ is regular.

Now we will prove that $(X, \mathscr{T})$ is 1-Lindelöff. For each $U \in \mathscr{T}$, let

$$X_U = \inf\{x \in X: U[(x, +\infty)] = \{0\}\},$$
$$F_U = \{x \in [0, x_U): U(x) = 0\};$$

here the infimum of the empty set is defined as $+\infty$. Let

$$\mathscr{B} = \left\{\bigcap \mathscr{A}: \mathscr{A} \subset \mathscr{P}, |\mathscr{A}| < \omega\right\},$$

then $\mathscr{B}$ is the base of $\mathscr{T}$. For each $U \in \mathscr{T}$, suppose that $U = \bigcup \mathscr{A}$, $\mathscr{A} \subset \mathscr{B}$, then from the properties of real number space we know that there exists a countable subfamily $\mathscr{A}_1$ of $\mathscr{A}$ such that

$$\sup\{x_v: V \in \mathscr{A}_1\} = x_U. \qquad (*)$$

Since $|F_v| < \omega$ for each $V \in \mathscr{B}$, it also does for each $V \in \mathscr{A}_1$. From the equality $(*)$ and $|\mathscr{A}_1| \leqslant \omega$ we have

$$|F_U| \leqslant |F_{\bigcup \mathscr{A}_1}| \leqslant \left|\bigcup \{F_v: V \in \mathscr{A}_1\}\right| \leqslant \omega. \qquad (**)$$

Let $\mathscr{U}$ be an open Q-cover of X; then $\sup\{x_U: U \in \mathscr{U}\} = +\infty$. Take a countable subfamily $\{U_i\}_{i \in N}$ of $\mathscr{U}$ such that $\sup\{x_{U_i}: i \in N\} = +\infty$; then from the inequality $(**)$ we have $|\bigcup \{F_{U_i}: i \in N\}| \leqslant \omega$ and from the definition of F_U we can easily find that $\mathscr{U}$ has a countable subfamily which is a Q-cover of X also, so that $(X, \mathscr{T})$ is 1-Lindelöff.

At last, we will prove that $(X, \mathscr{T})$ is not 1-paracompact. Let $\mathscr{V} = \{\frac{1}{2}\chi_{[0,x)}: x \in X\}$, then $\mathscr{V}$ is an open Q-cover of X. Suppose that an open refinement $\mathscr{W}$ of $\mathscr{V}$ is a Q-cover of X, then from the 1-Lindelöff property we have proved we can suppose that $\mathscr{W}$ is countable. For each $U \in \mathscr{T}$, let $C_U = \{x \in [0, x_U): U(x) < \frac{1}{2}\}$; then we can see that $|C_U| \leqslant \omega$ for every $U \in \mathscr{B}$. Similar to the proof of $(**)$ we can prove that $|C_U| \leqslant \omega$ for every $U \in \mathscr{T}$. Hence $C = \bigcup \{C_W: W \in \mathscr{W}\}$ is countable. Take $x^0 \in X \setminus C$; then for each $W \in \mathscr{W}$, when $x_W > x^0$, we have $W(x^0) = \frac{1}{2}$. So from $\sup\{x_W: W \in \mathscr{W}\} = +\infty$ and the fact that $\mathscr{W}$ is both an open refinement of $\mathscr{V}$ and a Q-cover of X, we know that there are an infinite number of members of $\mathscr{W}$ which take the value $\frac{1}{2}$ at point x and each Q-neighborhood of $(x^0)_{1/2}$ is quasi-coincident with an infinite number of members of $\mathscr{W}$.

4.13. THEOREM. *For every weakly induced fts $(X, \mathscr{T})$ the following conditions are equivalent*:

(i) $(X, \mathscr{T})$ *is S-Lindelöff.*

(ii) *There exists an* $\alpha \in (0, 1]$ *such that* $(X, \mathscr{T})$ *is* α-*Lindelöff.*

(iii) $(X, [\mathscr{T}])$ *is Lindelöff.*

Proof. (i)$\Rightarrow$(ii) Obvious.

(ii)$\Rightarrow$(iii) Let $\mathscr{U} \subset [\mathscr{T}]$ be a cover of X, then $\{\chi_U: U \in \mathscr{U}\} \subset \mathscr{T}$ is a Q-cover of 1_α; it has a countable subfamily $\{\chi_{U_i}: i \in N\}$ which is a Q-cover of 1_α also. Then $\{U_i\}_{i \in N}$ is a countable subcover of $\mathscr{U}$.

(iii)$\Rightarrow$(i) For each $\alpha \in (0, 1]$, let $\mathscr{U} \subset \mathscr{T}$ be a Q-cover of 1_α; then $\{U_{(1-\alpha)}: U \in \mathscr{U}\} \subset [\mathscr{T}]$ is a cover of X and it has a countable subcover $\{(U_i)_{(1-\alpha)}: i \in N\}$, here $\{U_i\}_{i \in N} \subset \mathscr{U}$. Hence $\{U_i\}_{i \in N}$ is a Q-cover of 1_α too. ∎

From 4.13, 4.10, and 3.6 we have

4.14. THEOREM. *If there exists an* $\alpha \in (0, 1]$ *such that a weakly induced regular fts* $(X, \mathscr{T})$ *is* α-*Lindelöff, then* $(X, \mathscr{T})$ *is* S^*-*paracompact.*

5. PARACOMPACT IN METRIC SPACES

5.1. DEFINITION [12] Let L be a completely distributive lattice with order reversing involution $a \mapsto a'$:

(a) A mapping $p: L^X \times L^X \to [0, \infty]$ is called a fuzzy $p \cdot q \cdot$ metric on X if p satisfies following conditions:

[M1] $p(\phi, \lambda) = \infty, \quad \lambda \in L^X \backslash \{\phi\},$

$p(\lambda, \lambda) = 0, \quad \lambda \in L^X,$

$p(\lambda, \phi) = 0, \quad \lambda \in L^X;$

[M2] $p(\lambda, \mu) \leqslant p(\lambda, \beta) + p(\beta, \mu), \quad \lambda, \beta, \mu \in L^X;$

[M3] (i) if $\lambda \subset \mu$, then $p(\lambda, \beta) \geqslant p(\mu, \beta)$ for every $\beta \in L^X$,
(ii) $p(\beta, \bigcup_\alpha \lambda_0) = \bigvee_\alpha p(\beta, \lambda_\alpha)$;

[M4] let r be a positive real number, $\mu \in L^X$, Δ is a set of indexes, and $\lambda_\alpha \in L^X$ for every $\alpha \in \Delta$. If

$$\alpha \in \Delta, \qquad \beta \in L^X, \qquad p(\lambda_\alpha, \beta) < r \Rightarrow \beta \subset \mu,$$

then

$$\gamma \in L^X, \qquad p\left(\bigcup_\alpha \lambda_\alpha, \gamma\right) < r \Rightarrow \gamma \subset \mu.$$

(b) Let p be a fuzzy $p \cdot q \cdot$metric on X. For each $r > 0$ and each $\lambda \in L^X$, let

$$D_r(\lambda) = \bigcup \{\mu \in L^X : p(\lambda, \mu) < r\}.$$

$\{D_r : r > 0\}$ is called the associated neighborhood maps of p and (L^X, p, D_r) is called a fuzzy $p \cdot q \cdot$metric space.

We often omit the word "fuzzy" in the discussion about metric.

5.2. THEOREM [13]. *Let (L^X, p, D_r) be a $p \cdot q \cdot$metric space, then the associated neighborhood map $\{D_r : r > 0\}$ satisfies:*

$$\langle \mathrm{A1} \rangle \quad D_r(\phi) = \varnothing;$$

$$\langle \mathrm{A2} \rangle \quad \lambda \subset D_r(\lambda);$$

$$\langle \mathrm{A3} \rangle \quad D_r\left(\bigcup_\alpha \lambda_\alpha\right) = \bigcup_\alpha D_r(\lambda_\alpha);$$

$$\langle \mathrm{A4} \rangle \quad D_r \circ D_s \leqslant D_{r+s};$$

$$\langle \mathrm{A5} \rangle \quad D_r = \bigvee_{s<r} D_s.$$

Proof. See also [13 or 12]. ▮

5.3. THEOREM [13]. *Let $\{D_r \mid D_r : L^X \to L^X,\ r > 0\}$ be a family of mappings which satisfies conditions $\langle \mathrm{A1} \rangle$–$\langle \mathrm{A5} \rangle$. For each $\lambda, \mu \in L^X$, let*

$$p(\lambda, \mu) = \bigwedge \{r : \mu \subset D_r(\lambda)\},$$

then p is a $p \cdot q \cdot$metric on X and $\{D_r : r > 0\}$ is exactly the associated neighborhood map of p.

Proof. See also [13 or 12]. ▮

5.4. *Remark.* From 5.2 and 5.3 we know that a $p \cdot q \cdot$metric and its associated neighborhood maps are decided by each other, so we can consider only one of them. For example, a $p \cdot q \cdot$metric space can be considered as (L^X, D_r), where the D_r's satisfy $\langle \mathrm{A1} \rangle$–$\langle \mathrm{A5} \rangle$.

5.5. DEFINITION [12]. A $p \cdot q \cdot$metric space (L^X, D_r) is called a pseudometric space if $\{D_r : r > 0\}$ better satisfies the condition:

$$\langle \mathrm{A6} \rangle \quad D_r = D_r^{-1},$$

where D_r^{-1} is defined as

$$D_r^{-1}(\lambda) = \bigcap \{\mu: D_r(\mu') \subset \lambda'\}.$$

5.6. DEFINITION [12]. For a $p \cdot q \cdot$ metric space (L^X, D_r), a topology on X which is generated from the base $\{D_r(\lambda): \lambda \in L^X, r > 0\}$ is called the $p \cdot q \cdot$ metric topology of (L^X, D_r). Fts $(X, \mathscr{T})$ is called $p \cdot q \cdot$ metrizable (resp. pseudometrizable) if there exist a $p \cdot q \cdot$ metric (resp. pseudometric) on X such that the topology it genertes is exactly $\mathscr{T}$.

5.7. *Remark.* As [12] has proved, in a $p \cdot q \cdot$ metric space (L^X, D_r), the family $\{D_r(\lambda): \lambda \in L^X, r > 0\}$ is a base for a topology indeed.

5.8. LEMMA [13]. *Let (I^X, D_r) be a $p \cdot q \cdot$ metric space and x_λ is a fuzzy point, then*

$$\mathscr{Q}(x_\lambda) = \{D_r(x_\alpha): r > 0, \alpha \in (1 - \lambda, 1]\}$$

is a base of the Q-neighborhood system of x_λ.

Proof. From $\langle A2 \rangle$ and 5.6 we have $\mathscr{Q}(x_\lambda) \subset Q(x_\lambda)$. For each $U \in Q(x_\lambda)$, from [12] we know $U = \bigcup \{A \in I^X: \exists r > 0, D_r(A) \subset U\}$, so we have an $A \in I^X$ and an $r > 0$ such that $A(x) > 1 - \lambda$ and $D_r(A) \subset U$. Take $\alpha = A(x)$, then from $\langle A3 \rangle$ we have

$$D_r(x_\alpha) \subset D_r(A) \subset U. \quad \blacksquare$$

5.9. LEMMA. *In every pseudometric space (I^X, D_r) we have*

$$A \bar{q} D_r(B) \Rightarrow D_{r/2}(A)\, \bar{q} D_{r/2}(B).$$

Proof. Since

$$D_{r/2}(D_{r/2}(B)) = D_{r/2} \circ D_{r/2}(B) \subset D_r(B),$$

so from $A \subset D_r(B)'$ we have

$$D_{r/2}(A) \subset D_{r/2}(D_r(B)') = D_{r/2}^{-1}(D_r(B)') = \bigcap \{C': D_{r/2}(C)$$

$$\subset D_r(B)\} \subset D_{r/2}(B'). \quad \blacksquare$$

5.10. DEFINITION. *Let $\mathscr{A}$ be a family of sets and B be a set in fts* $(X, \mathscr{T})$. We say that $\mathscr{A}$ is discrete (resp. $*$-discrete) in B if for each point e in B, there exists a $U \in Q(e)$ such that U is quasi-coincident (resp. intersects) with at most one member of $\mathscr{A}$; we say that $\mathscr{A}$ is σ-discrete (resp. σ^*-discrete (resp. σ^*-discrete) in B if $\mathscr{A}$ can be represented as a countable union of subfamilies and each of these subfamilies is discrete (resp. $*$-discrete) in B; we often omit the word "in B" when $B = X$.

5.11. THEOREM. *If a weakly induced fts $(X, \mathscr{T})$ is pseudometrizable, then for each $\alpha \in (0, 1]$, each α-open Q-cover of X has an open refinement which is both σ-discrete and locally finite in X, and it is an α-Q-cover of X also.*

Proof. Let $\{U_t\}_{t \in T}$ be an open Q-cover of 1_α. Taking a wel-ordering relation $<$ on the index set T, we can take a mapping

$$f: X \to T, \qquad f(x) = \min\{t \in T: x_\alpha q U_t\}.$$

Let $\mathscr{A} = \{x_\lambda : x \in X, \quad \lambda \in (0, 1)\}$, then for each $x_\lambda \in \mathscr{A}$ we have $(U_{f(x)})_{(1-\alpha)} \in Q(x_{1-\lambda})$, so from 5.8 we know that there exist an $r > 0$ and a $\mu \in (\lambda, 1]$ such that $D_r(x_\lambda) \subset D_r(x_\mu) \subset (U_{f(x)})_{(1-\alpha)}$; we can take a mapping

$$g: \mathscr{A} \to N, \qquad g(x_\lambda) = \min\{n \in N: D_{2/2^n}(x_\lambda) \subset (U_{f(x)})_{(1-\alpha)}\}.$$

Let $\beta = \frac{1}{2} \vee (1-\alpha)$, then $\beta \in [\frac{1}{2}, 1)$. We also denote $D(A, r)$ $D_r(A)$ for convenience. For each $i \in N$ and each $t \in T$, let

$$W_{i,t} = \bigcup \{D(x_\lambda, 1/2^i)_{(\beta)}: x_\lambda \in \mathscr{A}, f(x) = t, g(x_\lambda) = i\};$$

then for the same reason used to prove the existence of g, we can take the mapping

$$h_{i,t}: \{x_\lambda \in \mathscr{A}: x_\lambda \in W_{i,t}\} \to N,$$
$$h_{i,t}(x_\lambda) = \min\{n \in N: D(x_\lambda, 1/2^n) \subset W_{i,t}\}.$$

For $i, j \in N$, we inductively define:

$$C_{i,t,j} = \{x_\lambda \in \mathscr{A}: x_\lambda \in W_{i,t}, f(x) = t, g(x_\lambda) = i, h_{i,t}(x_\lambda) = j, \forall s \in T,$$
$$\forall k < i \vee j, \forall l < i \vee j, x_\alpha \bar{q} V_{k,s,l}\},$$

$$V_{i,t,j} = U_t \cap \bigcup \{D(x_\lambda, 1/2^{i+j}): x_\lambda \in C_{i,t,j}\}, \qquad i, j \in N, \quad t \in T,$$

$$\mathscr{V}_{i,j} = \{V_{i,t,j}\}_{t \in T}, \qquad i, j \in N,$$

$$\mathscr{V} = \bigcup_{i,j \in N} \mathscr{V}_{i,j}.$$

Then we have following results:

(1) $\mathscr{V}$ is an open refinement of $\mathscr{U}$.

(2) $\mathscr{V}$ is an α-open Q-coer of X. In fact, for every $x \in X$, let $t = f(x)$, since $x_\alpha q U_t$, so $x \in (U_t)_{(1-\alpha)}$, $(U_t)_{(1-\alpha)} \in Q(x_{1-\beta})$, and there exists an $i' \in N$ and a $\lambda \in (\beta, 1)$ such that $D(x_\lambda, 2/2^{i'}) \subset (U_t)_{(1-\alpha)}$. Then let $i = g(x_\lambda)$ and we have $D(X_\lambda, 2/2^i) \subset (U_t)_{(1-\alpha)}$, $x_\lambda \in D(x_\lambda, 1/2^i)_{(\beta)} \subset W_{i,t}$. Let $j = h_{i,t}(x_\lambda)$; now

if there exists $s \in T$, $k < i \vee j$, $1 < i \vee j$ such that $x_\alpha q V_{k,s,l}$, then we have no thing to prove. If not, then $x_\lambda \in C_{i,t,j}$, $D(x_\lambda, 1/2_{i+j})(x) \geqslant \lambda > \beta \geqslant 1-\alpha$, $V_{i,t,j}(x) \geqslant U_t(x) \wedge D(x_\lambda, 1/2^{i+j})(x) > 1-\alpha$, $x_\alpha q V_{i,t,j}$.

(3) For every $i \in N$ and every pair of t, $t' \in T$, and $t \neq t'$, we have $W_{i,t} \cap W_{i,t'} = \varnothing$. Suppose this is not so, and there is no harm in supposing $t < t'$, then there exist $x_\lambda \in \mathscr{A}$, $y_\mu \in \mathscr{A}$, and $z \in X$ such that $g(x_\lambda) = g(y_\mu) = i$, $f(x) = t$, $f(y) = t'$, $z_1 \in D(x_\lambda, 1/2^i)_{(\beta)} \cap D(y_\mu, 1/2^i)_{(\beta)}$. Since $\beta \geqslant \frac{1}{2}$,

$$D(X_\lambda, 1/2^i)(z) + D(y_\mu, 1/2^i)(z) > \beta + \beta \geqslant \tfrac{1}{2} + \tfrac{1}{2} = 1.$$

$D(x_\lambda, 1/2^i)$ and $D(y_\mu, 1/2^i)$ are quasi-coincident at the point z. But from $f(y) > f(x) = t$ we have $(U_t)_{(1-\alpha)}(y) = 0$, from $D(x_\lambda, 2/2^i) \subset (U_t)_{(1-\alpha)}$, and from 5.9 we know that $D(x_\lambda, 1/2^i)$ i not quasi-coincident with $D(y_\mu, 1/2^i)$; this is a contradiction.

(4) For each $x_\lambda \in \mathscr{A}$ such that $D(x_\lambda, 1/2^m) \subset (V_{k,s,l})_{(1-\alpha)}$ $(k, l, m = 1, 2, ...)$, we have that if $i, j \in N$, $i \vee j \leqslant k + l + m$, $t \in T$, then $D(x_\lambda, 1/2^{k+l+m})\, \bar{q} V_{i,t,j}$. In fact, for each $y_\mu \in C_{i,t,j}$, we have $y_\alpha \bar{q} V_{k,s,l}$ and $y \bar{\in} (V_{k,s,l})_{(1-\alpha)}$. Then from $D(x_\lambda, 1/2^m) \subset (V_{k,s,l})_{(1-\alpha)}$ we know $y_\mu \bar{q} D(x_\lambda, 1/2^m)$ and from 5.9 we know $D(x_\lambda, 1/2^{k+l+m})\, \bar{q} D(y_\mu, 1/2^{i+j})$. So from the definition of $V_{i,t,j}$ we know $D(x_\lambda, 1/2^{k+l+m})\, \bar{q} V_{i,t,j}$.

(5) $\mathscr{V}$ is σ-discrete. Let $i, j \in N$, $x \in X$, and $\lambda \in (0, 1]$, take a $\mu \in (1-\lambda, 1)$; then from (3) we know that x_μ intersects with at most one member of the family of sets

$$\left\{ \bigcup \{ D(z_\delta, 1/2^j) : z_\delta \in C_{i,t,j} \} : t \in T \right\}.$$

So from 5.9 we know that $D(x_\mu, 1/2^{i+j}) \in Q(x_\lambda)$ is quasi-coincident with at most one member of the family of sets

$$\left\{ \bigcup \{ D(z_\delta, 1/2^{i+j}) : z_\delta \in C_{i,t,j} \} : t \in T \right\}.$$

Hence from the definition of $V_{i,t,j}$ we know that the Q-neighborhood $D(x_\mu, 1/2^{i+j})$ of x_λ is quasi-coincident with at most one member of $\mathscr{V}_{i,j}$.

(6) $\mathscr{V}$ is locally finite. Let $x \in X$, $\lambda \in (0, 1]$, from (2) we know that there exist $k, l \in N$, $s \in T$ such that $V_{k,s,l}(x) > -\alpha$, so $(V_{k,s,l})_{(1-\alpha)} \in Q(x_\lambda)$. From 5.8 we know that there exists a $\mu \in (1-\alpha, 1)$ and an $m \in N$ such that $D(x_\mu, 1/2^m) \subset (V_{k,s,l})_{(1-\alpha)}$, so from (4) and (5) we know that only at that time $i \vee j < k + l + m$, $D(x_\mu, 1/2^{k+l+m})$ has the possibility to be quasi-coincident with one and at most one member of $\mathscr{V}_{i,j}$. Hence the Q-neighborhood $D(x_\mu, 1/2^{k+l+m})$ of x_λ is quasi-coincident with at most

$$|\{(i, j) \in N \times N : i \vee j < k + l + m\}| = (k + l + m - 1)^2$$

members of $\mathscr{V}$ and $\mathscr{V}$ is locally finite. ∎

From 5.11 and 3.6 we have

5.12. THEOREM. *If weakly induced fts $(X, \mathscr{T})$ is pseudometrizable, then $(X, \mathscr{T})$ is S^*-paracompact.*

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