Computing the Castelnuovo–Mumford regularity of some subschemes of $\mathbb{P}^n_K$ using quotients of monomial ideals

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Abstract

Given a homogeneous ideal $I \subset K[x_0,\ldots,x_n]$ defining a subscheme $X$ of projective $n$-space $\mathbb{P}^n_K$, we provide an effective method to compute the Castelnuovo–Mumford regularity of $X$ in the following two cases: when $X$ is arithmetically Cohen–Macaulay, and when $X$ is a not necessarily reduced projective curve. In both cases, we compute the Castelnuovo–Mumford regularity of $X$ by means of quotients of zero-dimensional monomial ideals.

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1. Introduction

Let $S := K[x_0,\ldots,x_n]$ be a polynomial ring over an infinite field $K$, and let $I$ be a homogeneous ideal of $S$ defining a subscheme $X$ of projective $n$-space $\mathbb{P}^n_K$. Among the several equivalent definitions for the Castelnuovo–Mumford regularity of $I$ (see [1]), we shall use the following: If

$$0 \to \bigoplus_{j=1}^{\beta_0} S(-e_{0j}) \xrightarrow{\phi_1} \bigoplus_{j=1}^{\beta_1} S(-e_{1j}) \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{\rho}} \bigoplus_{j=1}^{\beta_{\rho}} S(-e_{\rho j}) \xrightarrow{\phi_{\rho}} I \to 0$$

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is a minimal graded free resolution of $I$, then the Castelnuovo–Mumford regularity of $I$ is

$$
\text{reg} I := \max \left\{ \frac{e_i}{\text{FF}_i} - i; \ 0 \leq i \leq p \right\},
$$

where generators of the successive free modules in (1) have been ordered in such a way that $e_{i1} \leq e_{i2} \leq \cdots \leq e_{ip}$, for all $i$: $0 \leq i \leq p$. Being $I^\text{sat} = I : (x_0, \ldots, x_n)^\infty$ the saturation of $I$, the Castelnuovo–Mumford regularity of $I^\text{sat}$ is called the regularity of $\mathcal{X}$ and denoted by $\text{reg} \mathcal{X}$ [1, Section 1]. In this paper, we are interested in computing the regularity of some subschemes of $\mathbb{P}^n_K$, thus we shall always assume that the ideal $I$ is saturated.

Following the philosophy introduced by Bayer and Stillman in [2] as in our previous paper [3], our aim is to provide an effective method to compute $\text{reg} \mathcal{X}$ for some specific subschemes $\mathcal{X}$ of $\mathbb{P}^n_K$, avoiding the computation of a minimal graded free resolution of $I$. The knowledge of $\text{reg} I$ beforehand avoids unnecessary computations in large degrees while obtaining the minimal graded free resolution of $I$ through Buchberger’s syzygy algorithm.

In this paper, we shall essentially be concerned with the following two classes of subschemes of $\mathbb{P}^n_K$: arithmetically Cohen–Macaulay subschemes of $\mathbb{P}^n_K$ of any dimension (Section 2), and nonarithmetically Cohen–Macaulay subschemes of $\mathbb{P}^n_K$ of dimension one (Section 3). In both cases, denoting by $d := \dim S/I - 1$ the dimension of $\mathcal{X}$, and assuming that $K[x_{n-d}, \ldots, x_n]$ is a Noether normalization of $S/I$, we shall give an algorithm to compute $\text{reg} \mathcal{X}$ that does not require the knowledge of a minimal graded free resolution of $I$ (Theorems 2.5 and 3.3, respectively). These algorithms are a sequel of the one that computes the regularity of a not necessarily reduced projective curve in [3].

When $I$ defines an arithmetically Cohen–Macaulay subscheme $\mathcal{X}$ of $\mathbb{P}^n_K$ of any dimension, we show that $\text{reg} \mathcal{X}$ coincides with $\text{reg} \text{in}(I)$, where $\text{in}(I)$ is the initial ideal of $I$ with respect to the reverse lexicographic order. When $\mathcal{X}$ is a nonarithmetically Cohen–Macaulay subscheme of $\mathbb{P}^n_K$ of dimension one, $\text{reg} \mathcal{X}$ can be strictly smaller than $\text{reg} \text{in}(I)$. In this case, we associate to our ideal $I$ a monomial ideal $M(I)$ of $S$ such that $\text{reg} \mathcal{X} = \text{reg} M(I)$. In both cases, the computation of the regularity is reduced to the computation of the Castelnuovo–Mumford regularity of a monomial ideal determined through one Gröbner basis computation with respect to the reverse lexicographic order. We shall show that $\text{reg} \text{in}(I)$ in the first case, and $\text{reg} M(I)$ in the second one, can then be computed ‘by hand’ with no extra Gröbner basis computation, by means of quotients of zero-dimensional monomial ideals.

All the results have been implemented by the authors and Greuel in the specific library [4] of SINGULAR [7]. We shall make some comments on the implementation of our results in Section 4, and give significant examples to illustrate our methods along the paper. In particular, in Section 3 we give an example of an ideal $I$ defining a nonarithmetically Cohen–Macaulay projective monomial curve whose regularity was obtained in less than 1 second using [4], and whose minimal graded free resolution could not be computed using the command mres of [7]. This shows how important it
may be to know \( \text{reg} I \) beforehand to make the construction of a minimal graded free resolution of \( I \) more efficient using Buchberger’s syzygy algorithm.

2. Arithmetically Cohen–Macaulay subschemes of \( \mathbb{P}^n_K \)

Let \( I \) be a homogeneous ideal of \( S \) defining a projective subscheme \( \mathcal{X} \) of \( \mathbb{P}^n_K \) of dimension \( d = \dim S/I - 1 \). Assume that \( K[x_{n-d}, \ldots, x_n] \) is a Noether normalization of \( S/I \), i.e. that \( K[x_{n-d}, \ldots, x_n] \hookrightarrow S/I \) is an integral ring extension. Denote by \( \text{in}(I) \) the initial ideal of \( I \) with respect to the reverse lexicographic order. Monomials in \( S \) will be denoted by \( x^{\nu} := x^{\nu_0}_0 \cdots x^{\nu_n}_n \), with \( \nu = (\nu_0, \ldots, \nu_n) \in \mathbb{N}^{n+1} \).

The following is a criterion to determine whether \( S/I \) is Cohen–Macaulay. It implies that \( S/I \) is Cohen–Macaulay if and only if \( S/\text{in}(I) \) is Cohen–Macaulay.

**Proposition 2.1.** \( S/I \) is Cohen–Macaulay if and only if none of the minimal generators of \( \text{in}(I) \) is divisible by the variables \( x_{n-d}, \ldots, x_n \).

**Proof.** Using that \( S/I \) is Cohen–Macaulay if and only if \( x_{n-d}, \ldots, x_n \) is a regular sequence on \( S/I \) [8, Chapter 3, Proposition 4.4], the proof of [3, Proposition 2.1] can easily be generalized to any dimension. \( \square \)

Assume that \( S/I \) is Cohen–Macaulay. Denote by \( H(I) \) the regularity of the Hilbert function \( H_I \) of \( S/I \), i.e. the smallest integer \( s_0 \) such that for \( s \geq s_0 \), \( H_I(s) = P_I(s) \), being \( P_I(T) \) the Hilbert polynomial of \( S/I \).

We shall give an effective method to compute \( \text{reg} I \) (and \( H(I) \)) that does not require the knowledge of a minimal graded free resolution of \( I \). For this, we show that \( \text{reg} I = \text{reg} \text{in}(I) \), that can also be obtained from [2, Theorem 2.4(b)], and reduce the problem to the computation of the Castelnuovo–Mumford regularity of a zero-dimensional monomial ideal.

**Lemma 2.2.** If \( S/I \) is Cohen–Macaulay, then \( \text{in}(I) \cap K[x_0, \ldots, x_{n-d-1}] \) is a zero-dimensional monomial ideal, and

\[
\text{reg} I = \text{reg} \text{in}(I) = \text{reg} \text{in}(I) \cap K[x_0, \ldots, x_{n-d-1}].
\]

**Proof.** For all \( i: 0 \leq i \leq n-d-1 \), \( \exists \alpha_i \in \mathbb{N} - \{0\}/x_i^{\alpha_i} \in \text{in}(I) \). Thus, \( K[x_0, \ldots, x_{n-d-1}]/\text{in}(I) \cap K[x_0, \ldots, x_{n-d-1}] \) is artinian.

Since \( x_{n-d}, \ldots, x_n \) is a regular sequence on \( S/I \) and \( S/\text{in}(I) \), then \( \text{reg} I = \text{reg} \text{in}(I) \). Indeed, \( \text{reg} I = \text{reg} (I, x_{n-d}, \ldots, x_n) = H(I, x_{n-d}, \ldots, x_n) = H(I) + d + 1 = H(\text{in}(I)) + d + 1 = \text{reg} \text{in}(I), x_{n-d}, \ldots, x_n = \text{reg} \text{in}(I) \) by Eisenbud [6, Proposition 20.20]; Bayer and Stillman [2, Lemma 1.7] and the equality \( H(I) = H(\text{in}(I)) \).

Finally, minimal generators of the monomial ideals in \( \text{in}(I) \cap K[x_0, \ldots, x_{n-d-1}] \) and \( \text{in}(I) \) coincide by Proposition 2.1. Thus, one has the equality \( \text{reg} \text{in}(I) = \text{reg} \text{in}(I) \cap K[x_0, \ldots, x_{n-d-1}] \). \( \square \)
Remark 2.3. Note that in the proof of the previous lemma, we have got the equality $\text{reg } I = H(I) + d + 1$. If $I = (f_1, \ldots, f_{n-d})$ is a complete intersection, where $f_i$ is a homogeneous polynomial of degree $\delta_i$ for all $i: 1 \leq i \leq n - d$, then $\text{reg } I = \delta_1 + \cdots + \delta_{n-d} - n + d + 1$ because $H(I) = \delta_1 + \cdots + \delta_{n-d} - n$ ([8, Chapter 3, Remark 3.2.2]). In this case, we have nothing to compute.

For any homogeneous ideal $J$, we shall denote by $\delta(J)$ the maximum of the degrees of a minimal set of generators of $J$. The following is an effective method to compute the Castelnuovo–Mumford regularity of a zero-dimensional monomial ideal.

**Proposition 2.4.** Let $J \subset S = K[x_0, \ldots, x_n]$ be a zero-dimensional monomial ideal. Then, $\text{reg } J = \delta(J : (x_0, \ldots, x_n)) + 1$.

**Proof.** Since $\dim S/J = 0$, $\text{reg } J = H(J)$ by Bayer and Stillman [2, Lemma 1.7]. One can easily check that $H(J) = H(J : (x_0, \ldots, x_n)) + 1$.

Let us prove now that $H(J : (x_0, \ldots, x_n)) = \delta(J : (x_0, \ldots, x_n))$. Indeed, as $J : (x_0, \ldots, x_n)$ is a zero-dimensional ideal, one has that $H(J : (x_0, \ldots, x_n)) \geq \delta(J : (x_0, \ldots, x_n))$.

On the other hand, assume that there exists a monomial $M$ in $S$ of degree $\delta := \delta(J : (x_0, \ldots, x_n))$ such that $M \notin J : (x_0, \ldots, x_n)$. Then, $x_i M \notin J$ for some $i$. Let $i_0$ be the smallest integer $i: 0 \leq i \leq n$ such that $x_i M \notin J$, and let $z_0$ be the highest integer $\geq 1$ such that $x_{i_0}^z M \notin J$. Then, $M_0 := x_{i_0}^{-z_0} M \notin J$ and $x_i M_0 \in J$ for all $i \leq i_0$. If $x_i M_0 \in J$ for all $i \leq n$, set $N := M_0$. Otherwise, let $i_1$ denote the smallest integer $i: i_0 < i \leq n$ such that $x_i M_0 \notin J$, and let $z_1$ be the highest integer $\geq 1$ such that $x_{i_1}^z M_0 \notin J$. Setting $M_1 := x_{i_1}^{-z_1} M_0 = x_{i_0}^{z_0} x_{i_1}^{z_1} M$, one has that $M_1 \notin J$ and $x_i M_1 \in J$ for all $i \leq i_1$. As $i_0 < i_1 \leq n$, we obtain recursively a monomial $N := x_{i_0}^{z_0} \cdots x_{i_1}^{z_1} M \notin J$ such that $x_i N \in J$, for all $i: 0 \leq i \leq n$. Thus, $N \notin J : (x_0, \ldots, x_n)$. Moreover, $\deg N = \deg M + z_0 + \cdots + z_{i_1} > \delta$. Since $N \notin J$, $N$ should be a minimal generator of $J : (x_0, \ldots, x_n)$ which is a contradiction.

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**Theorem 2.5.** Let $I \subset S = K[x_0, \ldots, x_n]$ be a homogeneous ideal such that $S/I$ is Cohen–Macaulay, and assume that $K[x_{n-d}, \ldots, x_n]$ is a Noether normalization of $S/I$. Let $\mathcal{X}$ be the subscheme of $\mathbb{P}^n$ of dimension $d = \dim S/I - 1$ defined by $I$. If $\text{in}(I)$ is the initial ideal of $I$ with respect to the reverse lexicographic order, then

$$\text{reg } \mathcal{X} = \delta(\text{in}(I) : (x_0, \ldots, x_{n-d-1})) + 1.$$  

Moreover, the regularity of the Hilbert function of $S/I$ is

$$H(I) = \delta(\text{in}(I) : (x_0, \ldots, x_{n-d-1})).$$

**Proof.** By Proposition 2.1, minimal generators of in$(I) : (x_0, \ldots, x_{n-d-1})$ and $I \cap K[x_0, \ldots, x_{n-d-1}] : (x_0, \ldots, x_{n-d-1})$ coincide. Thus, the result follows from Lemma 2.2 and Proposition 2.4.

**Remark 2.6.** Observe that the above formula permits to compute the regularity of $\mathcal{X}$ (and $H(I)$) ‘by hand’ once we know in$(I)$. In fact, no extra Gröbner basis computation
is needed and one just has to estimate some least common multiples of monomials. This observation is based on the following fact: If $J$ is a monomial ideal minimally generated by the monomials $M_1, \ldots, M_t$, then $J : (x_0, \ldots, x_{n-d-1}) = \bigcap_{j=0}^{n-d-1} (J : (x_j))$.

For all $j$: $0 \leq j \leq n - d - 1$, one has that $J : (x_j) = (M'_1, \ldots, M'_t)$, where $M'_k = M_k / x_j$ if $x_j$ divides $M_k$, and $M'_k = M_k$ otherwise. Thus, one just has to compute some intersections of monomial ideals. If $J_1 = (M_1, \ldots, M_s)$ and $J_2 = (N_1, \ldots, N_t)$ are monomial ideals, then

$$J_1 \cap J_2 = (\text{lcm} (M_j, N_j)); \ 1 \leq i \leq s, \ 1 \leq j \leq t.$$  

Example 2.7. Consider the second Veronesean of $\mathbb{P}_K^3$, i.e. the projective monomial variety $X \subset \mathbb{P}^9_K$ whose defining ideal $I \subset S = K[x_0, \ldots, x_9]$ is the kernel of the morphism:

$$S \to K[a, b, c, d], \ x_0 \mapsto ab, \ x_1 \mapsto ac, \ x_2 \mapsto ad, \ x_3 \mapsto bc, \ x_4 \mapsto bd,$$

$$x_5 \mapsto cd, \ x_6 \mapsto a^2, \ x_7 \mapsto b^2, \ x_8 \mapsto c^2, \ x_9 \mapsto d^2.$$  

$K[x_6, x_7, x_8, x_9]$ is trivially a Noether normalization of $S/I$ and the ideal in $(I)$ is generated by all monomials of degree 2 in $x_0, \ldots, x_8$ except $x_0 x_5$. Thus, $X$ is arithmetically Cohen–Macaulay by Proposition 2.1, and in $(I) : (x_0, \ldots, x_5) = (x_0, \ldots, x_4, x_5^2) \cap (x_0^2, x_1, \ldots, x_3) = (x_1, \ldots, x_4, x_0^2, x_0 x_5)$ by Remark 2.6. So reg $X = 3$ and $H(I) = -1$ by Theorem 2.5.

3. Nonarithmetically Cohen–Macaulay projective curves

Let $I$ be a saturated ideal of $S$ defining a not necessarily reduced projective curve $\mathcal{C}$ in $\mathbb{P}^n_K$, such that $\mathcal{C}$ is not arithmetically Cohen–Macaulay. Assume that $K[x_{n-1}, x_n]$ is a Noether normalization of $S/I$, and denote by $(I)$ the initial ideal of $I$ with respect to the reverse lexicographic order. As in Section 2, using quotients of zero-dimensional ideals, we shall give an effective method to compute reg $I$ that does not require the knowledge of a minimal graded free resolution of $I$. In this case, we cannot reduce the computation of reg $I$ to the computation of reg in $(I)$ because reg $I$ may be strictly smaller than reg in $(I)$ (see the example in [3, Remark 2.10]). The problem here is that in $(I)$ may not even be saturated.

The following result reduces the computation of reg $I$ to the computation of the Castelnuovo–Mumford regularity of a monomial ideal $M(I)$ satisfying the same properties as $I$.

Lemma 3.1. Let $I \subset S = K[x_0, \ldots, x_n]$ be a saturated ideal and assume that $K[x_{n-1}, x_n]$ is a Noether normalization of $S/I$, and that $S/I$ is not Cohen–Macaulay. Let $K(t)$ be a simple transcendental extension of $K$, and let $S'$ denote the polynomial ring $K(t)[x_0, \ldots, x_n]$. Set $I' := \chi(I)S'$, where $\chi : S' \to S'$ is the morphism defined by $x_0 \mapsto x_0, \ldots, x_{n-1} \mapsto x_{n-1}, \ x_n \mapsto x_n 1$. Then, the monomial ideal $M(I)$ of $S$ generated by normalized generators of in $(I')$ is a saturated ideal, $K[x_{n-1}, x_n]$ is a Noether normalization of $S/M(I)$, $S/M(I)$ is not Cohen–Macaulay, and reg $I' = \text{reg} M(I)$.
Proof. Since the (monic) generators of the monomial ideals in \((I')\) and \(M(I)\) coincide, then \(\text{reg}(I') = \text{reg} M(I)\). Moreover, \(\text{reg} I = \text{reg} IS'\), so one has to prove that \(\text{reg} IS' = \text{reg} (I')\).

Since the field \(K\) is infinite, \(K[x_{n-1}, x_n]\) is a Noether normalization of \(S/I\), and \(I\) is a saturated ideal, then there exists a finite subset \(F\) of \(K\) such that \(x_n - \kappa x_{n-1}\) is a nonzero divisor on \(S/I\), for all \(\kappa \in K - F\). So \(x_n - tx_{n-1}\) is a nonzero divisor on \(S'/IS'\) and one gets the equality \(\text{reg} IS' = \text{reg} IS', x_n - tx_{n-1}\) from [6, Proposition 20.20].

Let \(\gamma : S' \to S'\) be the morphism defined by \(x_0 \mapsto x_0, \ldots, x_{n-1} \mapsto x_{n-1}, x_n \mapsto x_n + tx_{n-1}\). One has that \(\text{reg} IS', x_n - tx_{n-1}\) = \(\text{reg} ((IS'), x_n)\). Since the ideals \((\gamma(IS'), x_n)\) and \((I', x_n)\) coincide, and \(x_n\) is a nonzero divisor on \(S'/I'\), then the equality \(\text{reg} IS', x_n - tx_{n-1}\) = \(\text{reg} I'\) follows from [6, Proposition 20.20].

Finally, by Bermejo and Gimenez [3, Remark 2.10], one has the equality \(\text{reg} I' = \text{reg} (I')\). ☐

Definition 3.2. We shall call the monomial ideal \(M(I)\) in Lemma 3.1 the associated monomial ideal of \(I\). Observe that none of the minimal generators of \(M(I)\) is divisible by \(x_n\).

Let \(M_1, \ldots, M_\ell\) denote the minimal generators of \(M(I)\) which are divisible by \(x_{n-1}\) and let \(N_1, \ldots, N_\ell\) be the images of these monomials by the evaluation morphism which sends \(x_{n-1}\) to 1. Since we have assumed that \(S/I\) is not Cohen–Macaulay, then \(\ell \geq 1\) (Lemma 3.1 and Proposition 2.1). Let us assume that the monomials \(M_i\) have been ordered by increasing power of \(x_{n-1}\) i.e., setting \(d_i := \text{deg}_{x_{n-1}} M_i\), one has \(1 \leq d_1 \leq \cdots \leq d_\ell\).

Let \(M(I)_0 \subset S\) be the ideal generated by the image of \(M(I)\) by the evaluation morphism which sends \(x_{n-1}\) to 0. For all \(i = 1, \ldots, \ell\), define the monomial ideal \(M(I)_i := M(I)_0 + (N_1, \ldots, N_i)\). One gets a strictly increasing sequence of ideals in \(S\), all of them minimally generated by monomials which are not divisible by the variables \(x_{n-1}\) and \(x_n\) and defining arithmetically Cohen–Macaulay curves in \(\mathbb{P}^n_K\):

\[
M(I)_0 \subset M(I)_1 \subset \cdots M(I)_\ell \subset S.
\]

Observe that \(M(I)_\ell\) is the ideal generated by the image of \(M(I)\) by the evaluation morphism which sends \(x_{n-1}\) and \(x_n\) to 1.

Denote by \(h_i, 1 \leq i \leq \ell\), the maximum of the degrees of the minimal generators of \(M(I)_i - 1 : (x_0, \ldots, x_{n-2})\) which are divisible by \(N_i\).

Theorem 3.3. Let \(I \subset S = K[x_0, \ldots, x_n]\) be a saturated ideal defining a nonarithmetically Cohen–Macaulay projective curve \(\mathcal{C} \subset \mathbb{P}^n_K\) and assume that \(K[x_{n-1}, x_n]\) is a Noether normalization of \(S/I\). Let \(M(I)\) be the associated monomial ideal of \(I\). Defining \(M(I)_0\), and the integers \(d_1, \ldots, d_\ell\) and \(h_1, \ldots, h_\ell\) as above, one has

\[
\text{reg} \mathcal{C} = \max \{\delta(M(I)_0: (x_0, \ldots, x_{n-2})), 1, d_1 + h_1, \ldots, d_\ell + h_\ell\}.
\]

Moreover, if \(\text{reg} \mathcal{C} = \max \{d_1 + h_1, \ldots, d_\ell + h_\ell\}\),
then

\[ H(I) = \max \{d_1 + h_1, \ldots, d_\ell + h_\ell \} - 1. \]

**Proof.** \( \text{reg } \mathcal{G} = \text{reg } M(I) = \max \{ \text{reg } M(I_0), H(M(I)) + 1 \} \) from Lemma 3.1 and [3, Theorem 2.7]. By Theorem 2.5, \( \text{reg } M(I_0) = \delta (M(I)_0 : (x_0, \ldots, x_{n-2})) + 1. \) Thus, we have to prove that

\[ \max \{ \text{reg } M(I)_0, H(M(I)) + 1 \} = \max \{ \text{reg } M(I)_0, d_1 + h_1, \ldots, d_\ell + h_\ell \}. \]

Define \( F_i := \{ \alpha = (x_0, \ldots, x_{n-2}) \in \mathbb{N}^{n-1} \times \mathbb{X}^{(\alpha, 0, 0) \in M(I) - M(I)' - 1} \} \) for all \( i = 1 \leq i \leq \ell \), and consider the following partition which is a reformulation of the one introduced in [5, p. 3213]:

\[ \{ \alpha \in \mathbb{N}^{n+1} \times \mathbb{X}^2 \notin M(I) \} = \{ \alpha \in \mathbb{N}^{n+1} \times \mathbb{X}^{0} \ldots \mathbb{X}^{n-2} \notin M(I)' \} \cup \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_\ell, \]

where \( \mathcal{R}_i := \bigcup_{i \in F_i} \{ (x, x_{n-1}, x_n) \in \mathbb{N}^{n+1} \times \mathbb{X}^{0} < d_i \} \). Then, the value at \( s \in \mathbb{N} \) of the Hilbert function \( H_{M(I)} \) of \( S/M(I) \) is \( H_{M(I)}(s) = H_{M(I)'}(s) + \# \{ \beta \in \mathcal{R}_i \mid |\beta| = s \} \).

For all \( i = 1 \leq i \leq \ell \), one has that \( d_i + \max_{\alpha \in F_i} \{ |\alpha| \} - 1 \) is the smallest integer \( s_0 \) such that for \( s \geq s_0 \), \( \# \{ \beta \in \mathcal{R}_i \mid |\beta| = s \} = d_i \). \( F_i \).

It is easy to check that \( \max_{\alpha \in F_i} \{ |\alpha| \} = h_i \) for all \( i = 1 \leq i \leq \ell \). Thus, one has that

\[ H(M(I)) \leq \max \{ H(M(I)') + 1, d_1 + h_1 - 1, \ldots, d_\ell + h_\ell - 1 \}. \]

Since \( M(I)_0 \subseteq M(I)' \), one gets the inequality: \( H(M(I)) + 1 \leq \max \{ \text{reg } M(I)_0 - 1, d_1 + h_1, \ldots, d_\ell + h_\ell \}. \)

Let us now prove that \( \max \{ \text{reg } M(I)_0, H(M(I)) + 1 \} = \max \{ \text{reg } M(I)_0, d_1 + h_1, \ldots, d_\ell + h_\ell \}. \) Indeed, if \( \max \{ \text{reg } M(I)_0, d_1 + h_1, \ldots, d_\ell + h_\ell \} = \text{reg } M(I)_0 \) the result follows from the previous inequality. Otherwise, it is easy to check that \( H(M(I)) + 1 = \max \{ d_1 + h_1, \ldots, d_\ell + h_\ell \} \) and we are done.

Finally, let us prove that if \( \text{reg } \mathcal{G} = \max \{ d_1 + h_1, \ldots, d_\ell + h_\ell \} \), then \( H(I) = \max \{ d_1 + h_1, \ldots, d_\ell + h_\ell \} - 1. \) Indeed, since \( \max \{ d_1 + h_1, \ldots, d_\ell + h_\ell \} \geq \text{reg } M(I)_0 \geq \text{reg } M(I)_0 = H(M(I)') + 2 \), then \( H(M(I)) = \max \{ d_1 + h_1 - 1, \ldots, d_\ell + h_\ell - 1 \}. \) The result now follows from the equality \( H(I) = H(M(I)). \)

**Remark 3.4.** To determine \( M(I) \), we shall need a Gröbner basis computation with respect to the reverse lexicographic order. As in Remark 2.6, once \( M(I) \) has been determined, one can compute \( \text{reg } \mathcal{G} \) ‘by hand’ with no extra Gröbner basis computation: the integers \( \delta (M(I)_0 : (x_0, \ldots, x_{n-2})) \) and \( h_i \) in Theorem 3.3 can be obtained by least common multiples of monomials.

The Gröbner basis computation needed to determine \( M(I) \) has to be done over \( K(t) \), a simple transcendental extension of \( K \). For practical applications, a random choice of an element of the field \( K \) could replace the field extension. Nevertheless, if one knows beforehand that the ideal in \( (I) \) is saturated, we do not need to compute over \( K(t) \) to determine the regularity of \( I \). Indeed, the regularity of \( I \) coincides with \( \text{reg } (I) \) by [3, Remark 2.10]. Thus, \( \text{reg } I = \text{reg } M \text{in } (I) \) from Lemma 3.1, and \( M \text{in } (I) \) is the monomial ideal obtained replacing \( x_0 \) by \( x_{n-1} \) in \( (I) \). In particular, if \( x_n \) is a nonzero divisor on \( S/I \), \( \text{reg } I = \text{reg } M \text{in } (I) \) and moreover, \( M \text{in } (I) = (I) \).
Example 3.5. Consider the monomial ideal \( I \subset S = K[x_0, \ldots, x_3] \) generated by \( x_0^5, x_1^3, x_2^2x_1, x_0^2x_2^5 \) and \( x_0x_1x_3 \). It is easy to check that \( K[x_2, x_3] \) is a Noether normalization of \( S/I \). Since the quotient ideal \( I : (x_2, x_3) \) is equal to \( I \), then \( I \) is saturated. If \( \mathcal{C} \subset \mathbb{P}^3_K \) denotes the projective curve defined by \( I \), then \( \mathcal{C} \) is not arithmetically Cohen–Macaulay by Remark 3.4. The associated monomial ideal \( M(I) \) of \( I \) is equal to \( (x_0^5, x_1^3, x_2^2x_1^2, x_0x_1x_2^5) \) by Remark 3.4. Thus, \( M_1 = x_0x_1x_2, M_2 = x_0^5x_2, d_1 = 1, d_2 = 5, \) and \( N_1 = x_0x_1, N_2 = x_0^2 \). On the other hand, \( M(I)_0 : (x_0, x_1) = (x_0^5, x_1^3, x_0x_1) : (x_0, x_1) = (x_0^4, x_1^3, x_0x_1) \cap (x_0^2, x_1^2) = (x_0^4, x_1^3, x_0x_1, x_0x_1^7) \), so \( h_1 = 3 \), and \( M(I)_1 : (x_0, x_1) = (x_0^5, x_1^3, x_0x_1) : (x_0, x_1) = (x_0^4, x_1^3, x_0x_1), \) so \( h_2 = 4 \). Applying Theorem 3.3, \( \text{reg} \mathcal{C} = 9 \) and \( H(I) = 8 \).

Example 3.6 (Bermejo and Giménez [3, Remark 2.10]). Consider the homogeneous ideal \( I \subset S = K[x_0, \ldots, x_3] \) generated by \( x_0^3 - 3x_0x_1 + 5x_0x_3, x_0x_1 - 3x_1^2 + 5x_1x_3, x_0x_2 - 3x_1x_2, 2x_0x_3 - x_1x_3 \) and \( x_0^2 - x_1x_2 - 2x_1x_3 \). It is easy to check that \( K[x_2, x_3] \) is a Noether normalization of \( S/I \), and that \( I \) is saturated. Since \( x_2 \) is a zero divisor on \( S/I \), the projective curve \( \mathcal{C} \subset \mathbb{P}^3_K \) defined by \( I \) is not arithmetically Cohen–Macaulay. Computing (with [7] for example) the initial ideal of the image of \( I \) in \( K(t)[x_0, \ldots, x_3] \) by the morphism which sends \( x_3 \) to \( tx_2 \), one gets that \( M(I) = (x_0^2, x_0x_1, x_1^2, x_0x_2, x_1x_2) \subset S \). Applying Theorem 3.3, \( \text{reg} \mathcal{C} = 2 \) and \( H(I) = 1 \).

Example 3.7. Consider now the defining ideal \( I \subset S = K[x_0, \ldots, x_8] \) of the projective monomial curve \( \mathcal{C} \subset \mathbb{P}^8_K \) given by the parametrization:

\[
\begin{align*}
x_0 &= su^{24}, & x_1 &= s^2u^{31}, & x_2 &= s^3u^{22}, & x_3 &= s^9u^{16}, \\
x_4 &= s^{11}u^{14}, & x_5 &= s^{18}u^7, & x_6 &= s^{24}u, & x_7 &= u^{25}, & x_8 &= s^{25}.
\end{align*}
\]

One knows beforehand that \( I \) is saturated (it is prime), that \( K[x_7, x_8] \) is a Noether normalization of \( S/I \), and that \( x_8 \) is a nonzero divisor on \( S/I \). Computing \( \text{in}(I) \) with [8], one gets that it is generated by 35 elements of degree \( \leq 7 \), and that seven of them are divisible by \( x_7 \). Thus, \( \mathcal{C} \) is not arithmetically Cohen–Macaulay by Proposition 2.1. One can now compute \( \text{reg} \mathcal{C} \) by hand applying Theorem 3.3 to \( \text{in}(I) \) and using that \( M(\text{in}(I)) = \text{in}(I) \) (Remark 3.4). The regularity of \( \mathcal{C} \) is 8 and \( H(I) = 7 \).

Remark 3.8. In the previous example, the equality \( \text{reg} \mathcal{C} = 8 \) is obtained in less than one second using [4]. Nevertheless, a minimal graded free resolution of \( I \) could not be computed with the command \texttt{mres} of SINGULAR (Buchberger’s syzygy algorithm) in a Pentium III with 128 MB. It came out of memory after 2 h. We guess that, in this example, improving \texttt{mres} using that \( \text{reg} \mathcal{C} = 8 \), a minimal graded free resolution of \( I \) should be obtained.

4. About the implementation of the results

The algorithms derived from Theorems 2.5 and 3.3 can be easily implemented. This has been done by the authors and Greuel in the library “mregular.lib” [4] of SINGULAR.
Since we have always assumed that \( K[x_{n-d}, \ldots, x_n] \) is a Noether normalization of \( S/I \), with \( d = \dim S/I - 1 \), one can use the following criterion to check this hypothesis.

**Lemma 4.1 (Noether normalization test).** Let \( I \) be a homogeneous ideal of \( S = K[x_0, \ldots, x_n] \) such that \( d = \dim S/I - 1 \geq 0 \), and denote by \( \text{in}(I) \) the initial ideal of \( I \) with respect to the reverse lexicographic order. The following are equivalent:

(a) \( K[x_{n-d}, \ldots, x_n] \) is a Noether normalization of \( S/I \);
(b) \( \forall i : 0 \leq i \leq n - d - 1 \), there exists \( r_i \in \mathbb{N} - \{0\}/x_i^{r_i} \in \text{in}(I) \);
(c) \( \dim S/(I, x_{n-d}, \ldots, x_n) = 0 \);
(d) \( \dim S/(\text{in}(I), x_{n-d}, \ldots, x_n) = 0 \).

**Proof.** Condition (c) \( \Leftrightarrow \) (d) is a consequence of the equality in \((I, x_{n-d}, \ldots, x_n) = (\text{in}(I), x_{n-d}, \ldots, x_n) \). (b) \( \Leftrightarrow \) (d) and (a) \( \Rightarrow \) (b) are obvious. Thus, one has to prove that (b) \( \Rightarrow \) (a).

Suppose that \( \forall i : 0 \leq i \leq n - d - 1 \), there exists \( r_i \in \mathbb{N} - \{0\}/x_i^{r_i} \in \text{in}(I) \). Thus, \( \forall i : 0 \leq i \leq n - d - 1 \), there exists a homogeneous polynomial \( f_i \in I \) such that \( f_i = x_i^{r_i} + h_i \), where \( h_i \in (x_{i+1}, \ldots, x_n) \). It implies that the affine variety \( V_K(f_0, \ldots, f_{n-d-1}, x_{n-d}, \ldots, x_n) \) is equal to \((0)\), and so \( V_K(I, x_{n-d}, \ldots, x_n) \) is equal to \((0)\) too. The result now follows from [8, Proposition 5.4; 9, Remark 6.5.0].

We now sketch the way Theorems 2.5 and 3.3 have been implemented in the procedures \texttt{reg\_CM} and \texttt{reg\_curve} of [4], respectively.

**Algorithm 4.2.** Procedure \texttt{reg\_CM}:

**Input:** \( I \), a homogeneous ideal of \( S = K[x_0, \ldots, x_n] \) defining an arithmetically Cohen–Macaulay projective subscheme \( \mathcal{X} \) of \( \mathbb{P}_K^n \) (i.e. such that \( S/I^{\text{sat}} \) is Cohen–Macaulay), and satisfying that \( K[x_{n-d}, \ldots, x_n] \) is a Noether normalization of \( S/I \).

**Output:** The Castelnuovo–Mumford regularity of \( \mathcal{X} \), i.e. \( \text{reg} \mathcal{X} = \text{reg} I^{\text{sat}} \).

1. Compute the reduced Gröbner basis w.r.t. \( \text{dp} \) (reverse lexicographic order) of the saturation \( I^{\text{sat}} \) of \( I \) (using the command \texttt{sat}), and get \( \text{in}(I^{\text{sat}}) \) and \( d = \dim S/I - 1 \) (= \( \dim S/\text{in}(I^{\text{sat}}) - 1 \)).
2. If \( d = -1 \), return a WARNING message together with the value of the Castelnuovo–Mumford regularity of the ideal \( I \) (which coincides with the regularity of the Hilbert function of \( S/I \) obtained by Proposition 2.4).
3. Check that \( K[x_{n-d}, \ldots, x_n] \) is a Noether normalization of \( S/I \) (Lemma 4.1 applied to \( I^{\text{sat}} \)). If the answer is no, return a WARNING message.
4. Check that \( S/I^{\text{sat}} \) is Cohen–Macaulay (Proposition 2.1 applied to \( I^{\text{sat}} \)). If the answer is no, return a WARNING message.
5. Compute \( \text{reg} \mathcal{X} \) applying Theorem 2.5 to \( I^{\text{sat}} \).

**Note.** The algorithm also computes the regularity of the Hilbert function of \( S/I^{\text{sat}} \) by Theorem 2.5.
Algorithm 4.3. Procedure \texttt{reg\_curve}:

\textit{Input:} \( I \), a homogeneous ideal of \( S=K[x_0,\ldots,x_n] \) defining a not necessarily reduced projective curve \( C \) in \( \mathbb{P}^n_K \) (i.e. such that \( \dim S/I = 2 \)), and satisfying that \( K[x_{n-1},x_n] \) is a Noether normalization of \( S/I \).

\textit{Output:} The Castelnuovo–Mumford regularity of \( C \), i.e. \( \text{reg} C = \text{reg} I \).

1. Compute the Gröbner basis w.r.t. dp of \( I^{\text{sat}} \), and get in \( (I^{\text{sat}}) \) and \( d = \dim S/I - 1 \) (=\( \dim S/\text{in}(I^{\text{sat}}) - 1 \)).
2. Check that \( d = 1 \). If the answer is no, return a WARNING message.
3. Check that \( K[x_{n-1},x_n] \) is a Noether normalization of \( S/I \) (Lemma 4.1 applied to \( I^{\text{sat}} \)). If the answer is no, return a WARNING message.
4. Check if \( S/I^{\text{sat}} \) is Cohen–Macaulay or not (Proposition 2.1 applied to \( I^{\text{sat}} \)).
   - If the answer is YES, compute \( \text{reg} C \) applying Theorem 2.5 to \( I^{\text{sat}} \).
   - If the answer is NO, check if in \( (I^{\text{sat}}) \) is saturated or not.
     - If the answer is YES, replace \( x_n \) by \( x_{n-1} \) in in \( (I^{\text{sat}}) \) to obtain the ideal \( M(\text{in}(I^{\text{sat}})) \) (Remark 3.4).
     - Compute \( \text{reg} C \) applying Theorem 3.3 to the ideal in \( (I^{\text{sat}}) \).
     - If the answer is NO, compute in \( K(t )[x_0,\ldots,x_n] \) the initial ideal w.r.t. dp of the image of \( I^{\text{sat}} \) by the morphism which sends \( x_n \) to \( tx_{n-1} \) and get \( M(I^{\text{sat}}) \) (Lemma 3.1).
     - Compute \( \text{reg} C \) applying Theorem 3.3 to the ideal \( I^{\text{sat}} \).

Note. The algorithm also computes the regularity of the Hilbert function of \( S/I^{\text{sat}} \) in some cases (see Theorems 2.5 and 3.3).

Remark 4.4. In both algorithms, the last step requires the computation of quotients of monomial ideals. This can be done using the command \texttt{quotient}, or creating an easy procedure based on Remark 2.6. Both methods are very efficient.

In the special case of projective monomial curves, one can make the implementation more effective since it is not necessary, neither to check the hypothesis, nor to compute over the field extension \( K(t) \).

Algorithm 4.5. Procedure \texttt{reg\_moncurve}:

\textit{Input:} \( a_0 = 0 < a_1 < \cdots < d := a_n \), a strictly increasing sequence of integers whose first element is 0.

\textit{Output:} The Castelnuovo–Mumford regularity of the projective monomial curve \( C \subset \mathbb{P}^n_K \) parametrized by \( (s^d : s^{d-a_1}t^{a_1} : \cdots : s^{d-a_n-1}t^{a_n-1} : t^d) \).

1. Compute the defining ideal \( I \) of \( C \) using elimination.
2. Check if \( S/I \) is Cohen–Macaulay or not (Proposition 2.1).
   - If \( S/I \) is Cohen–Macaulay, compute \( \text{reg} C \) by Theorem 2.5.
   - Otherwise, compute \( \text{reg} C \) applying Theorem 3.3 to the ideal \( \text{in}(I) \) using that \( M(\text{in}(I)) = \text{in}(I) \) (Remark 3.4).
References