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Microstructure Diffusion Models with Secondary Flux*

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Totally fissured media in which the cells are isolated by the fissure system are effectively described by double porosity models with microstructure. These models contain the geometry of the individual cells or pores in the medium and the flux across their interface with the fissures which surrounds them. We extend these models to include the case of partially fissured media in which a secondary flux effect arises from cell-to-cell diffusion paths. These quasilinear problems are formulated in appropriate spaces for which the cells respond to the local linearization of the fissure pressure. It is shown that they are well-posed and that the solutions depend continuously on parameters that determine the models.

1. INTRODUCTION

The objective here is to develop and investigate a system of partial differential equations known as distributed microstructure models. These arise as models of flow through fractured porous media. A fractured medium consists of a large number of porous and permeable cells separated by a highly developed system of fractures. The advantage of microstructure models over more classical porous media models is that they include the additional information associated with the fine-scale structure of the fracture system. In such a medium the fractures account for a very small fraction of the total volume and most of the storage occurs in the porous cells. However, the bulk of the flow occurs in the fractures due to their very high relative permeability. An accurate model must describe the

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diffusion at two very different scales—the scale of the fissures and the scale of the cells.

One approach to describing this situation is the use of *double-porosity* models. For a region which consists of two finely interspersed materials, the double-porosity approach is to consider averaged properties of both materials existing everywhere in the region as if they were independent parallel flows. At each point in the region, two sets of material properties are defined. If we let u_1 represent the density of fluid in the first material system and u_2 the density in the second, the classical *parallel-flow double porosity model* would have the form

$$\frac{\partial}{\partial t}(au_1) - \nabla \cdot (A\nabla u_1) + \frac{1}{\delta}(u_1 - u_2) = f_1 \tag{1.1.a}$$

$$\frac{\partial}{\partial t}(bu_2) - \nabla \cdot (B\nabla u_2) + \frac{1}{\delta}(u_2 - u_1) = f_2. \tag{1.1.b}$$

Here a and A are functions representing the porosity and permeability respectively of the first equation, and the functions b and B represent the same quantities for the second material. The third term in each equation is a crude representation of the exchange across the intricate interface that separates the two media. See [8] for more information on such models.

One shortcoming of the classical double porosity approach is that it is unable to take into account the geometry of the cells. *Microstructure models* are a refinement of double porosity models in that they consider the fracture system as existing throughout the entire region Ω and containing a given continuous distribution of cells. At each point $x \in \Omega$ there is specified a cell Ω_x. One partial differential equation is specified to describe the global flow in the fracture system Ω and another is specified in each cell Ω_x for the flow internal to the cell Ω_x. The global fluid flow is described by a quasilinear equation of the form

$$\frac{\partial}{\partial t}(a(x)u(x, t)) - \nabla \cdot A(x, \nabla u) + q(x, t) = f(x, t), \qquad x \in \Omega, \tag{1.2.a}$$

and the local flow in each cell is described similarly by

$$\frac{\partial}{\partial t}(b(x, y)U(x, y, t)) - \nabla_y \cdot B(x, y, \nabla_y U) = F(x, y, t), \qquad y \in \Omega_x. \tag{1.2.b}$$

The subscript y on the gradient indicates that the gradient is with respect to the *local* variable y. A gradient operator without any subscript will mean that the gradient is taken with respect to the *global* variable x. For simplicity, Dirichlet boundary conditions are assumed for the global problem, although other conditions could easily be considered. Similarly, we shall set $a(x) = 1$, $b(x, y) = 1$, although any pair of non-negative functions could be obtained by standard techniques [27].

The boundary values for each cell problem are taken from information about the solution to the global equation in the vicinity of that point. The tacit assumption of the microstructure models is that the cells are so small that the global solution u may be effectively approximated over the cell boundary by an appropriate approximation to u. In the usual models with distributed microstructure, the approximation used for this purpose is merely the constant value of the global solution $u(x, t)$. The resulting boundary conditions are either of the "matched" or Dirichlet type in which the concentrations inside and out are assumed equal, or they are of the "regularized" or Robin type in which the difference between inside and outside concentrations drives the flux across the cell boundary. Our objective here is to refine this model in order to more accurately describe the flow through the cell system. If we consider the local coordinate system centered at the middle of the cell Ω_x, the best linear approximation to u on Γ_x is $u(x, t) + \nabla_x u(x, t) \cdot y$. This leads to boundary conditions of the form

$$B(x, s, \nabla_y U) \cdot \nu + \frac{1}{\delta} \partial m(U - u - \beta \nabla u \cdot s) \ni 0, \qquad s \in \Gamma_x, \quad (1.2.c)$$

where ∂m is a monotone function (or graph) and ν is the unit outward normal on Γ_x. When $\beta = 1$, this means that the flux across Γ_x is driven by the difference between the concentration on the inside of the cell and *the best linear approximation* to the concentration in the surrounding fractures. The *constant approximation* corresponds to $\beta = 0$. The monotone ∂m is a generalized Fourier or Newton type relation between the boundary flux and the concentration difference. It is usually a (single-valued) function, but it is useful to allow a multi-valued relation in order to include the case in which a given concentration difference permits a *range* of values of flux.

The term q in the global equation (1.2.a) is an exchange term to describe fluid flow between the fractures and cells, a function whose value at each point x is obtained from the solution to the boundary value problem in Ω_x. This exchange term q consists of two parts, the amount of fluid flowing into the cell Ω_x to be stored and the divergence of the *secondary flux* that is seen from the global region Ω as a result of fluid flowing across the cell Ω_x or, more generally, through the cell system. In the case of a symmetric cell Ω_x, the fluid flow *into* the cell is determined by the history of the *value* of the concentration at that point, and the fluid flow *across* the cell (or through the cell system due to bridging between cells) is driven by the concentration *gradient* in the surrounding global medium. In general, the combined effects of the value and the gradient of concentration on the cell comprise the best linear approximation of the global concentration that is used in the boundary condition (1.2.c). To be precise, the average amount of fluid flowing into this cell is given by

$$\frac{1}{|\Omega_x|}\int_{\Gamma_x} B(x, s, \nabla_y U)\cdot\nu\, ds,$$

where $|\Omega_x|$ denotes the Lebesgue measure of Ω_x, and this contributes to the *cell storage*. The additional contribution to the distributed source q, called the *cell flux* or *secondary flux*, arises from the vector function

$$\frac{1}{|\Omega_x|}\int_{\Gamma_x} B(x, s, \nabla_y U)\cdot\nu s\, ds.$$

This is the apparent flux seen at a point of the global medium due to the difference between the amounts of fluid entering and exiting at symmetric opposite points of the cell boundary. The total exchange term is then given by

$$\begin{aligned} q(x, t) = {} & \frac{1}{|\Omega_x|}\int_{\Gamma_x} B(x, y, \nabla_y U)\cdot\nu\, ds \\ & - \beta\nabla\cdot\left(\frac{1}{|\Omega_x|}\int_{\Gamma_x} B(x, y, \nabla_y U)\cdot\nu s\, ds\right). \end{aligned} \tag{1.2.d}$$

The effect of this new term with $\beta > 0$ is the main objective of this study. See [30] for the case with $\beta = 0$.

In summary, the microstructure model that we will consider consists of Eqs. (1.2.a) and (1.2.b) coupled by the interface boundary condition (1.2.c) and the distributed exchange (1.2.d). We shall refer to this as the *regularized microstructure model*. The limiting case "$\delta \to 0$" corresponds to the condition

$$U = u + \beta\nabla u\cdot s, \qquad s\in\Gamma_x. \tag{1.2.c$'$}$$

on the interface, and we shall include this case, which we call the *matched microstructure model*. We shall show that these nonlinear problems are well posed and that the solutions depend continuously on the regularization parameter $\delta > 0$ and on the secondary flux intensity $\beta > 0$.

Finally we mention that in the case of completely symmetric cells one can separate the effects of storage from those of the secondary flux. This corresponds to a decomposition of the exchange into its even and odd components, respectively. Furthermore, by means of a Green's function representation of the solution of the cell problem (1.2.b) and (1.2.c), the storage and secondary flux contributions in q can be independently expressed as convolutions in time of the values and gradients of concentration, respectively. This leads to a functional partial differential equation of the form

$$\begin{aligned}
&\frac{\partial}{\partial t}(a(x)u(x, t) + k_1(x, \cdot) * u(x, t)) \\
&\qquad - \nabla \cdot (A(x)\, \nabla u(x, t) + k_2(x, \cdot) * \nabla u(x, t)) \\
&\quad = f(x, t), \qquad x \in \Omega,\ t > 0.
\end{aligned}$$

which is known as Nunziato's equation [23]. For the case of $\beta = 0$, see [24] for a very thorough development of the model and its mathematical and numerical analysis.

The system (1.2) with $\beta = 0$, that is, the case of approximation of the global concentration by a function of time at each cell, is similar to those developed in heat conduction [14, see Section 148; 22], physical chemistry [25, 26, 16], soil science [9, 20], and in reservoir modeling [17, 2, 4]. See [28] for bibliographical remarks and perspective. Theory of related systems has been developed in [29, 30, 19]. For the derivation of such systems by homogenization from highly singular "exact" models see [31, 21, 5, 6, 1]. To our knowledge, the case $\beta = 1$ considered here appears for the first time in [3], although in an essentially equivalent discrete form of the linear case, and in [15] in the nonlinear case.

Our plan for the following is to regard (1.2) as an evolution equation in Hilbert space, and we shall construct the operator by appropriately restricting a monotone operator on Banach space. If V is a Banach space we denote its dual by V' and the action of $f \in V'$ on $u \in V$ by $\langle f, u\rangle$. The function $A : V \to V'$ is called *monotone* if $\langle A(u) - A(v), u - uX \geq 0$ for u, $v \in V$. We also consider multi-valued operators which arise as generalized derivatives of convex functions. Thus if $j : V \to \mathbb{R}_\infty \equiv \mathbb{R} \cup \{+\infty\}$ is a convex function, its *subgradient* is the operator ∂j, given by $\partial j(u) = \{f \in V' : f(v - u) \leq j(v) - j(u)$, for all $v \in V\}$. This gives a special class of maximal monotone operators. When V is Hilbert space and $\langle \cdot, \cdot \rangle$ is the scalar product on $V \cong V'$, we say the multi-valued $A : V \to 2^V$ is *monotone* if $\langle f_1 - f_2, u_1 - u_2 \rangle \geq 0$ for $f_i \in A(u_i)$, $i = 1, 2$. A monotone operator A is *maximal monotone* if, additionally, the range, $\text{Rg}(I + A)$, is all of V. See [7, 10] for an exposition of these operators and their applications to partial differential equations.

2. The Abstract Evolution Equation

In the physical description of the problem, a cell Ω_x and a boundary value problem is specified at each point of the global region Ω. In order to state this rigorously, we shall use the notion of a continuous direct sum of Banach spaces. Let Ω be any measurable subset of $\mathbb{R}^n$ and let $L^q(\Omega, L^q(\mathbb{R}^n))$ be the space of (equivalence classes of) Bochner q^{th} integrable functions from Ω into $L^q(\mathbb{R}^n)$. Consider any function $U \in L^q(\Omega \times \mathbb{R}^n)$. If (x, y) represents a pair in the product $\Omega \times \mathbb{R}^n$ then the function of x given by $U(x, \cdot)$ is in $L^q(\Omega, L^q(\mathbb{R}^n))$ by Fubini's theorem. This shows that $L^q(\Omega \times \mathbb{R}^n)$ is contained in $L^q(\Omega, L^q(\mathbb{R}^n))$, and a simple argument with step functions shows that equality holds.

Let Q be a measurable subset of $\Omega \times \mathbb{R}^n$ and let Ω_x be the x-section $\Omega_x = \{y \in \mathbb{R}^n : (x, y) \in Q\}$. It will be necessary to place some technical restrictions on Q. We shall assume that the function giving the Lebesgue measure of Ω_x, $x \mapsto |\Omega_x|$, is in $L^\infty(\Omega)$ and is uniformly bounded away from 0. Identify $L^q(Q)$ as a subspace of $L^q(\Omega \times \mathbb{R}^n) = L^q(\Omega, L^q(\mathbb{R}^n))$ and each $L^q(\Omega_x)$ as a subspace of $L^q(\mathbb{R}^n)$ by zero extension. Thus we can identify

$$L^q(Q) \cong \{U \in L^q(\Omega, L^q(\mathbb{R}^n)) : U(x) \in L^q(\Omega_x), \text{ a.e. } x \in \Omega\}.$$

Denote the right side by $L^q(\Omega, L^q(\Omega_x))$. This is a continuous direct sum of Banach spaces in that at each point $x \in \Omega$, a function in $L^q(\Omega, L^q(\Omega_x))$ takes values in a different Banach space.

In order to define Sobolev spaces and trace maps on the spaces developed above, we need some smoothness requirements on Ω and on the Ω_x's. Assume that Ω and each of the Ω_x's are bounded domains in $\mathbb{R}^n$ and that the boundaries $\partial\Omega_x = \Gamma_x$ are C^2 manifolds of dimension $n - 1$. Assume also that each Ω_x lies locally on one side of its boundary. Let $W^{1,p}(\Omega)$ be the Sobolev space of functions in $L^p(\Omega)$ whose first-order (distributional) derivatives also lie in $L^p(\Omega)$. Define $W^{1,q}(\Omega_x)$ similarly. Let $W_0^{1,p}(\Omega)$ be the closure of C_0^∞ in $W^{1,p}(\Omega)$. For $1 < q < \infty$ we also define

$$L^q(\Omega, W^{1,q}(\Omega_x)) \equiv \left\{U \in L^q(\Omega, L^q(\Omega_x)) : U(x) \in W^{1,q}(\Omega_x), \text{ a.e. } x \in \Omega \right.$$
$$\left. \text{and } \int_\Omega \|U(x)\|^q_{W^{1,q}(\Omega_x)}\, dx < \infty\right\}.$$

Let $\gamma_x : W^{1,q}(\Omega_x) \to L^q(\Gamma_x)$ be the trace maps from each cell to its boundary. Let γ denote the *distributed trace* on $L^q(\Omega, W^{1,q}(\Omega_x))$, defined as

$$\gamma(U)(x, s) \equiv \gamma_x(U(x))(s) \qquad \forall\, x \in \Omega, \forall\, s \in \Gamma_x.$$

Assume that the family of maps $\{\gamma_x\}$ is uniformly bounded so that the distributed trace belongs to $L^q(\Omega, L^q(\Gamma_x))$. It will be convenient to weight the norm in the space $L^q(\Omega, L^q(\Gamma_x))$ to include a scaling factor. Define a function w on Ω by

$$w(x) = 1/|\Omega_x|.$$

For $U \in L^q(\Omega, L^q(\Gamma_x))$, we define the norm of U as

$$\|U\|^q_{L^q(\Omega, L^q(\Gamma_x))} \equiv \int_\Omega \|U\|^q_{L^q(\Gamma_x)} \, w(x)\, dx.$$

Since w is bounded away from zero and is also bounded above, the same elements belong to $L^q(\Omega, L^q(\Gamma_x))$ with this norm as with the more standard norm (i.e., with $w \equiv 1$).

It is occasionally necessary to extend a function $u \in L^q(\Omega)$ as a function in $L^q(\Omega, L^q(\Gamma_x))$. We define an embedding λ between the two spaces given by constant extension, i.e., $(\lambda u)(x, s) = u(x)$, $x \in \Omega$, $s \in \Gamma_x$.

Assume that we are given a function $\hat{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ which is measurable in its first component and continuous in the second. Assume also that there exist constants $c, c_0 > 0$, $1 < p < \infty$, and functions $g_1 \in L^{p'}(\Omega)$, $g_0 \in L^1(\Omega)$ such that for almost every $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^n$,

$$|\hat{A}(x, \xi)| \leq c|\xi|^{p-1} + g_1(x), \tag{2.1.a}$$

$$\langle \hat{A}(x, \xi) - \hat{A}(x, \eta), \xi - \eta \rangle \geq 0, \tag{2.1.b}$$

$$\hat{A}(x, \xi) \cdot \xi \geq c_0|\xi|^p - g_0(x). \tag{2.1.c}$$

Define the operator $\mathcal{A} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ by

$$\mathcal{A}u(\varphi) \equiv \int_\Omega \hat{A}(x, \nabla u(x))\, \nabla \varphi(x)\, dx, \qquad u, \varphi \in W_0^{1,p}(\Omega).$$

Define Au to be the restriction of $\mathcal{A}u$ to $C_0^\infty(\Omega)$ so that $Au = -\nabla \cdot \hat{A}(\cdot, \nabla u)$ in the sense of distributions for each $u \in W_0^{1,p}(\Omega)$. It is well known that (2.1) implies that $\mathcal{A}$ is continuous, bounded, and monotone [7].

Next, we develop the operator $\mathcal{B}$ on $L^q(\Omega, W^{1,q}(\Omega_x))$ in a similar manner. Recall that Q is a subset of $\Omega \times \mathbb{R}^n$. Assume that we are given a function $\hat{B} : Q \times \mathbb{R}^n \to \mathbb{R}^n$ and a $1 < q \leq p$. Assume that $\hat{B}$ is measurable in its first two components and continuous in the third. Finally, assume that there exist functions $h_1 \in L^{q'}(Q)$ and $h_0 \in L^1(Q)$ such that $\hat{B}$ satisfies for almost every $(x, y) \in Q$ and all $\xi, \eta \in \mathbb{R}^n$:

$$|\hat{B}(x, y, \xi)| \leq c|\xi|^{q-1} + h_1(x, y), \tag{2.2.a}$$

$$\langle \hat{B}(x, y, \xi) - \hat{B}(x, y, \eta), \xi - \eta \rangle \geq 0, \tag{2.2.b}$$

$$\hat{B}(x, y, \xi) \cdot \xi \geq c_0|\xi|^q - h_0(x, y). \tag{2.2.c}$$

For each $x \in \Omega$ define the operator $\mathcal{B}_x : W^{1,q}(\Omega_x) \to W^{1,q}(\Omega_x)'$ by

$$\mathcal{B}_x w(v) \equiv \int_{\Omega_x} \hat{B}(x, y, \nabla_y w(y)) \cdot \nabla_y v(y)\, dy, \qquad w, v \in W^{1,q}(\Omega_x).$$

Define $B_x w$ to be the restriction of $\mathcal{B}_x w$ to $C_0^\infty(\Omega_x)$ so that

$$B_x w = -\nabla_y \cdot \hat{B}(x, \cdot, \nabla_y w)$$

in the sense of distributions on Ω_x for each $w \in W^{1,q}(\Omega_x)$. Define the corresponding distributed operator $\mathcal{B} : L^q(\Omega, W^{1,q}(\Omega_x)) \to L^{q'}(\Omega, W^{1,q}(\Omega_x)')$ by

$$\mathcal{B}U(\Phi) \equiv \int_\Omega \mathcal{B}_x(U(x))\Phi(x)\, dx, \qquad U, \Phi \in L^q(\Omega, W^{1,q}(\Omega_x)).$$

This operator is likewise continuous, bounded, and monotone.

Let $\mathcal{W} = W_0^{1,p}(\Omega) \times L^q(\Omega, W^{1,q}(\Omega_x))$. This will be the *energy space* for our problem. An element of this product space will usually be denoted by a letter overscored with a tilde. The first and second components of this pair will be denoted by the corresponding lower and upper case letters. For example, $\tilde{u}$ denotes the pair $[u, U]$. We shall regard $\mathcal{A}$ as an operator from $\mathcal{W}$ into $\mathcal{W}'$ by $\mathcal{A}[u, U] = [\mathcal{A}u, 0]$. Similarly, we define $\mathcal{B}$ from $\mathcal{W}$ into $\mathcal{W}'$ by $\mathcal{B}[u, U] = [0, \mathcal{B}U]$. Any reference to $\mathcal{A}$ or $\mathcal{B}$ as operators on $\mathcal{W}$ will be understood in this manner.

Next we construct the exchange term coupling the global and local equations. Let

$$T_\beta : \mathcal{W} \to L^q(\Omega, L^q(\Gamma_x))$$

be given by

$$T_\beta[u, U](x, s) = \gamma_x U(x, s) - \lambda u(x) - \beta s \cdot \lambda \nabla u(x), \qquad x \in \Omega,\ s \in \Gamma_x.$$

Note that the operator λ equals $\gamma \circ \lambda_0$, where $\lambda_0 u$ is the constant extension of $u(x)$ to all of Ω_x. The expression $U - \lambda_0 u - y \cdot \lambda_0 \nabla u$ is in $W^{1,q}(\Omega_x)$.

LEMMA 1. *Assume that $1 < q \leq p < \infty$. Then T_β is a continuous linear function from $W_0^{1,p}(\Omega) \times L^q(\Omega, W^{1,q}(\Omega_x))$ to $L^q(\Omega, L^q(\Gamma_x))$.*

Proof. First we note that (pointwise a.e.)

$$|T_\beta[u, U]|^q \leq k_1(|\gamma_x U|^q + |\lambda u|^q + \|\lambda \nabla u\|_{l_n^q}^q),$$

since $|a + b + c|^q \leq 4^q(|a|^q + |b|^q + |c|^q)$ and the diameters of the Ω_x's are uniformly bounded. Also there exists another constant k_2 such that

$$\int_\Omega \int_{\Gamma_x} |\gamma U|^q \, ds \, w(x) \, dx \leq k_2 \int_\Omega \|U\|_{W^{1,q}(\Omega_x)}^q \, dx$$

because the distributed trace γ is a bounded linear operator. Finally, there exists a constant k_3 such that

$$\int_\Omega \int_{\Gamma_x} (|\lambda u_n|^q + \|\lambda \, \nabla u_n\|_{l_n^q}^q) \, ds \, w(x) \, dx$$

$$\leq k_3 \int_\Omega (|u_n|^q + \|\nabla u_n\|_{l_n^q}^q) \, dx$$

$$= k_3 \|u\|_{W^{1,q}(\Omega)}^q,$$

since Ω is bounded and the $W^{1,q}$ norm is dominated by the $W^{1,p}$ norm. ∎

The exchange term will ge given by a function m defined as follows. Let $m : \mathbb{R} \to \mathbb{R}^+$ be convex and satisfy the growth conditions

$$c_0|x|^q \leq m(x) \leq C|x|^q, \qquad x \in \mathbb{R}. \tag{2.3}$$

For $g \in L^q(\Omega, L^q(\Gamma_x))$, define

$$M(g) = \int_\Omega \int_{\Gamma_x} m(g) \, ds \, w(x) \, dx.$$

LEMMA 2. *M is a proper, convex, continuous function on the space*

$$L^q(\Omega, L^q(\Gamma_x)).$$

Also, $f \in \partial M(g)$ if and only if $f \in L^{q'}(\Omega, L^{q'}(\Gamma_x))$ and it satisfies $f(x, s) \in \partial m(g(x, s))$ for almost every $x \in \Omega$ and $s \in \Gamma_x$.

Proof. M is proper and convex because m is proper and convex. The growth estimates (2.3) show that M is a bounded function. A lower semi-continuous convex function is continuous on the interior of its domain. Since the domain of M is all of $L^q(\Omega, L^q(\Gamma_x))$, M is continuous. The characterization of the subgradient is standard. ∎

Finally we define

$$M_\beta(\tilde{u}) = \int_\Omega \int_{\Gamma_x} m(T_\beta(\tilde{u})(x, s))\, ds\, w(x)\, dx.$$

Having established the continuity of T_β, we may apply the chain rule for subgradients (see [18]) to the composite map $M_\beta = M \circ T_\beta$ to obtain

$$\partial M_\beta = (T_\beta)^* \circ \partial M \circ T_\beta.$$

By using $\langle \cdot, \cdot \rangle$ to indicate duality pairing, the system (1.2) can be stated as follows. Suppose we are given a function $\bar{f}: [0, T] \to L^{p'}(\Omega) \times L^{q'}(Q)$. A solution to the regularized model is a function $\tilde{u}: [0, T] \to \mathcal{W}$ such that $\tilde{u}': [0, T] \to L^{p'}(\Omega) \times L^{q'}(Q)$ and for every $[\varphi, \Phi] \in \mathcal{W}$,

$$\begin{aligned} &\langle [u'(t), U'(t)], [\varphi, \Phi] \rangle + \mathcal{A}(u(t))\varphi + \mathcal{B}(U(t))\Phi \\ &\quad + \frac{1}{\delta}\, \partial M_\beta[u(t), U(t)][\varphi, \Phi] \ni \langle [f(t), F(t)], [\varphi, \Phi] \rangle. \end{aligned} \tag{2.4}$$

In order to characterize the *matched problem* with interface condition (1.2.c′) we shall use the space

$$\mathcal{W}_\beta = \{[u, U] \in \mathcal{W} : T_\beta[u, U] = 0\},$$

the closed subspace of $\mathcal{W}$ obtained as the kernel of T_β. Replacing $\mathcal{W}$ with $\mathcal{W}_\beta$ above gives the matched microstructure model with (1.2.c′). In Section 4 we show that the solutions to the regularized microstructure model converge to the solution of the mached model as $\delta \to 0$.

In order to show that the evolution equation (2.4) is an abstract version of the system (1.2), we expand the abstract formulation in terms of integrals:

$$\begin{aligned} &\int_\Omega \left\{ \frac{\partial}{\partial t} u(x, t)\varphi + \int_{\Omega_x} \frac{\partial}{\partial t} U(x, y, t)\Phi(x, y)\, dy \right. \\ &\qquad \left. + \hat{A}(x, \nabla u(x, t))\, \nabla\varphi(x) \right\} dx \\ &\qquad + \int_\Omega \int_{\Omega_x} \hat{B}(x, y, \nabla_y U(x, y, t))\, \nabla_y \Phi(x, y, t)\, dy\, dx \\ &\qquad + \frac{1}{\delta} \int_\Omega \frac{1}{|\Omega_x|} \int_{\Gamma_x} \partial m(T_\beta[u, U]) T_\beta[\varphi, \Phi]\, ds\, dx \\ &\quad = \int_\Omega \left\{ f(x, t)\varphi(x) + \int_{\Gamma_x} F(x, y, t)\Phi(x, y)\, dy \right\} dx. \end{aligned}$$

Here ∂m denotes any (pointwise a.e.) selection in the sense of Lemma 2. By setting $\Phi = 0$ we obtain

$$\begin{aligned}\int_\Omega \frac{\partial}{\partial t} u(x, t)\varphi + \hat{A}(x, \nabla u(x, t)) \nabla\varphi(x)\, dx \\ - \frac{1}{\delta}\int_\Omega \frac{1}{|\Omega_x|}\int_{\Gamma_x} \partial m(T_\beta[u, U])(\lambda\varphi(x) \\ + \beta s \cdot \lambda\nabla\varphi(x))\, ds\, dx \\ = \int_\Omega f(x, t)\varphi(x)\, dx.\end{aligned} \tag{2.5.a}$$

Setting $\varphi = 0$ shows that for almost every x,

$$\begin{aligned}\int_{\Omega_x} \frac{\partial}{\partial t} U(x, y, t)\Phi(x, y)\, dy \\ + \int_{\Omega_x} \hat{B}(x, y, \nabla_y U(x, y, t)) \nabla_y\Phi(x, y, t)\, dy \\ = + \frac{1}{\delta|\Omega_x|}\int_{\Gamma_x} \partial m(T_\beta[u, U])\gamma\Phi(x, s)\, ds \\ = \int_{\Omega_x} F(x, y, t)\Phi(x, y)\, dy.\end{aligned} \tag{2.5.b}$$

If $\hat{B}(x, \cdot, U)$ is sufficiently smooth (i.e., contained in $W^{1,q'}(\Omega_x)^n$) then the classical Green's theorem shows that

$$\begin{aligned}\int_{\Omega_x} \hat{B}(x, y, \nabla_y U(x, y, t)) \nabla_y\Phi(x, y, t)\, dy \\ = \int_{\Gamma_x} \hat{B}(x, y, \nabla_y U(x, y, t)) \cdot \nu_x\gamma_x\Phi(x, y, t)\, ds \\ - \int_{\Omega_x} \nabla_y\hat{B}(x, y, \nabla_y U(x, y, t))\Phi(x, y, t)\, dy.\end{aligned}$$

Such an equality still holds in the absence of the regularity required above if we denote the action of the abstract Green's operator on test functions by the above boundary integral. This convention will be used throughout our presentation. See Lemmas 1 and 2 of [30] for details of constructing the appropriate Green's operator for this problem. Applying this to Eq. (2.5.b) we obtain

$$\int_{\Omega_x} \frac{\partial}{\partial t} U(x, y, t)\Phi(x, y)\, dy - \int_{\Omega_x} \nabla_y \cdot \hat{B}(x, y, \nabla_y U(x, y, t))\Phi(x, y, t)\, dy$$
$$+ \frac{1}{\delta} \int_{\Gamma_x} \{\partial m(T_\beta[u, U]) + \hat{B}(x, y, \nabla_y U(x, y, t)) \cdot \nu_x\}\gamma_x \Phi(x, s)\, ds$$
$$= \int_{\Omega_x} F(x, y, t)\Phi(x, y)\, dy. \tag{2.6}$$

This yields the partial differential equation

$$\frac{\partial}{\partial t} U - \nabla_y \cdot \hat{B}(x, y, \nabla_y U) = F, \qquad y \in \Omega_x$$

and the boundary condition

$$\hat{B}(x, s, \nabla_y U) \cdot \nu + \frac{1}{\delta}\partial m(U - u - \beta \nabla u \cdot s) \ni 0, \qquad s \in \Gamma_x.$$

We use this in (2.5.a) to obtain

$$\int_\Omega \frac{\partial}{\partial t} u(x, t)\varphi + \hat{A}(x, \nabla u(x, t))\, \nabla\varphi(x)\, dx$$
$$+ \int_\Omega \frac{1}{|\Omega_x|} \int_{\Gamma_x} \hat{B}(x, y, \nabla_y U) \cdot \nu\, ds \varphi$$
$$+ \left\{\frac{\beta}{|\Omega_x|} \int_{\Gamma_x} \hat{B}(x, y, \nabla_y U) \cdot \nu s\, ds\right\} \cdot \nabla\varphi\, dx$$
$$= \int_\Omega f(x, t)\varphi(x)\, dx,$$

and this gives the partial differential equation

$$\frac{\partial}{\partial t} u - \nabla \cdot \hat{A}(x, \nabla u) + \frac{1}{|\Omega_x|} \int_{\Gamma_x} \hat{B}(x, y, \nabla_y U) \cdot \nu\, ds$$
$$-\beta \nabla \cdot \left\{\frac{1}{|\Omega_x|} \int_{\Gamma_x} \hat{B}(x, y, \nabla_y U) \cdot \nu s\, ds\right\} = f, \qquad x \in \Omega.$$

The boundary condition

$$u = 0 \qquad \text{on } \partial\Gamma$$

follows since $u \in W_0^{1,p}(\Omega)$. Conversely, a solution of (1.2) satisfies (2.4) and the above equivalence holds likewise for the matched problem.

3. Resolution of the Cauchy Problem

The initial-boundary problem for (1.2) will be resolved as an application of classical results on the Cauchy problem in Hilbert space. We shall show successively that the stationary problem is well-posed, that the operator obtained by restricting the stationary problem to L^2 is maximal monotone, and that this operator is a subgradient. The last result gives regularizing effects which show the evolution is of *parabolic* type. For reference we list hypotheses that are used below:

H_1. The measurable $Q \subset \Omega \times \mathbb{R}^n$ is given with sections $\Omega_x = \{y \in \mathbb{R}^n : (x, y) \in Q\}$ with smooth boundary Γ_x as in Section 2, measures $|\Omega_x|$ bounded and uniformly bounded above zero, and uniformly bounded traces $\gamma_x : W^{1,q}(\Omega_x) \to L^q(\Gamma_x)$.

H_2. The sections Ω_x are uniformly bounded in some direction, e.g., $\sup\{|y_n| : y \in \Omega_x, x \in \Omega\} < \infty$.

H_3. The functions $\hat{A}$, $\hat{B}$, m satisfy (2.1), (2.2), (2.3) with $1 < q \leq p < \infty$.

A consequence of H_2 is the Poincaré-type estimate

$$c_0\|\Phi\|^q_{L^q(Q)} \leq \|\gamma\Phi\|^q_{L^q(\Omega,L^q(\Gamma_x))} + \|\nabla_y\Phi\|^q_{L^q(Q)}$$

in which the positive c_0 depends on the constant in H_2.

The following result from [11] will be used for the proof of our first two theorems.

Theorem 0 (Brézis). *Let V be a separable, reflexive Banach space and let $A : V \to V'$ be a bounded, continuous, and monotone operator. Let $j : V \to \mathbb{R}_\infty$ be proper, convex, and lower semicontinuous. If there exists a v_0 in the domain of j such that*

$$\lim_{\|v\|\to\infty} \left\{\frac{Av(v - v_0) + j(v)}{\|v\|}\right\} = +\infty,$$

then the range of $A + \partial j$ is V'.

The Stationary Problem

Theorem 1. *Assume* H_1, H_2, *and* H_3. *The operator* $\mathcal{A} + \mathcal{B} + (1/\delta)\, \partial M_\beta$ *maps* $\mathcal{W}$ *onto its dual.*

Proof. The following is a generalization of a similar proof from [30]. Since $\mathcal{A} + \mathcal{B}$ is continuous, bounded, monotone, and ∂M_β is a subgradient, according to Theorem 0 it suffices to show that $\mathcal{A} + \mathcal{B} + (1/\delta)M_\beta$ satisfies an appropriate coercivity condition.

Let $[u, U]$ be a pair in $\mathcal{W} = W_0^{1,p}(\Omega) \times L^q(\Omega, W^{1,q}(\Omega_x))$. It suffices to show that

$$\frac{\mathcal{A}(u)u + \mathcal{B}(U)U + (1/\delta)M_\beta([u, U])}{\|u\|_{W_0^{1,p}(\Omega)} + \|U\|_{L^q(\Omega, W^{1,q}(\Omega_x))}} \to +\infty$$

as $\|u\|_{W_0^{1,p}(\Omega)} + \|U\|_{L^q(\Omega, W^{1,q}(\Omega_x))} \to +\infty$.

The *a priori* estimates on $\mathcal{A}$ and $\mathcal{B}$ show that there exists a constant $c_0 > 0$ such that

$$\begin{aligned}
&\mathcal{A}u(u) + \mathcal{B}U(U) + (1/\delta)M_\beta([u, U]) \\
&\quad \geq c_0\|\nabla u\|_{L^p(\Omega)}^p - \|g_0\|_{L^1(\Omega)} + c_0\|\nabla_y U\|_{L^q(Q)}^q - \|h_0\|_{L^1(Q)} \\
&\quad + (1/\delta)c_0\|\gamma U - \lambda u - \beta s \cdot \lambda \nabla u\|_{L^q(\Omega, L^q(\Gamma_x))}^q.
\end{aligned} \tag{3.1}$$

Suppose that the above ratio were bounded by a constant K. This would imply that the right side of the above inequality is bounded by

$$K(\|u\|_{W_0^{1,p}(\Omega)} + \|U\|_{L^q(\Omega, W^{1,q}(\Omega_x))})$$

which is in turn bounded by

$$\begin{aligned}
&K(\|u\|_{W_0^{1,p}(\Omega)} + \|\nabla_y U\|_{L^q(Q)} + \|\gamma U\|_{L^q(\Omega, L^q(\Gamma_x))}) \\
&\quad \leq K(\|u\|_{W_0^{1,p}(\Omega)} + \|\nabla_y U\|_{L^q(Q)} \\
&\qquad + \|\gamma U - \lambda u - \beta\lambda\nabla u\|_{L^q(\Omega, L^q(\Gamma_x))} + \|\lambda u + \beta\lambda\nabla u\|_{L^q(\Omega, L^q(\Gamma_x))}).
\end{aligned} \tag{3.2}$$

Since $\|\nabla u\|_{L^p(\Omega)}$ is equivalent to the standard norm on $W_0^{1,p}(\Omega)$, every term in (3.1) is bounded by a corresponding term in (3.2) raised to a power larger than one, which implies that each is bounded. ∎

Remark. Theorem 1 is also true with $\mathcal{W}$ replaced by $\mathcal{W}_\beta$, the kernel of T_β; since the coercivity estimate holds for every $[u, U] \in \mathcal{W}$, it trivially holds for every $[u, U] \in \mathcal{W}_\beta \subseteq \mathcal{W}$. Therefore Theorem 1 gives us the existence for the matched microstructure model, as well as the regularized model.

The Maximal Monotone Case

Let $H = L^2(\Omega, L^2(\Omega_x)) \cong L^2(Q)$ and $\mathcal{H} = L^2(\Omega) \times H$. The inner product on H is given by

$$(U, V)_H = \int_\Omega \int_{\Omega_x} U(x, y)\, V(x, y)\, dy\, dx$$

and the inner product on $\mathcal{H}$ is given by

$$([u, U], [v, V])_{\mathcal{H}} = \int_\Omega u(x)v(x)\, dx + (U, V)_H.$$

We will identify $\mathcal{H}$ with its dual through this inner product. Let $\mathcal{N} = \mathcal{A} + \mathcal{B} + (1/\delta)\partial M_\beta$; we define a relation N on $L^2(\Omega) \times L^2(Q)$ by $[f, F] \in N[u, U]$ iff $[u, U] \in (L^2(\Omega) \times L^2(Q)) \cap \mathcal{W}$ and $[f, F] \in \mathcal{N}[u, U] \cap (L^2(\Omega) \times L^2(Q))$. Each of the operators $\mathcal{A}$, $\mathcal{B}$, and ∂M_β is monotone from $\mathcal{W}$ to $\mathcal{W}'$ and thus N is $\mathcal{H}$-monotone on $D(N)$. We shall show that N is maximal monotone, and this will give the following [12].

THEOREM 2. *Assume* H_1, H_2, *and* H_3. *For each* $u_0 \in D(N)$ *and* $f \in W^{1,1}(0, T; \mathcal{H})$ *there exists a unique* $u \in W^{1,\infty}(0, T; \mathcal{H})$ *such that* $u(t) \in D(N)$ *for all* $0 \le t < T$, $u(0) = u_0$, *and*

$$(d/dt)u(t) + N(u(t)) \ni f(t)$$

for almost every in $(0, T)$. *Furthermore, if* d/dt *is replaced by the right-derivative* D^+, *the above equation holds for every* t *in* $[0, T)$.

Proof. Since N is the restriction of the monotone operator $\mathcal{N}: \mathcal{W} \to \mathcal{W}'$, N is monotone in $L^2(\Omega) \times L^2(Q)$. It remains to be shown that $I + N$ maps $D(N)$ onto $L^2(\Omega) \times L^2(Q)$. We follow the proof of Theorem 1 to show that $I + \mathcal{N}$ maps $(L^2(\Omega) \times L^2(Q)) \cap \mathcal{W}$ onto its dual $(L^2(\Omega) \times L^2(Q)) \oplus \mathcal{W}'$, which contains $L^2(\Omega) \times L^2(Q)$. We apply Theorem 0 with the convex function j defined by

$$j[u, U] = \frac{1}{\delta} M_\beta[u, U] + \tfrac{1}{2}\|u\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|U\|^2_{L^2(Q)}.$$

It is clear from the proof of Theorem 1 that

$$\frac{\mathcal{A}(u)u + \mathcal{B}(U)U + j([u, U])}{\|u\|_{W_0^{1,p}(\Omega)} + \|u\|_{L^2(\Omega)} + \|U\|_{L^q(\Omega, W^{1,q}(\Omega_x))} + \|U\|_{L^2(Q)}} \to +\infty$$

as the denominator goes to infinity. ∎

As is the case with Theorem 1, Theorem 2 applies to the matched model as well as to the regularized model; the proof consists of simply replacing $\mathcal{W}$ with $\mathcal{W}_\beta$ in the above arguments.

The Subgradient Case

In the preceding, one could add certain first-order terms and obtain the same results. Thus Theorem 1 and Theorem 2 can be extended and applied to various problems with convection. In the case of a subgradient flow the parabolic regularizing effects are known and one can extend these to apply to certain multi-valued operators and make use of the calculus of subgradients. We consider this direction now.

Suppose that we are given a function $\varphi_A : \Omega \times \mathbb{R}^n \to \mathbb{R}^+$ which satisfies the following:

For each x, the function $\xi \mapsto \varphi_A(x, \xi)$ is continuous and convex. (3.3.a)

For each ξ, the function $x \mapsto \varphi_A(x, \xi)$ is measurable. (3.3.b)

$$\varphi_A(x, \xi) \geq 0 \quad \text{and} \quad \varphi_A(x, 0) = 0. \tag{3.3.c}$$

There exists $g_0 \in L^{p'}(\Omega)$, $g_1 \in L^1(\Omega)$, and constants c_0 and c_1 such that

$$c_0|\xi|^p - g_0(x) \leq \varphi_A(x, \xi) \leq c_1|\xi|^p + g_1(x). \tag{3.3.d}$$

Define $\Phi_A : L^p(\Omega)^n \to \mathbb{R}$ by $\Phi_A(u) = \int_\Omega \varphi_A(x, u(x))\, dx$. Let $\partial_2\varphi_A$ denote the subgradient of φ_A with respect to its second component.

LEMMA 3. *Φ_a is a proper, continuous convex function of $L^p(\Omega)^n$ and*

$$\partial\Phi_A(u)v = \int_\Omega \partial_2\varphi_A(x, u(x))v(x)\, dx.$$

Proof. $\Phi_A(0) = 0$ and so Φ_A is proper. For each $u \in L^p(\Omega)^n$ the function $x \mapsto \varphi_A(x, u(x))$ is measurable. (To verify this, let u_n be a sequence of step functions converging pointwise to u. For each n, $\varphi_A(x, u_n(x))$ is measurable. Therefore the pointwise limit is measurable.) The *a priori* estimates on φ_A ensure that Φ_A is bounded and, thus (by convexity), continuous on the interior of its domain. But (3.3.d) shows that the domain of Φ_A is all of $L^p(\Omega)^n$. Thus, Φ_A is continuous.

Let $h(x, \xi)$ be a selection out of $\partial_2\varphi_A(x, \xi)$. For any pair $\xi, \eta \in \mathbb{R}^n$, the definition of subgradient states that

$$h(x, \xi)(\eta - \xi) \leq \varphi_A(x, \eta) - \varphi_A(x, \xi).$$

By integrating this over Ω with $\xi = u(x)$ and $\eta = v(x)$ we see that the function $x \mapsto \partial_2\varphi_A(x, u(x))$ is contained in $\partial\Phi_A(u)$. To show that all elements of $\partial\Phi_A$ are obtained this way, suppose that $h \in \partial\Phi_A(u) \subseteq L^{p'}(\Omega)^n$. Thus,

$$\int_\Omega h(x)(v(x) - (u(x))\, dx \leq \int_\Omega \{\varphi_A(x, v(x)) - \varphi_A(x, u(x))\}\, dx, \quad v \in L^p(\Omega)^n.$$

Let E be any measureable subset of Ω and define $w(x)$ as $v(x)$, if $x \in E$, and $u(x)$, otherwise. Now substitute w for v in the above inequality. This shows that

$$\int_E \{h(x)(v(x) - u(x)) - (\varphi_A(v(x)) - \varphi_A(u(x)))\}\, dx \leq 0$$

for every measurable $E \subseteq \Omega$ and, thus, the integrand is non-positive almost everywhere and so $h(x) \in \partial_2\varphi_A(x, u(x))$ for almost every x. ∎

COROLLARY. *If* $\hat{A} = \partial\varphi_A$, *then* $\mathcal{A} = \partial(\Phi_a \circ \nabla)$.

Proof. For any continuous linear operator Λ and any proper convex function φ continuous at some point in the range of Λ, the following *chain rule* holds:

$$\partial(\varphi \circ \Lambda) = \Lambda^* \circ \partial\varphi \circ \Lambda.$$

(See [18].) Apply this with Φ_A and ∇. ∎

The above development shows that with analogous assumptions on $\varphi_B : Q \times \mathbb{R}^n \mapsto \mathbb{R}$ we can construct an operator Φ_B whose subgradient equals $\mathcal{B}$. That is, we assume the following:

For each $(x, y) \in Q$, the function $\varphi_B(x, y, \cdot)$ is continuous and convex. (3.4.a)

For each ξ, the function $\varphi_B(\cdot, \cdot, \xi)$ is measurable on Q. (3.4.b)

$$\varphi_B(x, y, \xi) \geq 0 \quad \text{and} \quad \varphi_B(x, y, 0) = 0. \tag{3.4.c}$$

There exists a $g_2 \in L^1(Q)$ and positive constants c_0 and c_1 such that

$$c_0|\xi|^q - g_2(x, y) \leq \varphi_B(x, y, \xi) \leq c_1|\xi|^q + g_2(x, y). \tag{3.4.d}$$

Then we can define $\Phi_B : L^q(Q)^n \to \mathbb{R}$ by $\Phi_b(U) = \int_Q \varphi_B(x, y, U(x, y))\, dy\, dx$ and the chain rule gives $\partial(\Phi_B \circ \nabla_y) = -\nabla_y \cdot \partial\Phi_B \circ \nabla_y$ in $L^q(\Omega, W^{1,q}(\Omega_x))'$ as before. That is, $F \in \partial(\Phi_B \circ \nabla_y)(U)$ means there is an $H \in L^{q'}(Q)^n$ with

$$F(V) = \int_Q H(x, y) \cdot \nabla_y V(x, y)\, dy\, dx, \qquad V \in L^q(\Omega, W^{1,q}(\Omega_x)),$$

and $H(x, y) \in \partial\varphi_B(x, y, \nabla_y U(x, y))$, a.e. $(x, y) \in Q$. Since the sum rule for subgradients holds in our situation, this says that the operator $\mathcal{N}$ of Section 3.2 is a subgradient if $\mathcal{A}$ and $\mathcal{B}$ are constructed as above because $\mathcal{N} = \partial(\Phi_A + \Phi_B + (1/\delta)M_\beta)$. This allows us to obtain much stronger results about the regularity of the solution to the Cauchy problem $u'(t) + Nu(t) \ni f(t)$.

Let $\Phi = \Phi_A + \Phi_B + (1/\delta)M_\beta$. Φ can be extended to a convex function Φ_H from $\mathcal{H}$ to $\mathbb{R}_\infty$ by defining it to be infinite outside of $\mathcal{W} \cap \mathcal{H}$.

LEMMA 4. *Φ_H is a proper, lower semi-continuous convex function on $\mathcal{H}$ and the restriction of $\partial\Phi$ to $\mathcal{H}$ agrees with $\partial\Phi_H$ on $\mathcal{W} \cap \mathcal{H}$.*

Proof. Φ_H is proper because Φ is. Also, extending Φ to be infinite off of a convex set does not change its convexity. We will show that Φ_H is weakly lower semi-continuous. Suppose that $\tilde{u}_n \rightharpoonup \tilde{u}$ in $\mathcal{H}$. Assume that $\{\Phi_H(\hat{u}_n)\}$ is bounded; otherwise there is nothing to prove. Since weakly convergent subsequences are bounded, we know that

$$\{\|\tilde{u}_n\|^2_{\mathcal{H}} + \Phi_H(\tilde{u}_n)\}$$

is also bounded. The coercivity estimate in Theorem 2 shows that $\{\tilde{u}_n\}$ is bounded in $\mathcal{W} \cap \mathcal{H}$. Therefore $\{\tilde{u}_n\}$ has a subsequence $\{\tilde{u}_{u'}\}$ which converges weakly in $\mathcal{W} \cap \mathcal{H}$. The weak limit must be $\tilde{u}$ since $\tilde{u}_n \rightharpoonup \tilde{u}$ in $\mathcal{H}$. In fact, the original sequence $\{\tilde{u}_n\}$ must converge to $\tilde{u}$; otherwise, some subsequence stays outside of some weak neighborhood of $\tilde{u}$ and we would not be able to extract a further subsequence converging to $\tilde{u}$. Thus

$$\Phi_H(\tilde{u}) \le \liminf_{n \to 0} \Phi_H(\tilde{u}_n)$$

from the weak lower semi-continuity of Φ on $\mathcal{W} \cap \mathcal{H}$.

Next, we must establish that for any $\tilde{u} \in \mathcal{W} \cap \mathcal{H}$,

$$\partial\Phi(\tilde{u}) \cap \mathcal{H} = \partial\Phi_H(\tilde{u}).$$

Assume that $\tilde{w} \in \partial\Phi(\tilde{u}) \cap \mathcal{H}$. For every $\tilde{v} \in \mathcal{W} \cap \mathcal{H}$,

$$\langle \tilde{w}, \tilde{v} - \tilde{u} \rangle \leq \Phi_H(\tilde{v}) - \Phi_H(\tilde{u}). \tag{3.5}$$

If $\tilde{v} \in \mathcal{H}$ but $\tilde{v} \notin \mathcal{W}$, $\Phi_H(\tilde{v}) = \infty$, and thus Eq. (3.5) holds for all $\tilde{v} \in \mathcal{H}$.

Now pick $\tilde{u} \in \mathcal{H}$ and assume that $\tilde{w} \in \partial\Phi_H(\tilde{u}) \cap \mathcal{H}$ so that (3.5) holds for all $\tilde{v} \in \mathcal{H}$. Since Φ_H is proper, there exists $\tilde{v} \in \mathcal{H}$ such that $\Phi_H(\tilde{v})$ is finite and thus $\Phi_H(\tilde{u})$ is finite, i.e., $\tilde{u} \in \mathcal{W} \cap \mathcal{H}$. ▮

From Lemma 4 we know that the operator N is the subgradient of a proper, lower semi-continuous convex function on $\mathcal{H}$ and thus that the following result holds [12].

THEOREM 3. *For each $f \in L^2(0, T; \mathcal{H})$ and $\tilde{u}_0 \in \overline{\mathrm{dom}(\Phi_h)}$, there exists a unique $\tilde{u} \in C(0, T; \mathcal{H})$ for which $\tilde{u}(t) \in D(N)$ for almost every $t \in (0, T)$, $\sqrt{t}\, \tilde{u}' \in L^2(0, T; \mathcal{H})$, $\Phi_H(\tilde{u}(\cdot)) \in L^1(0, T)$, $\tilde{u}(0) = \tilde{u}_0$, and*

$$\frac{d\tilde{u}}{dt}(t) + N(\tilde{u}(t)) \ni f(t)$$

for almost every $t \in (0, T)$. If, in addition, $\tilde{u}_0 \in \mathrm{dom}(\Phi_H)$, then $\tilde{u} \in W^{1,2}(0, T; \mathcal{W})$.

The additional hypothesis that our differential operator is a subgradient allows us to start with less regular data and yet achieve smoother solutions. We drop the assumption of Theorem 2 that our forcing function f be absolutely continuous and only require that it be square integrable. Initial conditions $\tilde{u}_0$ may be chosen from $\mathcal{H}$ rather than from the more restricted $D(N) \subseteq \mathcal{W}$. Solutions that start outside of the domain of the operator N are drawn into its domain for almost every future time t. If f were as smooth as required for Theorem 2, the solution would be in the domain of N for all $t > 0$.

4. DEPENDENCE ON PARAMETERS

Dependence on δ

THEOREM 4. *As $\delta \to 0$, the solutions to the regularized microstructure model converge strongly in $C(0, T; \mathcal{H})$ to the solution to the matched model.*

By the convergence results of [13], it suffices to show that the solutions to the resolvent equation for the regularized model converge strongly to the solution of the resolvent equation for the matched model. This will be established in the following sequence of lemmas.

Let $\mathcal{V} = \mathcal{W} \cap \mathcal{H}$ and let $\mathcal{V}_\beta \cap \mathcal{H}$. For every $f \in \mathcal{H}$ and for every $\delta > 0$, there exist unique $\bar{u}_\delta = [u_\delta, U_\delta] \in \mathcal{V}$, $\bar{u}_0 = [u_0, U_0] \in \mathcal{V}_\beta$, and $\mu_\delta \in (1/\delta)\, \partial M_\beta \bar{u}_\delta$ such that

$$\bar{u}_\delta + \mathcal{A}\bar{u}_\delta + \mathcal{B}\bar{u}_\delta + \mu_\delta = f \qquad \text{in } \mathcal{V}' \tag{4.1}$$

$$\bar{u}_0 + \mathcal{A}\bar{u}_0 + \mathcal{B}\bar{u}_0 = f \qquad \text{in } \mathcal{V}_\beta'. \tag{4.2}$$

LEMMA 5. *The set* $\{\bar{u}_\delta\}$ *is bounded in* $\mathcal{V}$.

Proof. From the subgradient inequality, we know that for any $\bar{v} \in \mathcal{W}$,

$$\delta\mu_\delta(\bar{v} - \bar{u}_\delta) \leq M_\beta(\bar{v}) - M_\beta(\bar{u}_\delta).$$

Choosing $\bar{v} = 0$ shows that

$$\mu_\delta \bar{u}_\delta \geq (1/\delta) M_\beta(\bar{u}_\delta). \tag{4.3}$$

Therefore,

$$\langle f, \bar{u}_\delta \rangle = \langle \bar{u}_\delta + \mathcal{A}\bar{u}_\delta + \mathcal{B}\bar{u}_\delta + \mu_\delta, \bar{u}_\delta \rangle \tag{4.4}$$

$$\geq \|\bar{u}_\delta\|_{\mathcal{H}}^2 + \mathcal{A}(u_\delta)u_\delta + \mathcal{B}(U_\delta)U_\delta + (1/\delta)M_\beta(\bar{u}_\delta). \tag{4.5}$$

As long as $\bar{u}_\delta \neq 0$, we have

$$\frac{\langle f, \bar{u}_\delta \rangle}{\|\bar{u}_\delta\|_{\mathcal{V}}} \geq \frac{\|\bar{u}_\delta\|_{\mathcal{H}}^2 + \mathcal{A}(u_\delta)u_\delta + \mathcal{B}(U_\delta)U_\delta + \mu_\delta \bar{u}_\delta}{\|\bar{u}_\delta\|_{\mathcal{V}}}. \tag{4.6}$$

The left side of the above expression is bounded independent of δ. From the proof of Theorem 4, we know that the right side cannot be bounded unless the denominator is bounded. ∎

Since $\bar{u}_\delta$ is a bounded sequence in a reflexive space, there exists a weakly convergent sequence $\bar{u}_{\delta_i}$ with $\lim_{i\to\infty} \delta_i = 0$. We will denote the above sequence simply as $\bar{u}_\delta$. Let $\bar{u}$ denote the weak limit.

LEMMA 6. $\partial M_\beta \bar{u} \ni 0$.

Proof. It suffices to show that $M_\beta u = 0$, since the subgradient of a function at its minimum contains zero. M_β is weakly lower semicontinuous, since m is lower semicontinuous. Thus

$$M_\beta \bar{u} \leq \liminf_{\delta\to 0} M_\beta \bar{u}_\delta.$$

From Eq. (4.3),

$$M_\beta(\tilde{u}_\delta) \leq \delta \mu_\delta \tilde{u}_\delta .$$

Since $\mathcal{A}$ and $\mathcal{B}$ are bounded operators, $\{\mathcal{A}\tilde{u}_\delta\}$ and $\{\mathcal{B}\tilde{u}_\delta\}$ are bounded in $\mathcal{V}'$. Thus $\{\mu_\delta\}$ is bounded. Equation (4.3) shows that

$$\lim_{\delta \to 0} M_\beta \tilde{u}_\delta = 0$$

and so $M_\beta \tilde{u} \leq 0$. But M_β is always non-negative, and so $M_\beta \tilde{u} = 0$. ∎

LEMMA 7. *$(\mathcal{A} + \mathcal{B})\tilde{u}_\delta$ converges weakly to $(\mathcal{A} + \mathcal{B})\tilde{u}$ as $\delta \to 0$.*

Proof. Since $\mathcal{A} + \mathcal{B}$ is a bounded operator, the sequence $\{(\mathcal{A} + \mathcal{B})\tilde{u}_\delta\}$ is bounded in $\mathcal{V}'$. By passing to a subsequence if necessary, we may assume that it is weakly convergent. We must show that the weak limit is $(\mathcal{A} + \mathcal{B})\tilde{u}$.

Since $\mathcal{A} + \mathcal{B}$ is pseudomonotone, and thus type-M, we need only verify that

$$\limsup_{\delta \to 0} \langle (\mathcal{A} + \mathcal{B})\tilde{u}_\delta, \tilde{u}_\delta - \tilde{u} \rangle \leq 0,$$

or, equivalently,

$$\limsup_{\delta \to 0} \langle f - \tilde{u}_\delta - \mu_\delta, \tilde{u}_\delta - \tilde{u} \rangle \leq 0.$$

Since $\tilde{u}_\delta \rightharpoonup \tilde{u}$, $\langle f, \tilde{u}_\delta - \tilde{u} \rangle \to 0$. Using this, we reduce the problem to showing that

$$\liminf_{\delta \to 0} (\langle \tilde{u}_\delta, \tilde{u}_\delta - \tilde{u} \rangle + \langle \mu_\delta - 0, \tilde{u}_\delta - \tilde{u} \rangle) \geq 0.$$

The first term is non-negative by the weak lower semicontinuity of the norm and the second, by the monotonicity of ∂M_β. ∎

LEMMA 8. $\tilde{u} = \tilde{u}_0$.

Proof. Let $\tilde{v} \in \mathcal{V}_\beta$ be given. Since $\mathcal{V}_\beta \subseteq \mathcal{V}$,

$$\langle \tilde{u}_\delta + \mathcal{A}\tilde{u}_\delta + \mathcal{B}\tilde{u}_\delta + \mu_\delta, \tilde{v} \rangle = \langle f, \tilde{u}_\delta \rangle.$$

In fact,

$$\langle \tilde{u}_\delta + \mathcal{A}\tilde{u}_\delta + \mathcal{B}\tilde{u}_\delta, \tilde{v} \rangle = \langle f, \tilde{v} \rangle,$$

since $\tilde{v} \in \ker T_\beta$.

By letting $\delta \to 0$ we see that

$$\langle \tilde{u} + \mathcal{A}\tilde{u} + \mathcal{B}\tilde{u}, v\rangle = \langle f, \tilde{v}\rangle.$$

Since $\tilde{v}$ was arbitrary,

$$\tilde{u} + \mathcal{A}\tilde{u} + \mathcal{B}\tilde{u} = f \qquad \text{in } \mathcal{V}'_\beta.$$

By uniqueness of solutions to Eq. (4.2), $\tilde{u} = \tilde{u}_0$. ∎

LEMMA 9. *As $\delta \to 0$, $\tilde{u}_\delta$ converges strongly to $\tilde{u}_0$ in $\mathcal{H}$.*

Proof. From Eqs. (4.1) and (4.2) we have

$$\langle \tilde{u}_\delta + \mathcal{A}\tilde{u}_\delta + \mathcal{B}\tilde{u}_\delta + \mu_\delta, \tilde{u}_\delta - \tilde{u}_0\rangle = \langle f, \tilde{u}_\delta - \tilde{u}_0\rangle \tag{4.7}$$

$$\langle \tilde{u}_0 + \mathcal{A}\tilde{u}_0 + \mathcal{B}\tilde{u}_0, \tilde{u}_\delta - \tilde{u}_0\rangle = \langle \tilde{u}_0 + \mathcal{A}\tilde{u}_0 + \mathcal{B}\tilde{u}_0, \tilde{u}_\delta\rangle - \langle f, \tilde{u}_0\rangle. \tag{4.8}$$

Subtracting equations yields

$$\begin{aligned}\langle \tilde{u}_\delta - \tilde{u}_0 + \mathcal{A}\tilde{u}_\delta - \mathcal{A}\tilde{u}_0 + \mathcal{B}\tilde{u}_\delta - \mathcal{B}\tilde{u}_0, \tilde{u}_\delta - \tilde{u}_0\rangle + \langle \mu_\delta - 0, \tilde{u}_\delta - \tilde{u}_0\rangle \\ = \langle f - \tilde{u}_0 - \mathcal{A}\tilde{u}_0 - \mathcal{B}\tilde{u}_0, \tilde{u}_\delta\rangle.\end{aligned} \tag{4.9}$$

Using the monotonicity of $\mathcal{A} + \mathcal{B} + (1/\delta)\partial M_\beta$ and the fact that $\partial M_\beta \tilde{u}_0 \ni 0$, we have

$$\langle f - \tilde{u}_0 - \mathcal{A}\tilde{u}_0 - \mathcal{B}\tilde{u}_0, \tilde{u}_\delta\rangle \geq \langle \tilde{u}_\delta - \tilde{u}_0, \tilde{u}_\delta - \tilde{u}_0\rangle. \tag{4.10}$$

The left side of Eq. (4.10) goes to zero as $\tilde{u}_\delta \rightharpoonup \tilde{u}_0$ and the right side is $\|\tilde{u}_\delta - \tilde{u}_0\|^2_{\mathcal{H}}$. ∎

By the uniqueness of the weak limit, the original sequence $\{\tilde{u}_\delta\}$ converges to $\tilde{u}_0$.

Dependence on β

For this section, we assume that the monotone graph ∂m arising from the cell boundary condition is a Lipschitz function. For this reason we will denote ∂m by m'.

THEOREM 5. *As $\beta \to 0$, the solutions to the regularized microstructure model with β positive converge strongly in $C(0, T; \mathcal{H})$ to the solution with $\beta = 0$.*

Proof. As before, we will show that the solutions to the corresponding resolvent equations converge strongly in the pivot space $\mathcal{H}$ and apply [13].

For every $f \in \mathcal{H}$ and for every $\beta > 0$, there exist $\tilde{u}_\beta = [u_\beta, U_\beta]$ and $\tilde{u}_0 = [u_0, U_0]$ in $\mathcal{V}$ such that

$$\tilde{u}_\beta + \mathcal{A}\tilde{u}_\beta + \mathcal{B}\tilde{u}_\beta + (1/\delta)\partial M_\beta \tilde{u}_\beta = f \quad \text{in } \mathcal{V}' \tag{4.11}$$

$$\tilde{u}_0 + \mathcal{A}\tilde{u}_0 + \mathcal{B}\tilde{u}_0 + (1/\delta)\partial M_0 \tilde{u}_0 = f \quad \text{in } \mathcal{V}'. \tag{4.12}$$

Subtract equations and apply the test function $\tilde{u}_\beta - \tilde{u}_0$. We have

$$\begin{aligned}\|\tilde{u}_\beta - \tilde{u}_0\|_{\mathcal{H}}^2 &+ \langle \mathcal{A}\tilde{u}_\beta - \mathcal{A}\tilde{u}_0 + \mathcal{B}\tilde{u}_\beta - \mathcal{B}\tilde{u}_0, \tilde{u}_\beta - \tilde{u}_0\rangle \\ &+ (1/\delta)\langle \partial M_\beta \tilde{u}_\beta - \partial M_\beta \tilde{u}_0, \tilde{u}_\beta - \tilde{u}_0\rangle \\ &+ (1/\delta)\langle \partial M_\beta \tilde{u}_0 - \partial M_0 \tilde{u}_0, \tilde{u}_\beta - \tilde{u}_0\rangle = 0.\end{aligned} \tag{4.13}$$

By the monotonicity of $\mathcal{A} + \mathcal{B}$ and the fact that ∂M_β is a subgradient, the middle two terms are non-negative. Thus

$$\|\tilde{u}_\beta - \tilde{u}_0\|_{\mathcal{H}}^2 \leq -(1/\delta)\langle \partial M_\beta \tilde{u}_0 - \partial M_0 \tilde{u}_0, \tilde{u}_\beta - \tilde{u}_0\rangle.$$

The following lemmas will allow us to conclude that the right side goes to 0 as $\beta \to 0$. ∎

LEMMA 10. *The set $\{\tilde{u}_\beta\}$ is bounded in $\mathcal{V}$.*

Proof. This will be essentially the same proof used for Lemma 5. For each β,

$$\langle \tilde{u}_\beta + \mathcal{A}\tilde{u}_\beta + \mathcal{B}\tilde{u}_\beta + \mu_{\beta,\beta}, \tilde{u}_\beta\rangle = \langle f, \tilde{u}_\beta\rangle. \tag{4.14}$$

Following the same reasoning as Lemma 5, we may conclude that

$$\frac{\langle f, \tilde{u}_\beta\rangle}{\|\tilde{u}_\beta\|_{\mathcal{V}}} \geq \frac{\|\tilde{u}_\beta\|_{\mathcal{H}}^2 + \mathcal{A}(u_\beta)u_\beta + \mathcal{B}(U_\beta)U_\beta + \mu_{\beta,\beta}\tilde{u}_\beta}{\|\tilde{u}_\beta\|_{\mathcal{V}}}. \tag{4.15}$$

As before, the coercivity estimate in the proof of Theorem 4 allows us to conclude that $\{\tilde{u}_\beta\}$ must be bounded in $\mathcal{V}$. ∎

We must establish that

$$\lim_{\beta \to 0} \langle \partial M_\beta \tilde{u}_0 - \partial M_0 \tilde{u}_0, \tilde{u}_\beta - \tilde{u}_0\rangle = 0.$$

We know that

$$\langle \partial M_\beta \tilde{u}_0, \tilde{u}_\beta - \tilde{u}_0\rangle = \partial M(T_\beta \tilde{u}_0) T_\beta(\tilde{u}_\beta - \tilde{u}_0)$$

and

$$\langle \partial M_0 \tilde{u}_0, \tilde{u}_\beta - u_0 \rangle = \partial M(T_0\tilde{u}_0)T_0(\tilde{u}_\beta - \tilde{u}_0).$$

To show that the difference between the above expressions vanishes as $\beta \to 0$, we add and subtract $\partial M(T_0\tilde{u}_0)T_\beta(\tilde{u}_\beta - \tilde{u}_0)$ are thus break the task into the following two lemmas.

LEMMA 11. $\lim_{\beta\to 0} (\partial M(T_\beta\tilde{u}_0) - \partial M(T_0\tilde{u}_0))T_\beta(\tilde{u}_\beta - \tilde{u}_0) = 0.$

Proof. The difference $(\partial M(T_\beta\tilde{u}_0) - \partial M(T_0\tilde{u}_0))T_\beta(\tilde{u}_\beta - \tilde{u}_0)$ may be expanded in terms of integals as

$$\int_\Omega \int_{\Gamma_x} (m'(T_\beta(\tilde{u}_0)) - m'(T_0(\tilde{u}_0)))T_\beta(\tilde{u}_\beta - \tilde{u}_0)\, ds\, w(x)\, dx. \tag{4.16}$$

The absolute value of the expression (4.16) above is bounded by

$$\int_\Omega \int_{\Gamma_x} k\beta|s \cdot \nabla u|\, |T_\beta(\tilde{u}_\beta - \tilde{u}_0)| ds\, w(x)\, dx, \tag{4.17}$$

where k is the Lipschitz constant of m'. Since we are only concerned with small β's, we may assume that β is contained in $[0, 1]$ so that the operators T_β are uniformly bounded in the operator norm. We may conclude that the expression (4.17) is bounded by βK, where K is independent of β because the $\tilde{u}_\beta$'s are bounded in $\mathcal{V}$. ∎

LEMMA 12. $\lim_{\beta\to 0} \partial M(T_0\tilde{u}_0)(T_\beta - T_0)(\tilde{u}_\beta - \tilde{u}_0) = 0.$

Proof. We expand $\partial M(T_0\tilde{u}_0)(T_\beta - T_0)(\tilde{u}_\beta - \tilde{u}_0)$ as

$$-\beta \int_\Omega \int_{\Gamma_x} m'(T_0\tilde{u}_0)s \cdot \nabla(u_\beta - u_0)\, ds\, w(x)\, dx.$$

Since m' is Lipschitz and the u_β's are bounded in $H_0^1(\Omega)$, the above integral is bounded independently of β. ∎

REFERENCES

1. G. ALLAIRE, Homogenization and two-scale convergence, *SIAM J. Math. Anal.* **23** (1992), 1482–1518.
2. T. ARBOGAST, Analysis of the simulation of single phase flow through a naturally fractured reservoir, *SIAM J. Numer. Anal.* **26** (1989), 12–29.

3. T. ARBOGAST, The double porosity model for single phase flow in naturally fractured reservoirs, *Inst. Math. Appl.* **316** (1987).
4. T. ARBOGAST, Gravitational forces in dual-porosity models of single phase flow.
5. T. ARBOGAST, J. DOUGLAS, AND U. HORNUNG, Derivation of the double porosity model of single phase flow via homogenization theory, *SIAM J. Math. Anal.* **21** (1990), 823–836.
6. T. ARBOGAST, J. DOUGLAS, AND U. HORNUNG, Modeling of naturally fractured petroleum reservoirs by formal homogenization techniques, *in* "Frontiers in Pure and Applied Mathematics" (R. Dautray, Ed.), pp. 1–19, Elsevier, Amsterdam, 1991.
7. V. BARBU, "Nonlinear Semigroups and Differential Equations in Banach Spaces," Nordhoff, Groningen, 1976.
8. G. I. BARENBLATT, I. P. ZHELTOV, AND I. N. KOCHINA, Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks, *J. Appl. Math. Mech.* **24** (1960), 1286–1303.
9. J. A. BARKER, Block-geometry functions characterizing transport in densely fissured media, *J. Hydrology* **77** (1985), 263–279.
10. H. BREZIS, Monotonicity methods in Hilbert spaces and some application to nonlinear partial differential equations, *in* "Contributions to Nonlinear Functional Analysis" (E. H. Zarantonello, Ed.), Academic Press, New York, 1971.
11. H. BREZIS, Problemes unilateraux, *J. Math. Pures Appl.* **51** (1972), 1–164.
12. H. BREZIS, "Opérateurs Maximaux Monotones et Semigroupes de Contractions dans les Espaces de Hilbert," North Holland, Amsterdam, 1973.
13. H. BREZIS AND A. PAZY, Convergence and approximation of nonlinear semigroups in Banach spaces, *J. Funct. Anal.* **9** (1972), 63–74.
14. H. S. CARSLAW AND J. C. JAEGER, "Conduction of Heat in Solids," Oxford, London, 1947.
15. G. W. CLARK AND R. E. SHOWALTER, Fluid flow in a layered medium, *Quart. Appl. Math.*, to appear.
16. P. F. DIESLER, JR. AND R. H. WILHELM, Diffusion in beds of porous solids: Measurement by frequency response techniques, *Indust. Eng. Chem.* **45** (1953), 1219–1227.
17. J. DOUGLAS, P. J. PAES LEME, T. ARBOGAST, AND T. SCHMITT, Simulation of flow in naturally fractured reservoirs, *in* "Proceedings, Ninth SPEC Symposium on Reservoir Simulation, Dallas, 1987, pp. 271–279; Paper SPE 16019.
18. I. EKELAND AND R. TEMAM, "Convex Analysis and Variational Problems," North Holland, Amsterdam, 1976.
19. A. FRIEDMAN AND A. TZAVARAS, A quasilinear parabolic system arising in modelling of catalytic reactors, *J. Differential Equations* **70** (1987), 167–196.
20. M. TH. VAN GENUCHTEN AND F. N. DALTON, Models for simulating salt movement in aggregated field soils, *Geoderma* **38** (1986), 165–183.
21. U. HORNUNG AND W. JÄGER, Diffusion, convection, adsorption, and reaction of chemicals in porous media, *J. Differential Equations* **92** (1991), 199–225.
22. A. N. LOWAN, On the problem of the heat recuperator, *Philos. Mag.* **17** (1934), 914–933.
23. J. W. NUNZIATO, On heat conduction in materials with memory, *Quart. Appl. Math.* **29** (1971), 187–204.
24. M. PESZYNSKA, "Flow through Fissured Media; Mathematical Analysis and Numerical Approach," Dissertation, Augsburg, May 1992.
25. J. B. ROSEN, Kinetics of a fixed bed system for solid diffusion into spherical particles, *J. Chem. Phys.* **20** (1952), 387–394.
26. J. B. ROSEN AND W. E. WINSHE, The admittance concept in the kinetics of chromatography, *J. Chem. Phys.* **18** (1950), 1587–1592.

27. R. E. Showalter, "Hilbert Space Method for Partial Differential Equations," Pitman, New York, 1977.
28. R. E. Showalter, Diffusion models with micro-structure, *Transp. Porous Media* **6** (1991), 567–580.
29. R. E. Showalter and N. J. Walkington, Micro-structure models of diffusion in fissured media, *J. Math. Anal. Appl.* **155** (1991), 1–20.
30. R. E. Showalter and N. J. Walkington, Diffusion of fluid in a fissured medium with micro-structure, *SIAM J. Math. Anal.* **22** (1991), 1702–1722.
31. Ch. Vogt, A homogenization theorem leading to a Volterra integro-differential equation for permeation chromatography, Preprint 155, SFB **123,** Heidelberg, 1982.