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Note

P-immune sets with holes lack self-reducibility properties[☆]

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Abstract

No P-immune set having exponential gaps is positive-Turing self-reducible.
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1. Introduction

A set is P-immune if it is infinite yet has no infinite P subsets ([5], see also [4,13]). That is, each P-immune set is so hard that any polynomial-time machine that avoids accepting elements of its complement can in fact recognize only a finite number of its elements. Informally put, P sets cannot well-approximate it from the inside. P-immunity has been extensively studied in complexity theory.

We are interested in the following issue: Does P-immunity have any repercussions regarding self-reducibility properties? In particular, does P-immunity ever preclude certain self-reducibilities? We know of at least three papers in the literature that prove results in this direction.

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The paper “Strong self-reducibility precludes strong immunity” [9] proves that the complexity class NT is not P-balanced-immune. NT is a complexity class made up exactly of sets having a specific 1-query-fixed-truth-table self-reducibility property. That paper thus realizes the on-target intuition, due to Eric Allender (see the acknowledgments section of [9]), that “the restrictive self-reducibility structure of NT [should] constrain one’s ability to achieve strong separations from P” [9]. Also, Hemaspaandra and Silvestri [8] have proven a somewhat indirect but still interesting link between immunity and self-reducibility: If all sparse disjunctive-Turing self-reducible sets are in P, then $\text{FewP} \cap \text{coFewP}$ is not P-bi-immune.

Another paper exploring the extent to which P-immunity might conflict with self-reducibility properties is a 1990 paper by Kämper that proves that P-immune sets having *double-exponentially* large holes can never be *disjunctive-Turing* self-reducible [11]. Our work was motivated by Kämper’s paper and by the desire to see in what further ways P-immunity may preclude self-reducibility.

We prove that P-immune sets having *exponentially* large holes can never be *positive-Turing* self-reducible (or even locally left-positive-Turing *word-decreasing*-self-reducible). We prove also other related results.

2. Preliminaries

We assume the reader to be familiar with the basic concepts of complexity theory [16,3]. Throughout the paper all logarithms are base 2. The following reduction types will be used in this paper.

Definition 2.1. Let A and B be sets and M be a Turing machine.

- (1) We say that $A \leq_{\text{Turing}}^p B$ via M (“ A Turing reduces to B via M ”) if M is a deterministic polynomial-time Turing machine and $A = L(M^B)$ (see [14]).
- (2) We say that $A \leq_{\text{positive-Turing}}^p B$ via M (“ A positive-Turing reduces to B via M ”) if M is a deterministic polynomial-time Turing machine, $A = L(M^B)$, and for all sets C and D such that $C \subseteq D$ it holds that $L(M^C) \subseteq L(M^D)$ [19].
- (3) We say that $A \leq_{\text{disjunctive-Turing}}^p B$ via M (“ A disjunctive-Turing reduces to B via M ”) if M is a deterministic polynomial-time Turing machine, $A = L(M^B)$, and for all x , $x \in A$ if and only if $M^B(x)$ generates at least one query that is a member of B (see [14]).
- (4) We say that $A \leq_{\text{lpos-Turing}}^p B$ via M (“ A locally left-positive-Turing reduces to B via M ”) if M is a deterministic polynomial-time Turing machine, $A = L(M^B)$, and for all sets C , $L(M^{B-C}) \subseteq L(M^B)$ [7].

Self-reducibility is a central notion in complexity theory (see [10]). It appeared in concept first in Schnorr [18], and was formalized and extended by Meyer and Paterson [15], Balcázar [2], and others.

Definition 2.2. Let A and B be sets.

- (1) For any r for which “ $A \leq_r^p B$ via M ” has been defined, $A \leq_r^p B$ is defined as meaning there is a deterministic polynomial-time machine M such that $A \leq_r^p B$ via M .
- (2) For any $r \neq$ disjunctive-Turing for which “ $A \leq_r^p B$ via M ” has been defined, A is said to be r self-reducible if there is a deterministic polynomial-time machine M such that (see [2,3])
 - (a) $A \leq_r^p A$ via M , and
 - (b) on each input x , $M^A(x)$ queries only strings of length strictly less than $|x|$.
- (3) For any $r \neq$ disjunctive-Turing for which “ $A \leq_r^p B$ via M ” has been defined, A is said to be r word-decreasing-self-reducible if there is a deterministic polynomial-time machine M such that (see [2])
 - (a) $A \leq_r^p A$ via M , and
 - (b) on each input x , $M^A(x)$ queries only strings that are lexicographically strictly less than x .

Under the above definition, if we had not explicitly excluded the case of disjunctive-Turing reductions, only the empty set would be disjunctive-Turing self-reducible and only the empty set would be disjunctive-Turing word-decreasing self-reducible. The reason is that there is no way to get a “first” string into the set. Many textbooks are a bit careless on this point. However, careful definitions, such as that of Ambos-Spies and Kämper [1] of disjunctive-Turing self-reducibility, avoid this problem. (The same issue of course exists regarding disjunctive-truth-table self-reducibility, conjunctive-truth-table self-reducibility, and conjunctive-Turing self-reducibility, and is handled analogously.)

Definition 2.3. Let A and B be sets and M be a Turing machine.

- (1) We say that $A \leq_{\text{disjunctive}'\text{-Turing}}^p B$ via M (“ A disjunctive’-Turing reduces to B via M ”) if M is a deterministic polynomial-time Turing machine, $A = L(M^B)$, and M has the following acceptance behavior: on each input it accepts exactly if either
 - (a) it halted in an accepting state without asking any queries, or
 - (b) it asked at least one query and at least one query it asked received the answer “yes”.
- (2) (essentially [1]) A set B is said to be disjunctive-Turing self-reducible if there is a deterministic polynomial-time machine M such that
 - (a) $B \leq_{\text{disjunctive}'\text{-Turing}}^p B$ via M , and
 - (b) on each input x , $M^B(x)$ queries only strings of length strictly less than $|x|$.
- (b) A set B is said to be disjunctive-Turing word-decreasing-self-reducible if there is a deterministic polynomial-time machine M such that
 - (a) $B \leq_{\text{disjunctive}'\text{-Turing}}^p B$ via M , and
 - (b) on each input x , $M^B(x)$ queries only strings that are lexicographically strictly less than x .

Of course, for each r , every r self-reducible set is r word-decreasing-self-reducible. (For explicitness, in some of our theorems that apply to both we will mention both in the theorem statements.)

If one wishes to define conjunctive-Turing self-reducibility and conjunctive-Turing word-decreasing self-reducibility one, for reasons analogous to those outlined above, has to make the same type of special case as is done for disjunctive-Turing self-reducibility in Definition 2.3.

It is well known that $A \leq_{\text{disjunctive-Turing}}^p B$ implies $A \leq_{\text{positive-Turing}}^p B$ which in turn implies $A \leq_{\text{lpos-Turing}}^p B$. And there exist sets A', B', A'', B'' such that $A' \leq_{\text{lpos-Turing}}^p B'$ yet $A' \not\leq_{\text{positive-Turing}}^p B'$ [7], and $A'' \leq_{\text{positive-Turing}}^p B''$ yet $A'' \not\leq_{\text{disjunctive-Turing}}^p B''$ [14]. That is, $\leq_{\text{lpos-Turing}}^p$ is a more broadly applicable reduction than $\leq_{\text{positive-Turing}}^p$, which in turn is a more broadly applicable reduction than $\leq_{\text{disjunctive-Turing}}^p$.

Self-reducible sets have been intensively studied. It is well known that all disjunctive-Turing self-reducible sets are in NP and also in $E = \bigcup_{c>0} \text{DTIME}(2^{cn})$. Balcázar showed that in fact every Turing word-decreasing-self-reducible set is in E.

Theorem 2.4 (Balcázar [2]). *Every Turing word-decreasing-self-reducible set is in E.*

Immunity is a concept developed to study the degree of separation that can be achieved between classes (see [17]). In particular, P-immunity [5] (see also [4,13]) is a well-studied concept.

Definition 2.5 (see [17]). Let \mathcal{C} be any class. A set B is called \mathcal{C} -immune if B is infinite yet no infinite subset of B belongs to \mathcal{C} .

Definition 2.6. A set A has exponential-size gaps (E -gaps) if the following holds

$$(\exists c > 0)(\forall n \in \mathbb{N})(\exists m > n)[\{z \in A \mid m < |z| \leq 2^{cm}\} = \emptyset].$$

A set A has double-exponential-size gaps if $(\forall n \in \mathbb{N})(\exists m \geq n)[\{z \in A \mid m < |z| \leq 2^{2^m}\} = \emptyset]$. It has been shown by Kämper [11] that no P-immune set A having double-exponential-size gaps can be disjunctive-Turing self-reducible. Kämper proves his result for the model, different than that of this paper, in which self-reducibility is defined with respect to all polynomially well-founded orders.

In Section 3 we show that, in fact, no P-immune set having exponential-size gaps is positive-Turing self-reducible (or even locally left-positive-Turing word-decreasing-self-reducible). In Section 4, we study related issues such as consequences for SAT that would follow from NP-hardness for sets having exponential-sized gaps.

3. Immunity with holes versus self-reducibility

We now state our theorem about immunity, self-reducibility, and holes.

Theorem 3.1. *No P-immune set having E -gaps is locally left-positive-Turing word-decreasing-self-reducible.*

Corollary 3.2. (1) No P-immune set having E-gaps is positive-Turing self-reducible or positive-Turing word-decreasing-self-reducible.

(2) No P-immune set having E-gaps is disjunctive-Turing self-reducible or disjunctive-Turing word-decreasing-self-reducible.

(3) No P-immune set having E-gaps is locally left-positive-Turing self-reducible.

Proof of Theorem 3.1. Let A be a locally left-positive-Turing word-decreasing-self-reducible set having E-gaps. If A is finite, it is trivially not P-immune. So suppose that A is infinite. It suffices to show that A has an infinite subset in P.

Let $c > 0$ be a constant such that $(\forall n \in \mathbb{N})(\exists m > n)[\{z \in A \mid m < |z| \leq 2^{cm}\} = \emptyset]$. Let M be a deterministic polynomial-time Turing machine witnessing the locally left-positive-Turing word-decreasing-self-reducibility of A , in other words,

- (1) $L(M^A) = A$,
- (2) for all sets C , $L(M^{A-C}) \subseteq L(M^A)$, and
- (3) on each input x , $M^A(x)$ queries only strings that are lexicographically strictly less than x .

By Theorem 2.4, there exist a constant $d > 0$ and a deterministic 2^{dn} -time-bounded Turing machine M_e such that $L(M_e) = A$.

Consider the following deterministic Turing machine M' :

- (1) On input x simulate the action of $M(x)$ while answering the queries generated during that simulation as follows:
 - (a) Every query q with $|q| \leq (\log |x|)/c$ is answered according to the outcome of $M_e(q)$, i.e., if $|q| \leq (\log |x|)/c$ then $M_e(q)$ is simulated and the query q generated by $M(x)$ is answered “yes” if $M_e(q)$ accepts and is answered “no” otherwise.
 - (b) Every query q with $(\log |x|)/c < |q|$ is answered “no”.
- (2) Accept if and only if the simulation of $M(x)$, answering the queries (generated during the simulation of $M(x)$) as described above, accepts.

It is not hard to see that $M'(x)$ runs in time polynomial in $|x|$. Let $B = L(M')$. It follows that $B \in P$.

Claim 1. $B \subseteq A$.

Let x be a string such that $x \in B$, in other words, $x \in L(M')$. Since $L(M^A) = A$ and $L(M_e) = A$, $M'(x)$ gets the right answer to each query q that $M(x)$ generates that satisfies $|q| \leq (\log |x|)/c$. Since all other queries are answered “no”, the outcome of $M'(x)$ is identical to the outcome of $M^{(A^{\leq (\log |x|)/c})}(x)$. Thus $x \in L(M^{(A^{\leq (\log |x|)/c})})$. But note that due to the properties of M , $L(M^{(A^{\leq (\log |x|)/c})}) \subseteq L(M^A)$. Hence $x \in A$.

Claim 2. B is infinite.

Let $m_0 < m_1 < m_2 < \dots$ be an infinite sequence of natural numbers such that for all $i \geq 0$, both $\{z \in A \mid m_i < |z| \leq 2^{cm_i}\} = \emptyset$ and $\{z \in A \mid 2^{cm_i} < |z| \leq m_{i+1}\} \neq \emptyset$ hold. Such a sequence exists since A is infinite and has E-gaps. For all $i \geq 0$, define $z_i = \min\{z \in A \mid 2^{cm_i} < |z| \leq m_{i+1}\}$, where we minimize with respect to the lexicographical order

on strings. Note that for all $i \geq 0$, due to $(\log |z_i|)/c > m_i$ all strings y , $y <_{lex} z_i$, of length $|y| > (\log |z_i|)/c$ are not in A since they fall into the gap that extends at least down to the length $m_i + 1$ and that stretches at least up to just before z_i . Informally, z_i is the (lexicographically) first string in A beyond one of A 's exponential-size gaps.

It is clear that $\{z_i \mid i \geq 0\}$ is an infinite set. In order to show that B is infinite it certainly suffices to show that $\{z_i \mid i \geq 0\} \subseteq B$. But this follows from the fact that by construction for all $i \geq 0$ during the run of $M'(z_i)$ we correctly simulate the work of $M^A(z_i)$ since the answers to all queries q , $|q| \leq (\log |z_i|)/c$, generated by $M^A(z_i)$ are correctly found with the help of $M_e(q)$ and the answers to all queries q , $|q| > (\log |z_i|)/c$, generated by $M^A(z_i)$ are truly “no” since all those queries fall into the gap above the length m_i .

This completes the proof of Claim 2 and the proof of the theorem. \square

4. Emptiness testing and NP-hardness

In order to extend Theorem 3.1 to arbitrary Turing self-reducible sets—as opposed to requiring positivity properties for the self-reducibility—it appears crucial to have some knowledge of where the set has its holes.

Definition 4.1. A set B is said to be emptiness-testable if and only if

$$\{1^i \mid B^{=i} = \emptyset\} \in P.$$

Proposition 4.2. A set B is emptiness-testable if and only if

$$EmptyInterval_B = \{0^i 1^j \mid i \leq j \wedge \{z \in B \mid i \leq |z| \leq j\} = \emptyset\} \in P.$$

The proof of the proposition is immediate.

Theorem 4.3. No P-immune emptiness-testable set having E-gaps is Turing word-decreasing-self-reducible.

Proof. The proof is similar to the proof of Theorem 3.1. Let A be an infinite emptiness-testable Turing word-decreasing-self-reducible set having E-gaps. If A is finite then A is not P-immune, so we henceforth consider only the case that A is infinite. We will show that A has an infinite subset in P.

Let $c > 0$ be a constant such that $(\forall n \in \mathbb{N})(\exists m > n)[\{z \in A \mid m < |z| \leq 2^{cm}\} = \emptyset]$. Let M be a deterministic polynomial-time Turing machine witnessing the Turing word-decreasing-self-reducibility of A . By Theorem 2.4, there exist a constant $d > 0$ and a deterministic 2^{dn} -time-bounded Turing machine M_e such that $L(M_e) = A$. Let $T_e = \{1^i \mid A^{=i} = \emptyset\}$. By assumption $T_e \in P$.

Consider the following deterministic Turing machine M' :

- (1) On input x simulate the work of $M(x)$ while answering the generated queries as follows (initialize $u = 1$; the variable u will work as a flag to indicate whether the

answers to certain oracle queries are correct):

- (a) Every query q with $|q| \leq (\log |x|)/c$ is answered according to the outcome of $M_e(q)$, i.e., simulate $M_e(q)$ and answer “yes” to the query q if $M_e(q)$ accepts and answer “no” otherwise.
 - (b) Every query q with $(\log |x|)/c < |q|$ is answered “no”. If $1^{|q|} \notin T_e$, that is if $A^{|q|} \neq \emptyset$, set $u = u + 1$, otherwise leave u unchanged. (Informally put, we change the value of u if we answered “no” to a query that is of a length at which A is *not* empty.)
- (2) Accept if and only if both the simulation of $M(x)$ while answering the queries as described above accepts and $u = 1$.

From here on the proof proceeds in analogy to the proof of Theorem 3.1. \square

It is not hard to see that the information about the emptiness of a set A can also be present in form of one bit of advice *per length*. Thus, we have the following corollary. (Note, $P/1 = \{L \mid (\exists B \in P)(\exists f : 1^* \rightarrow \{0, 1\})(\forall x)[x \in L \Leftrightarrow \langle x, f(1^{|x|}) \rangle \in B]\}$, see [12].)

Corollary 4.4. *No P/1-immune set having E-gaps can be Turing word-decreasing-self-reducible or Turing self-reducible.*

We mention in passing that the proof of Theorem 4.3 shows something slightly stronger than claimed in the statement of Theorem 4.3, namely, that any self-reducible set having E-gaps can be P-immune only if its gaps are “hard to find”, in other words, no FP function should be able to recognize an infinite number of its gaps.

Can emptiness-testable sets having E-gaps be NP-hard? If this would be the case then it would follow from Theorem 4.3 that such an NP-hard set could not be P-immune. A few definitions will be helpful in studying the above question. By NP-hard we always mean \leq_m^P -hard for NP.

Definition 4.5. (1) Let \mathcal{C} be a complexity class. A set D has \mathcal{C} -easiness bands if, for every $\ell > 1$, there exists a set $B \in \mathcal{C}$ such that, for infinitely many $n \in \mathbb{N}$,

$$(D \Delta B) \cap \{z \mid n \leq |z| \leq n^\ell\} = \emptyset,$$

where Δ denotes the symmetric difference of sets, i.e., $D \Delta B = (D - B) \cup (B - D)$.

(2) Let \mathcal{C} be any complexity class. A set D is said to have obvious \mathcal{C} -easiness bands if, for every $\ell > 1$, there exist a set $B \in \mathcal{C}$ and an infinite tally set $T \in \mathcal{C}$ such that, for all $1^n \in T$,

$$(D \Delta B) \cap \{z \in |n \leq |z| \leq n^\ell\} = \emptyset.$$

If we wanted to apply part 2 of Definition 4.5 to classes much less nicely behaved than P, we would want to replace the tally-set- $T \in \mathcal{C}$ condition with a requirement that the tally set be \mathcal{C} -printable with respect to some natural printability notion corresponding to \mathcal{C} [6]. However, in this paper we will use the definition only as applied to P.

Theorem 4.6. (1) *If any set in E having E-gaps is NP-hard, then SAT (and indeed all positive-Turing word-decreasing-self-reducible NP sets) has P-easiness bands.*

(2) If any emptiness-testable set in E having E-gaps is NP-hard, then SAT (and indeed all positive-Turing word-decreasing-self-reducible NP sets) has obvious P-easiness bands.

Proof. Regarding part 1 of the theorem, let A be a set in E having E-gaps. Let $c > 0$ be a constant such that for infinitely many $m \in \mathbb{N}$ it holds that $\{z \in A \mid m < |z| \leq 2^{cm}\} = \emptyset$. Since $A \in E$ there exist a constant $d > 0$ and a 2^{dn} -time-bounded Turing machine M_e such that $L(M_e) = A$. Suppose that A is NP-hard. Let f be a polynomial-time computable such reduction, i.e., for all x , $x \in \text{SAT} \Leftrightarrow f(x) \in A$. Since SAT is positive-Turing self-reducible (even disjunctive-Turing self-reducible), there exists a deterministic polynomial-time machine M such that

- (1) $\text{SAT} = L(M^{\text{SAT}})$,
- (2) on each input x , $M^A(x)$ queries only strings of length strictly less than $|x|$, and
- (3) for all C and D such that $C \subseteq D$ it holds that $L(M^C) \subseteq L(M^D)$.

We will show that, for every $\ell > 1$, there exists a P set B such that, for infinitely many $n \in \mathbb{N}$,

$$(\text{SAT} \Delta B) \cap \{z \mid n \leq |z| \leq n^\ell\} = \emptyset.$$

We will do so by showing that, for every $k > 1$, there exist a P set $B' \subseteq \text{SAT}$ and an infinite tally set C such that for all $1^n \in C$,

$$\text{SAT} \cap \{z \mid n^{1/k} \leq |z| \leq n\} \subseteq B'.$$

Although the format here is $n^{1/k}$ versus n^1 rather than n^1 versus n^ℓ , it is not hard to see that this suffices. Let p be polynomial such that for all x and for all $n \in \mathbb{N}$, $|f(x)| \leq p(|x|)$ and $p(n) < p(n + 1)$.

Let $k > 1$. Consider the following deterministic Turing machine M' :

- (1) On input x , $|x| = n$, simulate $M(x)$ and each time $M(x)$ asks a query q to SAT compute $f(q)$ and answer the query “ $q \in \text{SAT}?$ ” as follows:
 - (a) If $|f(q)| \leq \log(n^k p(n^k))/c$ then answer “yes” if and only if $M_e(f(q))$ accepts and “no” otherwise.
 - (b) If $\log(n^k p(n^k))/c < |f(q)|$ then answer “no”.
- (2) Accept if and only if the simulation of $M(x)$, answering the queries as described above, accepts.

It is not hard to see that the above machine M' runs in time polynomial in n . Let $B' = L(M')$. Since M is globally positive and the above machine answers queries by exploiting the many-one reduction from SAT to A or by answering “no”, it follows that $B' \subseteq \text{SAT}$.

Define $C = \{1^n \mid (\exists m \in \mathbb{N})[\{z \in A \mid m < |z| \leq 2^{cm}\} = \emptyset \wedge p(n) < 2^{cm} \leq np(n)]\}$. Note that C is infinite. To see this let \hat{n} be such that, for all $n \geq \hat{n}$, $p(n + 1) < np(n)$. Such an \hat{n} clearly exists, since p is a monotonic polynomial of degree greater than zero. Now let m be any natural number such that $p(\hat{n}) < 2^{cm}$ and $\{z \in A \mid m < |z| \leq 2^{cm}\} = \emptyset$. Define $n_m = \max\{n' \mid p(n') < 2^{cm}\}$. Note $n_m \geq \hat{n}$ and $2^{cm} \leq p(n_m + 1) < n_m p(n_m)$. It follows that $n_m \in C$. Since there are infinitely many m satisfying both $p(\hat{n}) < 2^{cm}$ and $\{z \in A \mid m < |z| \leq 2^{cm}\} = \emptyset$, it follows that C is an infinite set.

We are now prepared to show that for all $1^n \in C$,

$$\text{SAT} \cap \{z \mid n^{1/k} \leq |z| \leq n\} \subseteq B'.$$

Let $1^n \in C$. In light of the definition of C , there exists some $m \in \mathbb{N}$ such that $\{z \in A \mid m < |z| \leq 2^{cm}\} = \emptyset$ and $p(n) < 2^{cm} \leq np(n)$. Choose such an m (which implicitly is m_n). Note that $2^{cm} \leq np(n)$ implies $m \leq \log(np(n))/c$. Hence any string y satisfying $\log(np(n))/c < |y| \leq p(n)$ cannot be in A .

Let z be such that $n^{1/k} \leq |z| \leq n$ and suppose that $z \in \text{SAT}$. Note that $n \leq |z|^k$. So, since \log and p are monotonic, $m \leq \log(|z|^k p(|z|^k))/c$, and of course $p(|z|) \leq p(n) < 2^{cm} \leq np(n) \leq |z|^k p(|z|^k)$. This implies that any string y satisfying $\log(|z|^k p(|z|^k))/c < |y| \leq p(|z|)$ cannot be in A .

Now consider the action of $M'(z)$. $M'(z)$ essentially simulates the work of $M(z)$. Note that for all queries q generated by $M(z)$, $|q| \leq |z|$ and hence $|f(q)| \leq p(|z|)$. Furthermore, any query q with $|f(q)| \leq \log(|z|^k p(|z|^k))/c$ is correctly answered during the simulation of $M(z)$ in our algorithm since $L(M_e) = A$. On the other hand, for all queries q with $|f(q)| > \log(|z|^k p(|z|^k))/c$ (recall that those queries are answered “no” by $M'(z)$ during the simulation of $M(z)$) $f(q)$ is in the gap associated with m (i.e., the gap that extends at least down to the length $m + 1$ and stretches at least up to the length 2^{cm}), in other words, $f(q) \notin A$ and consequently $q \notin \text{SAT}$. This shows that during the run of $M'(z)$ all queries generated in the simulation of $M(z)$ are answered correctly and hence $z \in \text{SAT}$ implies $z \in B'$.

So we showed that, under the assumption of part 1 of the theorem, SAT has P-easiness bands. The same proof works for any positive-Turing self-reducible NP set, or indeed, with the obvious minor change in the proof, for any positive-Turing word-decreasing-self-reducible NP set. This completes the proof of part 1.

Regarding the proof of part 2 we note that if A is emptiness-testable, then the above-defined set C is in P. This can be seen easily in light of the definition of C , using also Proposition 4.2. Although the set C of this proof marks upper ends of bands in contrast with part 2 of Definition 4.5 which requires the marking of the lower ends, it is not hard to see that this suffices, though due to rounding issues one has to be slightly careful. In particular, if we wish to prove bands of the form n -to- n^ℓ , we use the above proof for the value $k = \ell + 1$ to get bands of the form $n^{1/(\ell+1)}$ -to- n and to get an upper-edge-marking set $C \in \text{P}$. The set $C' = \{1^{\lceil j^{1/(\ell+1)} \rceil} \mid 1^j \in C \text{ and } \lceil j^{1/(\ell+1)} \rceil^\ell \leq j\}$ will also be in P, will be infinite, and will serve as the desired lower-edge-of-band marking tally set in the sense of part 2 of Definition 4.5. \square

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