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Asymptotic Expansions for Large Deviation Probabilities of Noncentral Generalized Chi-Square Distributions

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Asymptotic expansions for large deviation probabilities are used to approximate the cumulative distribution functions of noncentral generalized chi-square distributions, preferably in the far tails. The basic idea of how to deal with the tail probabilities consists in first rewriting these probabilities as large parameter values

for spherical measures to the multivariate domain of large deviations under consideration. At the so-called dominating point, the largest main curvature of the boundary of this domain tends to one as the large deviation parameter approaches infinity. Therefore, the dominating point degenerates asymptotically. For this reason the recent multivariate asymptotic expansion for large deviations in Breitung and Richter (1996, *J. Multivariate Anal.* **58**, 1–20) does not apply. Assuming a suitably parametrized expansion for the inverse $\tilde{g}^{-1}$ of the negative logarithm of the density-generating function, we derive a series expansion for the function f_k. Note that low-order coefficients from the expansion of $\tilde{g}^{-1}$ influence practically all coefficients in the expansion of the tail probabilities. As an application, classification probabilities when using the quadratic discriminant function are discussed.

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1. INTRODUCTION

A commonly made assumption in the theory of statistical models is the uncorrelatedness of the error variables. If the considered population follows a multivariate normal law then the uncorrelatedness of its components coincides with their independence. Many statisticians have been trying to allow sampling in cases of dependent but uncorrelated observations, just like in the class of spherically symmetric distributions. In the present paper we consider sampling from populations with distributions belonging to this

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class. Note that it includes both heavy and light tailed sampling distributions such as the Pearson-VII type and Kotz type distributions, respectively.

The slightly more general class of elliptically contoured distributions was initially studied by Schoenberg (1938) and Kelker (1970). Many authors after them contributed to the now quite complex and even matrix variate theory and gave numerous statistical applications which have been reviewed recently by Gupta and Varga (1993). Anderson and Fang (1982) considered quadratic forms for elliptically contoured distributions and studied their central distributions. Cacoullous and Koutras (1984) as well as Hsu (1990) discussed the corresponding noncentral distributions. In the present paper we consider large deviation probabilities for these noncentral distributions and derive asymptotic representations and expansions for large deviation probabilities of noncentral generalized chi-square distributions.

Asymptotic representations as well as saddlepoint approximations for large deviation probabilities are shown in several papers, e.g., Daniels (1987), to be good explicit approximations for tail probabilities of statistical distributions. Asymptotic expansions for large deviations in the noncentral generalized chi-square distribution are used in Ittrich *et al.* (2000) for making statistical inferences concerning the mean in multivariate elliptically contoured distributions. For another application of large deviations in the noncentral generalized chi-square distribution see the example at the end of this section.

An asymptotic expansion for large deviation of the ordinary central chi-square distribution with k degrees of freedom (d.f.), i.e., for $1-\mathrm{CQ}(k)(x)$ as $x\to\infty$, can easily be obtained by dealing with the asymptotic behavior of the one-dimensional Laplace integral

$$\int_1^\infty y^{k/2-1}e^{-(x/2)\,y}\,dy$$

using Laplace's method. General results in this direction can be found in Bleistein and Handelsman (1975) as well as Fedorjuk (1977). As an application it was shown in Richter and Schumacher (1990) that the cumulative chi-square distribution function satisfies the asymptotic expansion formula (in the notation of Bleistein and Handelsman)

$$1-\mathrm{CQ}(k)(c^2)\sim\frac{c^{k-2}e^{-c^2/2}}{2^{k/2-1}\Gamma\left(\frac{k}{2}\right)}\left[1+\sum_{l=1}^{\infty}\frac{2^l\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)c^{2l}}\right], \tag{1}$$

as $c \to \infty$, where for $k = 2m$, $m \in \mathbb{N}$, we put

$$\frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k}{2} - l\right)} = 0 \qquad \text{for} \quad l > \frac{k}{2}.$$

Recognize that if $\mathbf{X} = (X_1, ..., X_k)^{\mathrm{T}}$ follows the standard Gaussian distribution then

$$1 - \mathrm{CQ}(k)(c^2) = P(X_1^2 + \cdots + X_k^2 > c^2),$$

where $X_1^2, ..., X_k^2$ are i.i.d. random variables having a finite moment-generating function. A relation similar to (1) also could therefore be proved using a suitable saddlepoint technique as in, e.g., Daniels (1987). The necessary standard assumptions for applying such techniques, however, are far from being satisfied if $\mathbf{X}$ is distributed according to a non-Gaussian spherically symmetric probability law.

Let $\mathrm{CQ}(k; g)(x)$, $x \in \mathbb{R}$, denote the cumulative distribution function of the g-generalized chi-square distribution, i.e., the distribution of $\|\mathbf{X}\|^2$, where $\mathbf{X}$ follows a k-dimensional spherically symmetric distribution with density

$$p(\mathbf{x}; g) = C(k, g)\, g(\|\mathbf{x}\|^2), \qquad \mathbf{x} \in \mathbb{R}^k,$$

$$0 < \frac{1}{C(k, g)} = \omega_k \int_0^\infty r^{k-1} g(r^2)\, dr < \infty,$$

where $\omega_k = 2\pi^{k/2}/\Gamma(k/2)$ denotes the content of the surface area of the unit sphere in $\mathbb{R}^k$.

It is known that the density of $\|\mathbf{X}\|^2$ admits the representation

$$\frac{\partial}{\partial r} \mathrm{CQ}(k; g)(r) = K(k, g)\, r^{k/2 - 1} g(r), \qquad r > 0,$$

where $K(k, g)$ is a suitably chosen norming constant.

In the same way as was used in Richter and Schumacher (1990), one can show therefore that if g is of Kotz type, i.e.,

$$g(r) = r^{N-1} e^{-\beta r^\gamma}, \qquad r > 0,$$

for some constants $\gamma > 0$, $\beta > 0$, and $2N + k > 2$, then

$$1 - \mathrm{CQ}(k;\, g)(c^2) \sim \frac{\beta^{(k+2N-2)/(2\gamma)-1}}{\Gamma\left(\dfrac{k+2N-2}{2\gamma}\right)} c^{k+2N-2-2\gamma} e^{-\beta c^{2\gamma}} \times \left[1 + \sum_{l=1}^{\infty} d_l c^{-2l\gamma}\right] \tag{2}$$

as $c \to \infty$, where

$$d_l = \frac{(k+2N-2-2\gamma)(k+2N-2-4\gamma) \cdots (k+2N-2-2l\gamma)}{(2\gamma\beta)^l}.$$

The noncentral g-generalized chi-square distribution function with k d.f. and noncentrality parameter (n.c.p.) $\delta^2 > 0$ can be defined by

$$\mathrm{CQ}(k, \delta^2;\, g)(x) = P(\|\mathbf{X} + \mu\|^2 < x), \qquad x \in \mathbb{R}, \tag{3}$$

for arbitrary $\mu \in \mathbb{R}^k$ satisfying $\|\mu\|^2 = \delta^2$. This distribution was first studied by Cacoullos and Koutras (1984) and later by Hsu (1990). We start our considerations of large deviations for this distribution from the formula

$$1 - \mathrm{CQ}(k, \delta^2;\, g)(c^2) = \Phi(cA_c^\delta;\, g), \qquad c > 0, \tag{4}$$

where $\Phi(\,\cdot\,; g)$ denotes the probability measure corresponding to the density $p(\,\cdot\,; g)$,

$$A_c^\delta := \left\{\mathbf{x} \in \mathbb{R}^k : \left\| \mathbf{x} - \begin{pmatrix} \delta/c \\ \mathbf{o}_{k-1} \end{pmatrix} \right\| \geqslant 1 \right\}, \qquad c > 0,$$

and

$$cA = \{(cx_1, ..., cx_k)^T : (x_1, ..., x_k)^T \in A\}.$$

We have thus reformulated the original one-dimensional problem of evaluating the probabilities $1 - \mathrm{CQ}(k, \delta^2;\, g)(c^2)$ as the k-dimensional problem to determine the probabilities that a spherically distributed random vector $\mathbf{X}$ falls into the set cA_c^δ, $c > 0$.

Large deviation probabilities for random vectors have been studied by many authors. Asymptotic expansions for such probabilities can be deduced formally from corresponding expansions for multiple Laplace integrals for large parameters. The possibility of expansions of multiple integrals with boundary maxima is discussed in Bleistein and Handelsman (1975), Fedorjuk (1977), and Wong (1989), but no methods for obtaining higher-order expansion terms explicitly are outlined there. A geometric approach to an asymptotic expansion for a certain class of large deviation probabilities of Gaussian random vectors has been developed recently in Breitung and Richter (1996). This approach is devoted to large deviation domains with boundaries whose main curvatures near the so-called dominating points do not change when the large deviation parameter approaches infinity. The uniquely determined dominating point of the large deviation domain cA_c^δ considered above is $(c-\delta, \mathbf{o}_{k-1})^{\mathrm{T}}$. The largest main curvature $\kappa(c) = 1 - \delta/c$ of the boundary of cA_c^δ approaches 1 when c approaches infinity. At the same time, the leading term of the expansion in Breitung and Richter (1996) tends to infinity. Hence, this expansion is not applicable in the present case. We shall develop therefore a new type of asymptotic expansion for large deviations which will be new even when the underlying distribution is a Gaussian one.

EXAMPLE 1 (Classification Probabilities). Let a random vector $\mathbf{X}$ follow a k-dimensional elliptically contoured distribution such that for some $\mu \in \mathbb{R}^k$ and $\sigma > 0$,

$$\frac{1}{\sigma}(\mathbf{X} - \mu) \overset{d}{\sim} \Phi(\,\cdot\,; g)$$

holds. We assume that with $\mu_1 \neq \mu_2$, $\sigma_1^2 \neq \sigma_2^2$, the hypotheses

$$H_1\colon (\mu, \sigma^2) = (\mu_1, \sigma_1^2) \qquad \text{and} \qquad H_2\colon (\mu, \sigma^2) = (\mu_2, \sigma_2^2)$$

hold true with probabilities p and $1-p$, respectively. In accordance with Dorflo (1993), let us make the decision that $\mathbf{X}$ satisfies H_1 if for its observed value $\mathbf{x}$, the critical point $b = \ln(\sigma_1(1-p)^2/[\sigma_2 p^2])$ and the quadratic discriminant function

$$Q_0(\mathbf{x}) = \|\mathbf{x} - \mu_2\|^2/\sigma_2^2 - \|\mathbf{x} - \mu_1\|^2/\sigma_1^2$$

holds

$$Q_0(\mathbf{x}) > b.$$

Recognize that if $\sigma_1^2 > \sigma_2^2$ then $Q_0(\mathbf{x}) > b$ can be rewritten as

$$\|\mathbf{x} - \vartheta\|^2 > c(b),$$

where

$$\vartheta = (\sigma_1^2 - \sigma_2^2)^{-1}\,(\sigma_1^2\mu_2 - \sigma_2^2\mu_1)$$

and

$$c(b)=(\sigma_1^2-\sigma_2^2)^{-1}\,(b\sigma_1^2\sigma_2^2+\|\mu_1\|^2\,\sigma_2^2-\|\mu_2\|^2\,\sigma_1^2$$
$$+(\sigma_1^2-\sigma_2^2)^{-2}\,\|\sigma_1^2\mu_2-\sigma_2^2\mu_1\|^2).$$

Let $P_2(Q_0(\mathbf{X})>b)$ denote the conditional misclassification probability of deciding for H_1 when H_2 is actually true. Then

$$P_2(Q_0(\mathbf{X})>b)=P_2\left(\left\|\frac{1}{\sigma}(\mathbf{X}-\mu)+\frac{1}{\sigma}(\mu-\vartheta)\right\|^2>\frac{c(b)}{\sigma^2}\right)$$
$$=\Phi\left(\left\{\mathbf{z}\in\mathbb{R}^k:\|\mathbf{z}+\delta_2\|^2>\frac{c(b)}{\sigma_2^2}\right\};\,g\right)$$

with

$$\delta_2=\frac{1}{\sigma_2}(\mu_2-\vartheta).$$

It follows that $P_2(Q_0(\mathbf{X})>b)$ can be written in terms of the noncentral generalized chi-square distribution function as

$$P_2(Q_0(\mathbf{X})>b)=1-\mathrm{CQ}(k,\,\|\delta_2\|^2;\,g)\left(\frac{c(b)}{\sigma_2^2}\right)$$
$$=1-\mathrm{CQ}(k,\,\sigma_2^2\,\|\mu_1-\mu_2\|^2/(\sigma_1^2-\sigma_2^2)^2;\,g)\left(\frac{c(b)}{\sigma_2^2}\right).$$

Note that if p is "small' then b and $c(b)$ are "large." For sufficiently large $c(b)$, Theorem 3.5 below applies in the sense that the right side of relation (19) serves as a suitable approximation for $P_2(Q_0(\mathbf{X})>b)$ if the quantities c^2 and δ^2 in (19) are substituted by $c(b)/\sigma_2^2$ and $\sigma_2^2\,\|\mu_1-\mu_2\|^2/(\sigma_1^2-\sigma_2^2)^2$, respectively.

2. ESTIMATES FOR THE LARGE DEVIATION PROBABILITIES

Before deriving the announced asymptotic expansion we will give lower and upper bounds for the large deviation probabilities

$$1-\mathrm{CQ}(k,\,\delta^2;\,g)(c^2).$$

We shall restrict our attention to the case that the density-generating function g admits the representation

$$g(r) = e^{-\tilde{g}(r)}, \qquad r > 0, \tag{D1}$$

with $\tilde{g}$ being first-order continuously differentiable and invertible for large r $(r \geqslant r_0^2)$. In the main part of what follows we further assume that a parameter $\lambda = \lambda(\tilde{g}, c)$ can be chosen in such a way that $\tilde{g}^{-1}$ allows a power series expansion in the form

$$\frac{\tilde{g}^{-1}(\lambda z + \tilde{g}((c-\delta)^2))}{c^2} = \sum_{j=0}^{m} c_j z^j + O(z^{m+1}), \qquad z \to 0, \qquad (\text{D2}, m)$$

where m is a natural number, the coefficients $c_j = c_j(\lambda, c)$ approach certain constants c_j^* as c tends to infinity,

$$c_j = c_j(\lambda, c) \to c_j, \qquad c \to \infty,$$

and

$$c_1^* > 0.$$

From (D2, 0) it follows that

$$c_0 = \left(1 - \frac{\delta}{c}\right)^2 \to 1 = c_0^* \qquad \text{as} \quad c \to \infty.$$

Remark. The coefficients c_j depend on the derivatives of $\tilde{g}$ at the point $(c-\delta)^2$. A straightforward proof shows that for $m \geqslant 3$,

- $c_1 = \lambda / [c^2 \tilde{g}'((c-\delta)^2)] > 0$
- $c_2 = -(\lambda/2)\, c_1 \tilde{g}''((c-\delta)^2) / [\tilde{g}'((c-\delta)^2)]^2$
- $c_3 = -(\lambda^2/12)\, c_1 [\tilde{g}'''((c-\delta)^2) / [\tilde{g}'((c-\delta)^2)]^3 - 3[\tilde{g}''((c-\delta)^2)]^2 / [\tilde{g}'((c-\delta)^2)]^4]$.

EXAMPLE 2 (Kotz Type Density-Generating Function). Since the Kotz type density-generating function has the form

$$g(r) = r^{N-1} \exp\{-\beta r^{\gamma}\}, \qquad \gamma > 0, \quad \beta > 0, \quad 2N + k > 2,$$

it follows that

$$\tilde{g}(r) = \beta r^{\gamma} - (N-1) \ln r$$

and we have

$$c_1 = \frac{\lambda}{c^{2\gamma}} \frac{(1-\delta/c)^2}{\left(\gamma\beta\left(1-\frac{\delta}{c}\right)^{2\gamma} - \frac{N-1}{c^{2\gamma}}\right)},$$

$$c_2 = -\frac{\lambda^2}{2c^{4\gamma}} \frac{\left(1-\frac{\delta}{c}\right)^2 \left(\gamma(\gamma-1)\,\beta\left(1-\frac{\delta}{c}\right)^{2\gamma} + \frac{N-1}{c^{2\gamma}}\right)}{\left(\gamma\beta\left(1-\frac{\delta}{c}\right)^{2\gamma} - \frac{N-1}{c^{2\gamma}}\right)^3},$$

and

$$c_3 = -\frac{\lambda^3}{12c^{6\gamma}} \frac{\left(1-\frac{\delta}{c}\right)^2 \left(\gamma(\gamma-1)(\gamma-2)\,\beta\left(1-\frac{\delta}{c}\right)^{2\gamma} - 2\,\frac{N-1}{c^{2\gamma}}\right)}{\left(\gamma\beta\left(1-\frac{\delta}{c}\right)^{2\gamma} - \frac{N-1}{c^{2\gamma}}\right)^4}$$

$$+\frac{\lambda^3}{4c^{6\gamma}} \frac{\left(1-\frac{\delta}{c}\right)^2 \left(\gamma(\gamma-1)\,\beta\left(1-\frac{\delta}{c}\right)^{2\gamma} + \frac{N-1}{c^{2\gamma}}\right)^2}{\left(\gamma\beta\left(1-\frac{\delta}{c}\right)^{2\gamma} - \frac{N-1}{c^{2\gamma}}\right)^5}.$$

Thus $\tilde{g}$ satisfies assumption (D2, m), $m = 1, 2...$, with $\lambda = c^{2\gamma}$.

Example 3 (Pearson VIII Type Density Generating Function). Since the Pearson-VII type density-generating function has the form

$$g(r) = \left(1+\frac{r}{m}\right)^{-M}, \qquad r > 0,$$

for certain constants $M > k/2$, $m > 0$, it follows that

$$\tilde{g}(r) = M \ln\left(1+\frac{r}{m}\right)$$

and we have

$$c_1 = \frac{\lambda}{M}\left(\frac{m}{c^2} + \left(1-\frac{\delta}{c}\right)^2\right), \qquad c_2 = \frac{\lambda^2}{2M^2}\left(\frac{m}{c^2} + \left(1-\frac{\delta}{c}\right)^2\right),$$

$$c_3 = \frac{\lambda^3}{6M^3}\left(\frac{m}{c^2} + \left(1-\frac{\delta}{c}\right)^2\right).$$

Thus $\tilde{g}$ satisfies assumption (D2, m), $m=1, 2, \ldots$, with an arbitrary positive constant λ.

THEOREM 2.1. *If assumptions* (D1) *and* (D2, 1) *are satisfied then there exist positive constants* k_1 *and* K_1 *such that*

$$k_1 c^k \lambda^{-(k+1)/2} \exp\{-\tilde{g}((c-\delta)^2)\} \leqslant 1-\mathrm{CQ}(k, \delta^2; g)(c^2)$$
$$\leqslant K_1 c^k \lambda^{-1} \exp\{-\tilde{g}((c-\delta)^2)\}. \quad (5)$$

COROLLARY 2.2. *If* $\lambda = c^{2\gamma}$ *then*

$$k_1 c^{k-(k+1)\gamma} \leqslant \frac{1-\mathrm{CQ}(k, \delta^2; g)(c^2)}{e^{-\tilde{g}((c-\delta)^2)}} \leqslant K_1 c^{k-2\gamma}. \quad (6)$$

If g is the Kotz type density-generating function then

$$k_1 c^{2(N-1)+k-(k+1)\gamma} e^{-\beta(c-\delta)^{2\gamma}} \leqslant 1-\mathrm{CQ}(k, \delta^2; g)(c^2)$$
$$\leqslant K_1 c^{2(N-1)+k-2\gamma} e^{-\beta(c-\delta)^{2\gamma}}. \quad (7)$$

If λ *is a positive constant there exist positive constants* k_2 *and* K_2 *such that*

$$k_2 \leqslant \frac{1-\mathrm{CQ}(k, \delta^2; g)(c^2)}{c^k g((c-\delta)^2)} \leqslant K_2,$$

where, formally, $k_2 = k_1/\lambda^{(k+2)/2}$ *and* $K_2 = K_1/\lambda$. *If g is the Pearson-VII type density-generating function then*

$$k_3 m^M c^{k-2M} \leqslant 1-\mathrm{CQ}(k, \delta^2; g)(c^2) \leqslant K_3 m^M c^{k-2M},$$

where

$$k_3 = k_2 \left(\frac{m}{c^2} + \left(1-\frac{\delta}{c}\right)^2\right)^M \to k_2, \qquad K_3 \to K_2 \qquad \text{as} \quad c \to \infty.$$

This corollary follows from Theorem 2.1 by choosing λ as $c^{2\gamma}$ or as a positive constant, respectively.

Remark. Inequalities (7) generalize well-known results from the Gaussian case $N=1$, $\beta=\frac{1}{2}$, $\gamma=1$ to a much more complex situation. Note further the dominating role of γ (in comparison with the other parameters N and β) in choosing the "artificial" parameter λ.

The main tool for proving Theorem 2.1 as well as the asymptotic expansion of Theorem 3.5 below is a geometric representation formula for the probabilities under consideration. Let

$$S_k(r)=\{\mathbf{x}\in\mathbb{R}^k : \|\mathbf{x}\|=r\}, \qquad r>0.$$

Denoting by ω the uniform probability distribution on the unit sphere $S_k(1)$ we define the intersection-percentage function of a Borel set $A\subseteq\mathbb{R}^k$ as

$$F(A,r)=\omega((r^{-1}A)\cap S_k(1)), \qquad r>0. \tag{8}$$

Further, let

$$I_{k,\tilde{g}}=\int_0^\infty r^{k-1}e^{-\tilde{g}(r^2)}\,dr. \tag{9}$$

LEMMA 2.3 (Geometric Measure Representation Formula). *If the density-generating function g satisfies assumption* (D1) *then for $c\geqslant r_0/\sqrt{c_0}$ and all $\lambda>0$ it holds, that*

$$\begin{aligned}\Phi(cA_c^\delta;\,g)=&\frac{\lambda}{2I_{k,\tilde{g}}}\exp\{-\tilde{g}((c-\delta)^2)\}\\ &\times\int_0^\infty F\left(A_c^\delta,\sqrt{\frac{\tilde{g}^{-1}(\lambda z+\tilde{g}((c-\delta)^2))}{c^2}}\right)\\ &\times[\tilde{g}^{-1}(\lambda z+\tilde{g}((c-\delta)^2))]^{(k-2)/2}\\ &\times\tilde{g}^{-1\prime}(\lambda z+\tilde{g}((c-\delta)^2))\,e^{-\lambda z}\,dz.\end{aligned} \tag{10}$$

Remark. Note that the assumption (D2, m) is not necessary for the geometric measure representation formula (10) to hold.

Remark. The representation formula (10) essentially relies on the fact that the distance of the set A_c^δ from the origin is $1-\delta/c$. Hence analogous formulas can be derived for sets A sharing this property. This will be exploited below.

Proof. Using the general representation formula for spherical distributions from Richter (1995) we get

$$\Phi(cA_c^\delta;\,g)=\frac{c^k}{I_{k,\tilde{g}}}\int_0^\infty F(cA_c^\delta,cv)\,v^{k-1}e^{-\tilde{g}(c^2v^2)}\,dv.$$

Since

$$F(cA_c^\delta, cv) \equiv F(A_c^\delta, v), \qquad \forall c > 0,$$

by the definition of the intersection-percentage function F and because

$$F(A_c^\delta, v) \equiv 0 \qquad \text{for} \quad v \in \left[0, 1 - \frac{\delta}{c}\right)$$

we have

$$\Phi(cA_c^\delta; g) = \frac{c^k}{I_{k, \tilde{g}}} \int_{1-\delta/c}^{\infty} F(A_c^\delta, v)\, v^{k-1} e^{-\tilde{g}(c^2v^2)}\, dv.$$

For $v \geqslant 1 - \delta/c$ and $c \geqslant r_0/\sqrt{c_0}$, the relation

$$\tilde{g}(c^2v^2) = \lambda y$$

is invertible and the substitution $\tilde{g}(c^2v^2) = \lambda y$ yields

$$\Phi(cA_c^\delta; g) = \frac{\lambda}{2I_{k, \tilde{g}}} \int_{\tilde{g}((c-\delta)^2)/\lambda}^{\infty} F\left(A_c^\delta, \sqrt{\frac{\tilde{g}(\lambda y)}{c^2}}\right)$$
$$\times [\tilde{g}^{-1}(\lambda y)]^{(k-2)/2}\, \tilde{g}^{-1\prime}(\lambda y)\, e^{-\lambda y}\, dy.$$

The asserted representation now follows by substituting

$$z = y - \tilde{g}((c-\delta)^2)/\lambda. \quad \blacksquare$$

Proof of Theorem 2.1. Let

$$A_S := \{\mathbf{x} \in \mathbb{R}^k : \|\mathbf{x}\| \geqslant 1 - \delta/c\}$$

be the complement of a sphere with radius $1 - \delta/c$ and let

$$A_H := \{\mathbf{x} \in \mathbb{R}^k : x_1 \geqslant 1 - \delta/c\}$$

be a halfspace with the same distance from the origin as A_S and A_c^δ. Then

$$A_H \subseteq A_c^\delta \subseteq A_S$$

and consequently

$$\Phi(cA_H; g) \leqslant \Phi(cA_c^\delta; g) \leqslant \Phi(cA_S; g).$$

As remarked after Lemma 2.3, geometric measure representation formulae

$$\Phi(cA;\, g) = \frac{\lambda}{2I_{k,\,\tilde g}} \exp\{-\tilde g((c-\delta)^2)\}$$
$$\times \int_0^\infty F\left(A,\, \sqrt{\frac{\tilde g^{-1}(\lambda z + \tilde g((c-\delta)^2))}{c^2}}\right)$$
$$\times [\tilde g^{-1}(\lambda z + \tilde g((c-\delta)^2))]^{(k-2)/2}$$
$$\times \tilde g^{-1\prime}(\lambda z + \tilde g((c-\delta)^2))\, e^{-\lambda z}\, dz, \tag{11}$$

analogous to (10), hold true for the sets $A = A_H$ and $A = A_S$ as well. We will exploit now these formulae to determine the asymptotic behavior of the probabilities $\Phi(cA_H;\, g)$ and $\Phi(cA_S;\, g)$ as $c \to \infty$.

We start with the derivation of the upper bound. Since

$$F\left(A_S,\, \sqrt{\frac{\tilde g^{-1}(\lambda z + \tilde g((c-\delta)^2))}{c^2}}\right) \equiv 1 \qquad \text{for} \quad z \in [0, \infty],$$

we have

$$\Phi(cA_S;\, g) = \frac{c^k}{2I_{k,\,\tilde g}} \exp\{-\tilde g((c-\delta)^2)\}$$
$$\times \int_0^\infty \left[\frac{\tilde g^{-1}(\lambda z + \tilde g((c-\delta)^2))}{c^2}\right]^{(k-2)/2}$$
$$\times \frac{\partial}{\partial z}\left[\frac{\tilde g^{-1}(\lambda z + \tilde g((c-\delta)^2))}{c^2}\right] e^{-\lambda z}\, dz.$$

Using assumptions (D1) and (D2, 1), an application of Laplace's method to the last integral implies

$$\Phi(cA_S;\, g) \sim \frac{c^k}{2\lambda I_{k,\,\tilde g}} \exp\{-\tilde g((c-\delta)^2)\}\, c_0^{(k-2)/2} c_1, \qquad \lambda \to \infty.$$

Hence

$$\Phi(cA_S;\, g) \asymp \frac{c^k}{\lambda} \exp\{-\tilde g((c-\delta)^2)\} \qquad \text{as} \quad \lambda \to \infty. \tag{12}$$

We now deal with A_H. The intersection-percentage function for halfspaces was determined in Richter (1992) and generalized in Richter (1995). For more details we refer the reader to Section 3, formula (15) in the latter. It holds that

$$F\left(A_H, \sqrt{\frac{\tilde{g}^{-1}(\lambda z + \tilde{g}((c-\delta)^2))}{c^2}}\right) = \frac{\omega_{k-1}}{\omega_k} \int_0^{\alpha^*(z)} (\sin\alpha)^{k-2}\, d\alpha$$

with

$$\alpha^*(z) = \arctan \sqrt{\frac{\tilde{g}^{-1}(\lambda z + \tilde{g}((c-\delta)^2))}{(c-\delta)^2} - 1}.$$

Using

$$\arctan x \sim x, \qquad x \to 0,$$

we get

$$\alpha^*(z) \sim \sqrt{\frac{\tilde{g}^{-1}(\lambda z + \tilde{g}((c-\delta)^2))}{(c-\delta)^2} - 1}, \qquad z \to 0.$$

From

$$\int_0^{\varepsilon} (\sin\alpha)^{k-2}\, d\alpha \sim \frac{\varepsilon^{k-1}}{k-1}, \qquad \varepsilon \to 0,$$

it now follows that

$$F\left(A_H, \sqrt{\frac{\tilde{g}^{-1}(\lambda z + \tilde{g}((c-\delta)^2))}{c^2}}\right) \sim \frac{\omega_{k-1}}{\omega_k(k-1)} \left(\frac{\tilde{g}^{-1}(\lambda z + \tilde{g}((c-\delta)^2))}{(c-\delta)^2} - 1\right)^{(k-1)/2}.$$

From assumptions (D1) and (D2, 1) we have

$$\left[\frac{\tilde{g}^{-1}(\lambda z + \tilde{g}((c-\delta)^2))}{(c-\delta)^2} - 1\right]^{(k-1)/2} \asymp z^{(k-1)/2}, \qquad z \to 0$$

$$\left[\frac{\tilde{g}^{-1}(\lambda z + \tilde{g}((c-\delta)^2))}{c^2}\right]^{(k-2)/2} \asymp 1, \qquad z \to 0,$$

$$\frac{\tilde{g}^{-1\prime}(\lambda z + \tilde{g}((c-\delta)^2))}{c^2}\, \lambda \asymp 1, \qquad z \to 0.$$

These relations, together with an application of Laplace's method, lead to the asymptotic equivalence

$$\Phi(cA_H; g) \asymp c^k \lambda^{-(k+1)/2} \exp\{-\tilde{g}((c-\delta)^2)\}, \qquad c \to \infty. \tag{13}$$

This conclude the proof. ∎

3. MAIN RESULTS

The basic idea of how to deal with the large deviation probabilities of the noncentral generalized chi-square distributions consists in first rewriting these large deviation probabilities as large parameter values of the Laplace transform of a certain parameter-dependent function

$$y \to f_k(c, k, y)$$

defined below, second expanding $f_k(c, \lambda, y)$ into a series with respect to powers of $y^{1/2}$, and third applying a suitable modification of Watson's lemma.

LEMMA 3.1 (Laplace Integral Representation). *If the density-generating function g satisfies assumption* (D1) *then for $c \geqslant r_0/\sqrt{c_0}$ and all $\lambda > 0$ it holds that*

$$1 - \mathrm{CQ}(k, \delta^2; g)(c^2) = \frac{\lambda}{2cI_{k,\tilde{g}}} e^{-\tilde{g}((c-\delta)^2)} \int_0^\infty f_k(c, \lambda, y)\, e^{-(\lambda/c)\, y}\, dy \tag{14}$$

with

$$f_k(c, \lambda, y) = F\left(A_c^\delta, \sqrt{\frac{\tilde{g}^{-1}\left(\frac{\lambda}{c} y + \tilde{g}((c-\delta)^2)\right)}{c^2}}\right) \times \left[\tilde{g}^{-1}\left(\frac{\lambda}{c} y + \tilde{g}((c-\delta)^2)\right)\right]^{(k-2)/2} \tilde{g}^{-1\prime}\left(\frac{\lambda}{c} y + \tilde{g}((c-\delta)^2)\right).$$

Proof. The assertion of this lemma follows immediately from Lemma 2.3 by substituting $z = y/c$. ∎

The essential step in expanding f_k is to derive a series representation for the intersection-percentage function F.

In Ittrich *et al.* (2000) it is shown that the set A_c^δ belongs to the system $\mathfrak{A}(\text{dir}, \text{dist})$ of Borel sets defined in Richter (1995) (Fig. 1). In Richter (1995) it is shown that

$$F(A, r) = \frac{\omega_{k-1}}{\omega_k} \int_0^{\alpha^*(r)} (\sin \alpha)^{k-2}\, d\alpha \tag{15}$$

holds for arbitrary A from $\mathfrak{A}(\text{dir}, \text{dist})$, where

$$\alpha^*(r) = \arc\tan \left(\left(\frac{r}{R_A(r)} \right)^2 - 1 \right)^{1/2}$$

and the so-called distance-type function $r \to R_A(r)$ describes a certain geometric property of the set A. For the set $A = A_c^\delta$ the function R_A has been determined in Ittrich *et al.* (2000):

$$R_A(r) = \frac{c^2 - \delta^2 - c^2 r^2}{2\delta c}.$$

We are now in a position to expand F.

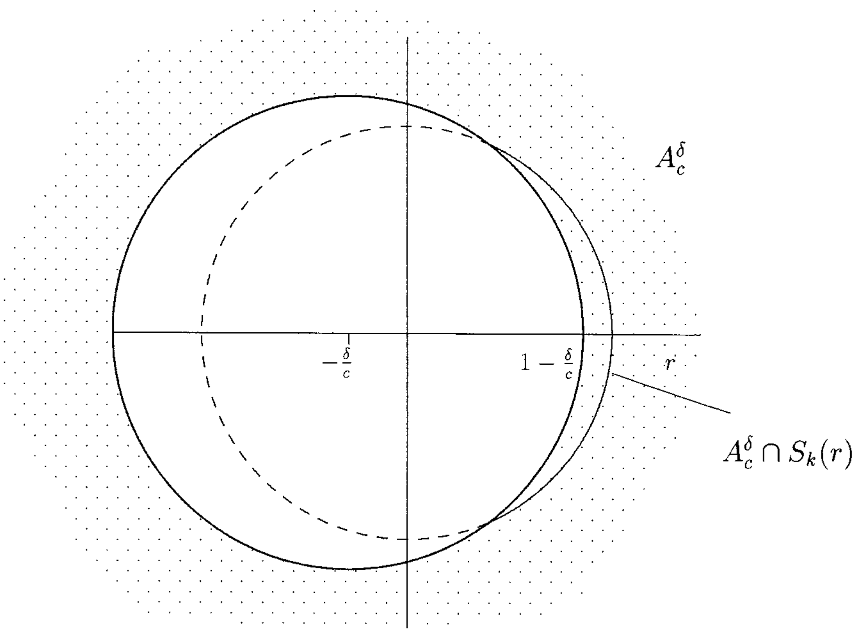

FIG. 1. The sets A_c^δ and $A_c^\delta \cap S_k(r)$.

LEMMA 3.2 (Expansion of the Intersection-Percentage Function). *Under the assumptions* (D1) *and* (D2, m) *the intersection-percentage function admits the representation*

$$F\left(A_c^\delta, \sqrt{\frac{\tilde{g}^{-1}(\lambda y/c+\tilde{g}((c-\delta)^2))}{c^2}}\right)$$

$$=\frac{\omega_{k-1}}{\omega_k}\sum_{j=0}^{m-1}\frac{B_j}{2j+k-1}\,y^{(2j+k-1)/2}+O(y^{(k-1)/2+m}), \tag{16}$$

as $y\to 0$, *for some well-defined constants* B_j *given explicitly in formula* (28) *below.*

The proof of this lemma is quite technical, and we will therefore leave it to Section 5.

Expansion (16) for the intersection-percentage function combined once more with the assumed expansion (D2, m) for $\tilde{g}^{-1}$ leads to an expansion for the whole function f_k.

LEMMA 3.3. *It holds that*

$$f_k(c,\lambda,y)=\frac{\omega_{k-1}}{\omega_k}\frac{c^k}{\lambda}\sum_{j=0}^{m-1}b_{j+1}\,y^{(k-1)/2+j}$$

$$+O(y^{(k-1)/2+m}),\qquad y\to 0, \tag{17}$$

with the first three coefficients b_j *given explicitly in the proof of* (17) *in Section* 5.

Remarks. Note that low-order coefficients from the expansion of $\tilde{g}^{-1}$ influence all coefficients b_j, $j=j_0, ..., m$, starting from some index j_0.

Although the aim of this paper is to derive an expansion in terms of c we are still dealing here with an expansion in terms of y. Consequently, the coefficients b_j occurring in Lemma 3.3 are not yet ordered with respect to the powers of c.

LEMMA 3.4 (Modified Watson's Lemma). *Let* $f\colon (0,\infty)^{\times 2}\to\mathbb{R}$ *satisfy the following assumptions*:

(i) $f(\lambda,\cdot)$ *is locally integrable for every* $\lambda>0$ *and uniformly* (*with respect to* λ) *bounded on finite intervals*;

(ii) $f(\lambda,y)=O(e^{ay})$, $y\to\infty$ *uniformly in* λ;

(iii) *for $y \to 0+$ the function f allows the expansion*

$$f(\lambda, y) = \sum_{j=0}^{m} c_j y^{a_j} + O(y^{a_{m+1}})$$

uniformly with respected to λ, where the sequence (a_j) increases monotonically to $+\infty$ as $j \to \infty$, $a_0 > -1$, and

$$c_j = c_j(\lambda) = O(1), \qquad \lambda \to \infty.$$

Then it holds that

$$\int_0^\infty f(\lambda, t)\, e^{-\lambda t}\, dt = \sum_{j=0}^{m} c_j \frac{\Gamma(a_j+1)}{\lambda^{a_j+1}} + O(\lambda^{-(a_{m+1}+1)}), \qquad \lambda \to \infty. \tag{18}$$

The proof repeats the arguments of the proof for the original lemma, given for example in Bleistein and Handelsman (1975, pp. 103–104). One only has to ensure that all estimates are uniformly valid with respect to λ.

THEOREM 3.5. *If g and λ satisfy the assumptions* (D1) *and* (D2, m) *and $\lambda/c \to \infty$ as $c \to \infty$, then*

$$\begin{aligned} 1 - \mathrm{CQ}(k, \delta^2; g)(c^2) &= \frac{\Gamma(k/2)}{2\sqrt{\pi}\, I_{k,\tilde{g}}}\, c^{k-1} e^{-\tilde{g}((c-\delta)^2)} \\ &\quad \times \left[b_1 \frac{k-1}{2} \left(\frac{c}{\lambda}\right)^{(k+1)/2} + b_2 \frac{(k-1)(k+1)}{4} \left(\frac{c}{\lambda}\right)^{(k+3)/2} + \cdots \right. \\ &\quad + b_m \frac{(k-1)\cdots(k+2m-1)}{2^m} \left(\frac{c}{\lambda}\right)^{(k-1)/2+m} \\ &\quad \left. + O\left(\left(\frac{c}{\lambda}\right)^{(k+1)/2+m}\right)\right] \end{aligned} \tag{19}$$

as $c \to \infty$ with constants b_j from Lemma 3.3.

Proof. Since we have the expansion (17) we can apply the modification of Watson's Lemma to the Laplace integral representation (14) with parameter λ/c instead of λ. This yields

$$
\begin{aligned}
&1-\mathrm{CQ}(k,\delta^2;\,g)(c^2)\\
&\quad=\frac{c^{k-1}\omega_{k-1}}{2\omega_k I_{k,\tilde g}}\,e^{-\tilde g((c-\delta)^2)}\left[\sum_{j=0}^{m-1} b_{j+1}\,\frac{\Gamma((k+1)/2+j)}{(\lambda/c)^{(k+1)/2+j}}\right.\\
&\qquad\left.+\,O\left(\left(\frac{c}{\lambda}\right)^{(k+1)/2+m}\right)\right]\\
&\quad=\frac{c^{k-1}}{2\sqrt{\pi}\,I_{k,\tilde g}}\,e^{-\tilde g((c-\delta)^2)}\,\frac{\Gamma(k/2)}{\Gamma((k-1)/2)}\\
&\qquad\times\left[b_1\Gamma\left(\frac{k+1}{2}\right)\left(\frac{c}{\lambda}\right)^{(k+1)/2}+b_2\Gamma\left(\frac{k+3}{2}\right)\left(\frac{c}{\lambda}\right)^{(k+3)/2}\right.\\
&\qquad+b_3\Gamma\left(\frac{k+5}{2}\right)\left(\frac{c}{\lambda}\right)^{(k+5)/2}+\cdots+b_m\Gamma\left(\frac{k-1}{2}+m\right)\left(\frac{c}{\lambda}\right)^{(k-1)/2+m}\\
&\qquad\left.+\,O\left(\left(\frac{c}{\lambda}\right)^{(k+1)/2+m}\right)\right].
\end{aligned}
$$

The assertion follows from using properties of the Γ-function. ▮

Note again that low-order coefficients from the expansion of $\tilde g^{-1}$ influence practically all coefficients in the expansion of the tail probabilities $1-\mathrm{CQ}(k,\delta^2;\,g)(c^2)$ starting from some index.

COROLLARY 3.6. *If g is the Kotz type density-generating function with $\gamma>1/2$ then it holds that*

$$
\begin{aligned}
&1-\mathrm{CQ}(k,\delta^2;\,g)(c^2)\\
&\quad=\frac{\beta^{k/2\gamma+(N-1)/\gamma-(k+1)/2}\,\Gamma\left(\dfrac{k}{2}\right)}{2\sqrt{\pi}\,\gamma^{(k-1)/2}\delta^{(k-1)/2}\,\Gamma\left(\dfrac{k}{2\gamma}+\dfrac{N-1}{\gamma}\right)}\\
&\qquad\times c^{(3k-1)/2-\gamma(k+1)+2N-2}e^{-\beta(c-\delta)^{2\gamma}}\\
&\qquad\times\left[1+D_{1-2\gamma}c^{1-2\gamma}+D_{-2\gamma}c^{-2\gamma}+D_{-1-2\gamma}c^{-1-2\gamma}+D_{2-4\gamma}c^{2-4\gamma}\right.\\
&\qquad+D_{1-4\gamma}c^{1-4\gamma}+D_{-4\gamma}c^{-4\gamma}+D_{-1-4\gamma}c^{-1-4\gamma}+D_{-2-4\gamma}c^{-2-4\gamma}\\
&\qquad\left.+\,O(c^{3-6\gamma})\right],\qquad c\to\infty,
\end{aligned}
\tag{20}
$$

with coefficients D_j given explicitly in the proof of (20) *in Section* 5.

Remarks. In accordance with the remarks after Lemma 3.3 and Theorem 3.5 the coefficients occurring in the expansions of Theorem 3.5

and Corollary 3.6 are not ordered with respect to the powers of c^{-1}. Actually, the value of the parameter γ influences the ordering of the expansion terms with respect to their rate of convergence to zero as c approaches infinity. Moreover, the orders of two or more expansion terms can coincide for special choices of γ. Therefore, there is formally no uniqueness in the notation of the coefficients D_ψ.

If, e.g., $\gamma = 1$ then

$$1 - 2\gamma > -2\gamma = 2 - 4\gamma,$$

$$\max\{-1-2\gamma, 1-4\gamma, -4\gamma, -1-4\gamma, -2-4\gamma\} = 3 - 6\gamma,$$

and $D_{-2\gamma}$ and $D_{2-4\gamma}$ are both coefficients of c^{-2}. If in this case we put $D^*_{-2} = D_{-2\gamma} + D_{2-4\gamma}$ then the result of Corollary 3.6 can be reformulated as

$$\begin{aligned} 1 - \mathrm{CQ}(k, \delta^2; g)(c^2) &= \frac{\beta^{N-3/2}\Gamma\left(\frac{k}{2}\right)}{2\sqrt{\pi}\,\delta^{(k-1)/2}\Gamma\left(\frac{k}{2}+N-1\right)} c^{(k-3)/2+2N-2} e^{-\beta(c-\delta)^2} \\ &\quad \times [1 + D_{-1}c^{-1} + D^*_{-2}c^{-2} + O(c^{-3})]. \end{aligned}$$

If $\gamma = 0.9$ then $1 - 2\gamma > 2 - 4\gamma > -2\gamma$ and the result from Corollary 3.6 can be reformulated as

$$\begin{aligned} 1 - \mathrm{CQ}(k, \delta^2; g)(c^2) &= \frac{\beta^{k/2\gamma+(N-1)/\gamma-(k+1)/2}\Gamma\left(\frac{k}{2}\right)}{2\sqrt{\pi}\,\gamma^{(k-1)/2}\delta^{(k-1)/2}\Gamma\left(\frac{k}{2\gamma}+\frac{N-1}{\gamma}\right)} \\ &\quad \times c^{(3k-1)/2-\gamma(k+1)+2N-2} e^{-\beta(c-\delta)^{2\gamma}} \\ &\quad \times [1 + D_{-0.8}c^{-0.8} + D_{-1.6}c^{-1.6} + D_{-1.8}c^{-1.8} + O(c^{-2.4})]. \end{aligned}$$

In the case $(N, \beta, \gamma) = (1, \frac{1}{2}, 1)$ of a Gaussian density generator the leading term in (20) is

$$\frac{1}{2\sqrt{2\pi}\,\delta^{(k-1)/2}} c^{(k-3)/2} e^{-(c-\delta)^2/2},$$

which is in accordance with Theorem 2.1 of order between those of the k-dimensional standard Gaussian measure of the half space A_H an the complement of the sphere A_S.

4. NUMERICAL EXPERIENCES

Because of the available computing techniques, today many statistical distributions can be sufficiently precisely estimated or numerically determined by simulating a suitable sample or by evaluating possibly complicated multiple integrals, respectively. Nevertheless, mathematicians and statisticians will continue to be interested in explicit exact or approximative analytical representations of these distributions for different reasons. Explicit formulae facilitate both quantitative and qualitative discussions on how different parameters of a distribution affect on a probability under this distribution. Explicit representation or approximation formulae for statistical distributions make possible discussions on monotonicity and about

TABLE I

$\delta^2 = 1.00,\ N = 1.00,\ \gamma = 1.00,\ \beta = 0.50$

k	c^2	$1 - \mathrm{CQ}(k, \delta^2; g)$	A_1 rel. err.	A_2 rel.error	A_3 rel. error
2	10.00	0.02995061	0.03167661 0.0576	0.03435470 0.1470	0.03029226 0.0114
	17.00	0.00193875	0.00197664 0.0195	0.00208108 0.0734	0.00195393 0.0078
	24.00	0.00011242	0.00011322 0.0071	0.00011778 0.0477	0.00011302 0.0053
4	14.00	0.02235200	0.02456210 0.0988	0.01997715 0.1062	0.02334853 0.0445
	22.00	0.00112615	0.00121140 0.0757	0.00105495 0.0632	0.00114929 0.0205
	28.00	0.00010629	0.00011336 0.0664	0.00010115 0.0484	0.00010775 0.0136
5	15.00	0.02696826	0.03359910 0.2458	0.01783363 0.3387	0.03004551 0.1141
	24.00	0.00102545	0.00122769 0.1972	0.00083206 0.1885	0.00107433 0.0476
	30.00	0.00010075	0.00011863 0.1774	0.00008621 0.1442	0.00010397 0.0319

TABLE II

$\delta^2 = 1.00,\ N = 2.00,\ \gamma = 0.90,\ \beta = 1.00$

k	c^2	$1 - \mathrm{CQ}(k, \delta^2; g)$	A_1 rel. err.	A_2 rel.error	A_3 rel. error
3	12.00	0.01945001	0.02130089 0.0951	0.02130089 0.0951	0.01856359 0.0455
	17.00	0.00192154	0.00197658 0.0286	0.00197658 0.0286	0.00192685 0.0027
	23.00	0.00010378	0.00010453 0.0072	0.00010453 0.0072	0.00010450 0.0069
4	12.00	0.02930958	0.03459325 0.1802	0.02828663 0.0349	0.02691409 0.0817
	19.00	0.00124264	0.00132998 0.0702	0.00117400 0.0552	0.00125370 0.0089
	24.00	0.00011373	0.00011946 0.0503	0.00010781 0.0520	0.00011504 0.0115
5	13.00	0.02922505	0.03801759 0.3008	0.02107394 0.2789	0.02826410 0.0328
	20.00	0.00134058	0.00156775 0.1694	0.00109777 0.1811	0.00138699 0.0346
	25.00	0.00012903	0.00014708 0.1398	0.00010999 0.1475	0.00013306 0.0312

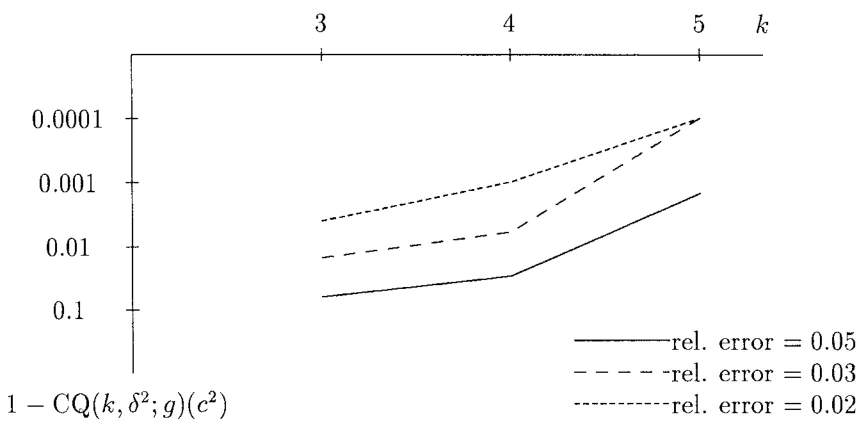

FIG. 2. Relative approximation error ($\delta^2 = 1.00$, $n = 1.00$, $\gamma = 1.00$, $\beta = 0.50$)

least-favorable parameter situations. Explicit approximation formulae are sometimes used in exact numerical methods for generating certain initial values. The explicit asymptotic approximations for large deviation probabilities from Theorem 3.5 and Corollary 3.6 are used in Ittrich *et al.* (2000) for deriving more or less explicit asymptotic quantile approximation formulae and iteration procedures.

In the tables of this section we compare exact tail probabilities with approximations for them based on the results of Corollary 3.6. To this end we take into account approximation results using one, two, and three terms of the asymptotic expansion in the columns A_1, A_2, and A_3, respectively. In the evaluation of the leading term of the expansion we actually used b_1 instead of the asymptotically equal term $1/(2\delta^{(k-1)/2}(\gamma\beta)^{(k+1)/2})$, because this proved to lead to more accurate approximations. The exact tail probabilities in the column $1-\mathrm{CQ}(k, \delta^2; g)$ are determined using a numerical algorithm given in Ittrich *et al.* (2000).

Table I gives numerical results for the usual noncentral chi-square distribution, whereas Table II deals with the noncentral generalized chi-square distribution when the density-generating function is of Kotz type with constants $(N; \gamma; \beta) = (2; 0.9; 1)$.

These parameter configurations are chosen to reflect a certain typical behavior of the approximations.

Both tables show the decrease of relative error for increasing values of c. The approximation with two terms is not always superior to the one-term approximation. Furthermore, the approximation becomes worse for increasing dimension (d.f.). This indicates that an asymptotic expansion including additionally the asymptotic for $k \to \infty$ might result in better numerical approximations for larger k.

Figure 2 illustrates how the degree of freedom k influence the values $1-\mathrm{CQ}(k, \delta^2; g)(c^2)$ for which relative approximation errors of 0.05, 0.03, and 0.02, respectively, can be guaranteed.

5. PROOFS AND AUXILIARY RESULTS

For $y \leqslant y_0$ it holds that

$$F\left(A_c^\delta, \sqrt{\frac{\tilde{g}^{-1}\left(\frac{\lambda}{c}\, y + \tilde{g}((c-\delta)^2)\right)}{c^2}}\right) = \frac{\omega_{k-1}}{\omega_k} \int_0^{\arctan \eta(y)} (\sin \alpha)^{k-2}\, d\alpha$$

with

$$\eta(y) = \sqrt{\frac{4\delta^2 \tilde{g}^{-1}\left(\frac{\lambda}{c}\, y + \tilde{g}((c-\delta)^2)\right)}{\left(c^2 - \delta^2 - \tilde{g}^{-1}\left(\frac{\lambda}{c}\, y + \tilde{g}((c-\delta)^2)\right)\right)^2} - 1}.$$

For proving Lemma 3.2 we start by expanding $\eta(y)$ for $y \to 0$.

LEMMA 5.1. *Under assumptions* (D1) *and* (D2, m) *it holds that*

$$\eta(y) = \sum_{j=0}^{m-1} a_{j+1}\, y^{j+1/2} + O(y^{m+1/2}) \tag{21}$$

as $y \to 0$, *where the first coefficients are*

$$a_1 = \sqrt{c_1}\, \frac{1}{\sqrt{\delta}\,(1-\delta/2)},$$

$$a_2 = c_1^{3/2} \frac{3+\delta/c}{8\delta^{3/2}(1-\delta/c)^2} + \frac{c_2}{\sqrt{c_1}} \frac{1}{2\sqrt{\delta}\, c(1-\delta/c)},$$

and

$$a_3 = c_1^{5/2} \frac{23 + 10\delta/c - \delta^2/c^2}{128\delta^{5/2}(1-\delta/c)^3} + \sqrt{c_1}\, c_2 \frac{3(3+\delta/c)}{16\delta^{3/2}c(1-\delta/c)^2}$$
$$- \frac{c_2^2}{c_1^{3/2}} \frac{1}{8\sqrt{\delta}\, c^2(1-\delta/c)} + \frac{c_3}{\sqrt{c_1}} \frac{1}{2\sqrt{\delta}\, c^2(1-\delta/c)}.$$

Proof. We start with considering

$$\frac{4\delta^2 \tilde{g}^{-1}\left(\frac{\lambda}{c}\, y + \tilde{g}((c-\delta)^2)\right)}{\left(c^2 - \delta^2 - \tilde{g}^{-1}\left(\frac{\lambda}{c}\, y + \tilde{g}((c-\delta)^2)\right)\right)^2}$$

$$= \frac{4\delta^2}{c^2} \frac{\tilde{g}^{-1}\left(\frac{\lambda}{c}\, y + \tilde{g}((c-\delta)^2)\right)\Big/ c^2}{\left(1 - \frac{\delta^2}{c^2} - \frac{\tilde{g}^{-1}(\frac{\lambda}{c} y + \tilde{g}((c-\delta)^2))}{c^2}\right)^2}.$$

From (D2, m) we have

$$\frac{4\delta^2}{c^2}\frac{\tilde{g}^{-1}\left(\frac{\lambda}{c}\,y+\tilde{g}((c-\delta)^2)\right)\Big/c^2}{\left(1-\frac{\delta^2}{c^2}-\frac{\tilde{g}^{-1}(\frac{\lambda}{c}\,y+\tilde{g}((c-\delta)^2))}{c^2}\right)^2}$$

$$=\frac{4\delta^2}{c^2}\frac{\sum_{j=0}^{m} c_j\left(\frac{y}{c}\right)^j+O\left(\left(\frac{y}{c}\right)^{m+1}\right)}{\left[1-\frac{\delta^2}{c^2}-\sum_{j=0}^{m} c_j\left(\frac{y}{c}\right)^j+\left(\left(\frac{y}{c}\right)^{m+1}\right)\right]^2}$$

$$=\frac{1}{\left(1-\frac{\delta}{c}\right)^2}\frac{\sum_{j=0}^{m} c_j\left(\frac{y}{c}\right)^j+O\left(\left(\frac{y}{c}\right)^{m+1}\right)}{\left[1-\frac{1}{2\delta(1-\delta/c)}\sum_{j=1}^{m} c_j\frac{y^j}{c^{j-1}}+O\left(\frac{y^{m+1}}{c^m}\right)\right]^2}.$$

Since

$$\frac{1}{(1-x)^2}=\sum_{j=0}^{\infty}(j+1)\,x^j \qquad \text{for} \quad |x|<1,$$

we obtain for sufficiently large c and small y,

$$\frac{4\delta^2\tilde{g}^{-1}\left(\frac{\lambda}{c}\,y+\tilde{g}((c-\delta)^2)\right)}{\left(c^2-\delta^2-\tilde{g}^{-1}\left(\frac{\lambda}{c}\,y+\tilde{g}((c-\delta)^2)\right)\right)}$$

$$=\frac{1}{\left(1-\frac{\delta}{c}\right)^2}\left[\sum_{j=0}^{m} c_j\left(\frac{y}{c}\right)^j+O\left(\left(\frac{y}{c}\right)^{m+1}\right)\right]$$

$$\times\left[\sum_{j=0}^{m}(j+1)\left(\frac{1}{2\delta(1-\delta/c)}\sum_{l=1}^{m} c_l\frac{y^l}{c^{l-1}}\right)^j+O(y^{m+1})\right]$$

$$=\frac{1}{\left(1-\frac{\delta}{c}\right)^2}\left[\sum_{j=0}^{m} c_j\left(\frac{y}{c}\right)^j+O\left(\left(\frac{y}{c}\right)^{m+1}\right)\right]\left[\sum_{j=0}^{m}\tilde{A}_j\,y^j+O(y^{m+1})\right],$$

where

$$\tilde{A}_0 = 1,$$

$$\tilde{A}_1 = \frac{c_1}{\delta(1-\delta/c)},$$

$$\tilde{A}_2 = \frac{3c_1^2}{4\delta^2(1-\delta/c)^2} + \frac{c-2}{c\delta(1-\delta/c)},$$

$$\tilde{A}_3 = \frac{c_1^3}{2\delta^3(1-\delta/c)^3} + \frac{3c_1c_2}{2c\delta^2(1-\delta/c)^2} + \frac{c_3}{c^2\delta(1-\delta/c)},$$

and

$$\tilde{A}_j = \tilde{A}_j(c, \lambda) = O(1) \qquad \text{as} \quad c \to \infty, \quad j = 4, ..., m.$$

This leads to

$$\frac{4\delta^2 \tilde{g}^{-1}\left(\frac{\lambda}{c} y + \tilde{g}((c-\delta)^2)\right)}{\left(c^2 - \delta^2 - \tilde{g}^{-1}\left(\frac{\lambda}{c} y + \tilde{g}((c-\delta)^2)\right)\right)^2} = \sum_{j=0}^{m} A_j y^j + O(y^{m+1}), \qquad y \to 0,$$

with

$$A_0 = 1,$$

$$A_1 = \frac{c_1}{\delta(1-\delta/c)^2},$$

$$A_2 = c_1^2 \frac{3+\delta/c}{4\delta^2(1-\delta/c)^3} + c_2 \frac{1}{c\delta(1-\delta/c)^2},$$

$$A_3 = \frac{c_1^3(2+\delta/c)}{4\delta^3(1-\delta/c)} + \frac{c_1c_2(3+\delta/c)}{2c\delta^2(1-\delta/c)^3} + \frac{c_3}{c^2\delta(1-\delta/c)^2},$$

and

$$A_j = A_j(c, \lambda) = O(1), \qquad c \to \infty, \quad j = 4, ..., m.$$

We are now able to expand η. It is

$$\eta(y) = \sqrt{\sum_{j=1}^{m} A_j y^j + O(y^{m+1})}, \qquad y \to 0,$$

$$= \sqrt{A_1}\; y^{1/2} \sqrt{1 + \sum_{j=2}^{m} \frac{A_j}{A_1} y^{j-1} + O(y^m)}, \qquad y \to 0.$$

Using

$$\sqrt{1+x} = \sum_{j=0}^{m-1} v_j x^j + O(x^m), \qquad |x| < 1,$$

we get for sufficiently small y that

$$\eta(y) = \sqrt{A_1}\; y^{1/2} \left[\sum_{j=0}^{m-1} v_j \left[\sum_{l=2}^{m} \frac{A_l}{A_1}\, y^{l-1} \right]^j + O(y^m) \right].$$

The proof of the lemma is finished by rearranging the terms in brackets according to the ascending powers of y. ▮

We now put

$$\tilde{\eta}(y) = \eta(y^2)$$

and consider

$$\Psi_k(y) = \int_0^{\arctan \tilde{\eta}(y)} (\sin \alpha)^{k-2}\, d\alpha. \tag{22}$$

Note that

$$F\left(A_c^\delta, \sqrt{\frac{\tilde{g}^{-1}\left(\frac{\lambda}{c}\, y + \tilde{g}((c-\delta)^2)\right)}{c^2}} \right) = \frac{\omega_{k-1}}{\omega_k}\, \Psi_k(\sqrt{y}). \tag{23}$$

It follows that

$$\Psi'_k(y) = \frac{\tilde{\eta}(y)^{k-2}\, \tilde{\eta}'(y)}{[1+\tilde{\eta}^2(y)]^{k/2}}. \tag{24}$$

This representation enables us to derive an expansion for Ψ_k by first expanding Ψ'_k using Lemma 5.1 and then applying termwise integration.

Proof of Lemma 3.2. Using the continuous differentiability of η, from (21) one can derive an expansion for $\tilde{\eta}'$,

$$\tilde{\eta}'(y) = \sum_{j=0}^{m-1} a_{j+1}\, y^{2j}(2j+1) + O(y^{2m}).$$

Inserting this expansion, together with (21), into the relation (24) gives

$$\Psi'_k(y) = \frac{[\sum_{j=0}^{m-1} a_{j+1} y^{2j+1} + O(y^{2m+1})]^{k-2} [\sum_{j=0}^{m-1} a_{j+1} y^{2j}(2j+1) + O(y^{2m})]}{[1+[\sum_{j=0}^{m-1} a_{j+1} y^{2j+1} + O(y^{2m+1})]^2]^{k/2}}. \tag{25}$$

It holds that

$$\left[\sum_{j=0}^{m-1} a_{j+1} y^{2j+1} + O(y^{2m+1})\right]^{k-2} = a_1^{k-2} y^{k-2} \left[\sum_{j=0}^{m-1} \mu_j y^{2j} + O(y^{2m})\right], \tag{26}$$

where

$$\mu_0 = 1,$$

$$\mu_1 = (k-2)\frac{a_2}{a_1},$$

$$\mu_2 = (k-2)\frac{a_3}{a_1} + \frac{(k-2)(k-3)}{2}\frac{a_2^2}{a_1^2},$$

and

$$\mu_j = \mu_j(\lambda, c) = 0(1), \qquad c \to \infty, \quad j = 3, ..., m-1.$$

Furthermore, we have

$$\left[1 + \left[\sum_{j=0}^{m-1} a_{j+1} y^{2j+1} + O(y^{2m+1})\right]^2\right]^{k/2} = \sum_{l=0}^{m-1} \binom{k/2}{l} \left(\sum_{j=0}^{m-1} a_{j+1} y^{2j+1}\right)^{2l} + O(y^{2m}) = 1 + \sum_{j=1}^{m-1} v_j y^{2j} + O(y^{2m})$$

with

$$v_1 = \frac{k}{2} a_1^2,$$

$$v_2 = k a_1 a_2 + \frac{k(k-2)}{8} a_1^4,$$

and

$$v_j = v_j(\lambda, c) = 0(1), \qquad c \to \infty, \quad j = 3, ..., m-1.$$

Because of

$$\frac{1}{1+x} = \sum_{l=0}^{m-1} (-1)^l x^l + O(x^m), \qquad |x| < 1,$$

this leads to

$$\frac{1}{[1+\tilde{\eta}^2(y)]^{k/2}} = \sum_{j=0}^{m-1} \xi_j y^{2j} + O(y^{2m}), \qquad y \to 0, \tag{27}$$

with

$$\xi_0 = 1,$$

$$\xi_1 = -\frac{k}{2} a_1^2,$$

$$\xi_2 = -k a_1 a_2 + \frac{k(k+2)}{8} a_1^4,$$

and

$$\xi_j = \xi_j(\lambda, c) = O(1), \qquad c \to \infty, \quad j = 3, ..., m-1.$$

Combining (25), (26), and (27) yields:

$$\Psi'_k(y) = a_1^{k-2} y^{k-2} \left[\sum_{j=0}^{m-1} \mu_j y^{2j} + O(y^{2m}) \right]$$
$$\times \left[\sum_{j=0}^{m-1} a_{j+1} y^{2j}(2j+1) + O(y^{2m}) \right] \left[\sum_{j=0}^{m-1} \xi_j y^{2j} + O(y^{2m}) \right]$$
$$= \sum_{j=0}^{m-1} B_j y^{k-2+2j} + O(y^{k-2+2m}),$$

with

$$B_0 = a_1^{k-1},$$
$$B_1 = -\frac{k}{2} a_1^{k+1} + (k+1)\, a_1^{k-2} a_2, \tag{28}$$
$$B_2 = a_1^{k-2} a_3 (k+3) + \frac{(k-2)(k+3)}{2}\, a_1^{k-3} a_2^2 - \frac{k(k+3)}{2}\, a_1^k a_2 + \frac{k(k+2)}{8}\, a_1^{k+3},$$

and

$$B_j = B_j(\lambda, c) = O(1), \qquad c \to \infty, \quad j = 3, ..., m-1.$$

Termwise integrating this relation with respect to y and inserting the resulting expansion into (23) completes the proof. ▮

Proof of Lemma 3.3. To expand f_k we insert the expansion of the intersection-percentage function and that from assumption (D2, m) into the relation

$$f_k(c, \lambda, y) = F\left(A_c^\delta, \sqrt{\frac{\tilde{g}^{-1}\left(\frac{\lambda}{c}\, y + \tilde{g}((c-\delta)^2)\right)}{c^2}} \right) \times \left[\tilde{g}^{-1}\left(\frac{\lambda}{c}\, y + \tilde{g}((c-\delta)^2)\right)\right]^{(k-2)/2} \tilde{g}^{-1'}\left(\frac{\lambda}{c}\, y + \tilde{g}((c-\delta)^2)\right).$$

This yields

$$f_k(c, \lambda, y) = \frac{\omega_{k-1}}{\omega_k} \frac{c^{k+1}}{\lambda} \left[\sum_{j=0}^{m-1} \frac{B_j}{k-1+2j}\, y^{(k-1)/2+j} + O(y^{(k-1)/2+m}) \right] \times \left[\sum_{j=0}^{m} c_j \left(\frac{y}{c}\right)^j + O\left(\left(\frac{y}{c}\right)^{m+1}\right)\right]^{(k-2)/2} \times \left[\sum_{j=1}^{m} c_j j \frac{y^{j-1}}{c^j} + O\left(\frac{y^m}{c^{m+1}}\right)\right].$$

We again make use of the binomial expansion

$$\left[\sum_{j=0}^{m} c_j \left(\frac{y}{c}\right)^j + O\left(\left(\frac{y}{c}\right)^{m+1}\right)\right]^{(k-2)/2}$$
$$= c_o^{(k-2)/2} \left[\sum_{j=0}^{m} \varrho_j \left(\frac{y}{c}\right)^j + O\left(\left(\frac{y}{c}\right)^{m+1}\right)\right]$$

with

$$\varrho_0 = 1,$$
$$\varrho_1 = \frac{(k-2)\, c_1}{2c_0},$$
$$\varrho_2 = \frac{(k-2)\, c_2}{2c_0} + \frac{(k-2)(k-4)\, c_1^2}{8c_0^2},$$

and

$$\varrho_j = \varrho_j(c, \lambda) = O(1), \qquad c \to \infty.$$

This finally gives

$$f_k(c, \lambda, y) = \frac{\omega_{k-1}}{\omega_k} \frac{c^{k+1}}{\lambda} \left(\sum_{j=0}^{m-1} \frac{B_j}{k-1+2j}\, y^{(k-1)/2+j} + O(y^{(k-1)/2+m}) \right)$$
$$\times c_0^{(k-2)/2} \left[\sum_{j=0}^{m} \varrho_j \left(\frac{y}{c}\right)^j + O\left(\left(\frac{y}{c}\right)^{m+1}\right)\right]$$
$$\times \left[\sum_{j=1}^{m} c_j j \frac{y^{j-1}}{c^j} + O\left(\frac{y^m}{c^{m+1}}\right)\right]$$
$$= \frac{\omega_{k-1}}{\omega_k} \frac{c^k}{\lambda} \sum_{j=0}^{m-1} b_{j+1}\, y^{(k-1)/2+j} + O(y^{(k-1)/2+m})$$

with

$$b_1 = \frac{c_1^{(k+1)/2}}{(k-1)\, \delta^{(k-1)/2}(1-\delta/c)},$$
$$b_2 = \frac{c_1^{(k+3)/2}}{\delta^{(k+1)/2}(1-\delta/c)^3} \left[-\frac{k-3}{8(k+1)} + \frac{k-3}{4(k-1)} \frac{\delta}{c} - \frac{\delta^2}{8c^2} \right]$$
$$+ \frac{c_1^{(k-1)/2} c_2}{c\delta^{(k-1)/2}(1-\delta/c)} \frac{k+3}{2(k-1)},$$

and

$$b_3 = \frac{c_1^{(k+5)/2}}{\delta^{(k+3)/2}(1-\delta/c)^5}\left[\frac{(k-5)(k-3)}{128(k+3)}\right.$$
$$\left. - \frac{(k-5)(k-3)}{32(k+1)}\frac{\delta}{c} + \frac{3(k-5)(k-3)}{64(k-1)}\frac{\delta^2}{c^2} - \frac{k-5}{32}\frac{\delta^3}{c^3} + \frac{k-3}{128}\frac{\delta^4}{c^4}\right]$$
$$+ \frac{c_1^{(k+1)/2}c_2}{c\delta^{(k+1)/2}(1-\delta/c)^3}\left[\frac{(k+5)(k-3)}{16(k+1)} - \frac{(k+5)(k-3)}{8(k-1)}\frac{\delta}{c} + \frac{k+5}{16}\frac{\delta^2}{c^2}\right]$$
$$+ \frac{c_1^{(k-1)/2}c_3}{c^2\delta^{(k-1)/2}(1-\delta/c)}\frac{k+5}{2(k-1)} + \frac{c_1^{(k-3)/2}c_2^2}{c^2\delta^{(k-1)/2}(1-\delta/c)}\frac{k+5}{8}. \quad \blacksquare$$

Proof of Corollary 3.6. From Example 2 we know that the Kotz type density-generating function satisfies the assumption (D2, m) with $\lambda = c^{2\gamma}$. Thus for $\gamma > \frac{1}{2}$ the assumptions of Theorem 3.5 are fulfilled. Furthermore,

$$I_{k,\tilde{g}} = \frac{\Gamma\left(\frac{2N+k-2}{2\gamma}\right)}{2\gamma\beta^{(2N+k-2)/(2\gamma)}}.$$

To obtain the coefficients of the asymptotic expansion we evaluate

$$\frac{b_1(k-1)}{2} = \frac{c_1^{(k+1)/2}}{2\delta^{(k-1)/2}(1-\delta/c)}$$
$$= \frac{(1-\delta/c)^k}{2\delta^{(k-1)/2}\left(\gamma\beta(1-\delta/c)^{2\gamma} - \frac{N-1}{c^{2\gamma}}\right)^{(k+1)/2}},$$

$$\frac{b_2(k-1)(k+1)}{4} = -\frac{c_1^{(k+3)/2}(k-1)(k-3)}{32\delta^{(k+1)/2}(1-\delta/c)^3} + \frac{c_1^{(k+3)/2}(k+1)(k-3)}{16c\delta^{(k-1)/2}(1-\delta/c)^3}$$
$$- \frac{c_1^{(k+3)/2}(k-1)(k+1)}{32c^2\delta^{(k-3)/2}(1-\delta/c)^3} + \frac{c_1^{(k-1)/2}c_2(k+3)(k+1)}{8c\delta^{(k-1)/2}(1-\delta/c)},$$

and

$$b_3\frac{(k-1)(k+1)(k+3)}{8} = \frac{c_1^{(k+5)/2}(k-5)(k-3)(k-1)(k+1)}{1024\delta^{(k+3)/2}\left(1-\frac{\delta}{c}\right)^5}$$
$$- \frac{c_1^{(k+5)/2}(k-5)(k-3)(k-1)(k+3)}{256c\delta^{(k+1)/2}\left(1-\frac{\delta}{c}\right)^5}$$

$$+\frac{3c_1^{(k+5)/2}(k-5)(k-3)(k+1)(k+3)}{512c^2\delta^{(k-1)/2}\left(1-\frac{\delta}{c}\right)^5}$$

$$-\frac{c_1^{(k+5)}(k-5)(k-1)(k+1)(k+3)}{256c^3\delta^{(k-3)/2}\left(1-\frac{\delta}{c}\right)^5}$$

$$+\frac{c_1^{(k+5)/2}(k-3)(k-1)(k+1)(k+3)}{1024c^4\delta^{(k-5)/2}\left(1-\frac{\delta}{c}\right)^5}$$

$$+\frac{c_1^{(k+1)/2}c_2(k-3)(k-1)(k+3)(k+5)}{128c\delta^{(k+1)/2}\left(1-\frac{\delta}{c}\right)^3}$$

$$-\frac{c_1^{(k+1)/2}c_2(k-3)(k+1)(k+3)(k+5)}{64c^2\delta^{(k-1)/2}\left(1-\frac{\delta}{c}\right)^3}$$

$$+\frac{c_1^{(k+1)/2}c_2(k-1)(k+1)(k+3)(k+5)}{128c^3\delta^{(k-3)/2}\left(1-\frac{\delta}{c}\right)^3}$$

$$+\frac{c_1^{(k-1)/2}c_3(k+1)(k+3)(k+5)}{16c^2\delta^{(k-1)/2}\left(1-\frac{\delta}{c}\right)}$$

$$+\frac{c_1^{(k-3)/2}c_2^2(k-1)(k+1)(k+3)(k+5)}{64c^2\delta^{(k-1)/2}\left(1-\frac{\delta}{c}\right)}.$$

Rearranging the terms according to powers of c, putting the leading term outside the brackets, and using

$$b_1 \sim \frac{1}{(k-1)\,\delta^{(k-1)/2}(\gamma\beta)^{(k+1)/2}}, \qquad c\to\infty,$$

lead to expansion (20), where

$$D_{1-2\gamma} = -\frac{c_1^{(k+3)/2}}{b_1 \delta^{(k+1)/2}\left(1-\frac{\delta}{c}\right)^3}\frac{k-3}{16}$$

$$D_{-2\gamma} = \frac{c_1^{(k+3)/2}}{b_1 \delta^{(k-1)/2}\left(1-\frac{\delta}{c}\right)^3}\frac{(k+1)(k-3)}{8(k-1)} + \frac{c_1^{(k-1)/2} c_2}{b_1 \delta^{(k-1)/2}\left(1-\frac{\delta}{c}\right)}\frac{(k+1)(k+3)}{4(k-1)}$$

$$D_{-1-2\gamma} = -\frac{c_1^{(k+3)/2}}{b_1 \delta^{(k-3)/2}\left(1-\frac{\delta}{c}\right)^3}\frac{k+1}{16}$$

$$D_{2-4\gamma} = \frac{c_1^{(k+5)/2}}{b_1 \delta^{(k+3)/2}\left(1-\frac{\delta}{c}\right)^5}\frac{(k-5)(k-3)(k+1)}{512}$$

$$D_{1-4\gamma} = -\frac{c_1^{(k+5)/2}}{b_1 \delta^{(k+1)/2}\left(1-\frac{\delta}{c}\right)^5}\frac{(k-5)(k-3)(k+3)}{128} + \frac{c_1^{(k+1)/2} c_2}{b_1 \delta^{(k+1)/2}\left(1-\frac{\delta}{c}\right)^3}\frac{(k-3)(k+3)(k+5)}{64}$$

$$D_{-4\gamma} = \frac{c_1^{(k+5)/2}}{b_1 \delta^{(k-1)/2}\left(1-\frac{\delta}{c}\right)^5}\frac{3(k-5)(k-3)(k+1)(k+3)}{256(k-1)} - \frac{c_1^{(k+1)/2} c_2}{b_1 \delta^{(k-1)/2}\left(1-\frac{\delta}{c}\right)^3}\frac{(k-3)(k+1)(k+3)(k+5)}{32(k-1)} + \frac{c_1^{(k-1)/2} c_3}{b_1 \delta^{(k-1)/2}\left(1-\frac{\delta}{c}\right)}\frac{(k+1)(k+3)(k+5)}{8(k-1)} + \frac{c_1^{(k-3)/2} c_2^2}{b_1 \delta^{(k-1)/2}\left(1-\frac{\delta}{c}\right)}\frac{(k+1)(k+3)(k+5)}{32}$$

$$D_{-1-4\gamma} = -\frac{c_1^{(k+5)/2}}{b_1 \delta^{(k-3)/2}\left(1-\frac{\delta}{c}\right)^5} \frac{(k-5)(k+1)(k+3)}{128}$$

$$+\frac{c_1^{(k+1)/2} c_2}{b_1 \delta^{(k-3)/2}\left(1-\frac{\delta}{c}\right)^3} \frac{(k+1)(k+3)(k+5)}{64}$$

$$D_{-2-4\gamma} = \frac{c_1^{(k+5)/2}}{b_1 \delta^{(k-5)/2}\left(1-\frac{\delta}{c}\right)^5} \frac{(k-3)(k+1)(k+3)}{512},$$

where

$$b_1 = \frac{\left(1-\frac{\delta}{c}\right)^k}{(k-1)\,\delta^{(k-1)/2}\left(\gamma\beta\left(1-\frac{\delta}{c}\right)^{2\gamma} - \frac{N-1}{c^{2\gamma}}\right)^{(k+1)/2}}$$

and the coefficients c_j, $j = 1, 2, 3$, are given in Example 2. ∎

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REFERENCES

1. T. W. Anderson and K. T. Fang, Tech. Rep. 53, Department of Statistics, Stanford University, Stanford, CA, 1982.
2. N. Bleistein and R. A. Handelsman, "Asymptotic Expansions of Integrals," Holt, Rinehart, and Winston, New York, 1975.
3. K. Breitung and W.-D. Richter, A geometric approach to an asymptotic expansion for large deviation probabilities of Gaussian random vectors, *J. Multivariate Anal.* **58** (1996), 1–20.
4. T. Cacoullous and M. Koutras, Quadratic forms in spherical random variables: generalized noncentral χ^2 distribution, *Naval Res. Logistics Quart.* **31**, No. 3 (1984), 447–461.
5. H. E. Daniels, Tail probability approximations, *Internat. Statist. Rev.* **55** (1987), 37–48.
6. A. S. S. Dorflo, A unified method for computing probabilities of misclassification when the covariance matrices are known, *Comm. Statist. Simula.* **22**, No. 3 (1993), 679–687.
7. M. V. Fedorjuk, "Saddlepoint Method," Nauka, Moscow, 1977. [In Russian]
8. A. K. Gupta and T. Varga, "Elliptically Contoured Models in Statistics," Kluwer Academic, Dordrecht/Boston/London, 1993.

9. H. Hsu, Noncentral distributions of quadratic forms for elliptically contoured distributions, *in* “Statistical Inference in Elliptically Contoured and Related Distributions” (K. T. Fang and T. W. Anderson, Eds.), pp. 97–102, Allerton Press, New York, 1990.
10. C. Ittrich, D. Krause, and W.-D. Richter, Probabilities and large quantiles of noncentral generalized chisquare distributions, *Statistics* **34** (2000), 53–101.
11. D. Kelker, Distribution theory of spherical distributions and a location-scale parameter generalization, *Sankhya A* **32** (1970), 419–430.
12. W.-D. Richter, Eine geometrische Methode in der Stochastik, *Rostock Math. Kolloq.* **44** (1991), 63–72.
13. W.-D. Richter, Eine geometrische und eine asymptotische Methode in der Statistik, preprint, Universität Bremen, 1992.
14. W.-D. Richter, A geometric approach to the Gaussian law, *in* “Symposia Gaussiana, Conference B” (Mammitzsch and Schneeweiß, Eds.), pp. 25–45, de Gruyter, Berlin/New York, 1995.
15. W.-D. Richter and J. Schumacher, Laplace method in large deviation theory, *Rost. Math. Koll.* **44** (1990), 47–52.
16. I. J. Schoenberg, Metric spaces and completely monotone functions, *Ann. Math.* **39** (1938), 811–841.
17. R. Wong, “Asymptotic Approximations of Integrals,” Academic Press, Boston, 1989.