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Asymptotic Behavior and Uniqueness for an Ultrahyperbolic Equation with Variable Coefficients*

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This paper describes the asymptotic behavior of solutions of a class of semi-linear ultrahyperbolic equations with variable coefficients. One consequence of the general analysis is a uniqueness theorem for a mixed boundary-value problem. Another demonstrates unique continuation at infinity. These results extend previous work by M. H. Protter, [Asymptotic decay for ultrahyperbolic operators, *in* "Contributions to Analysis" (Lars Ahlfors *et al.*, Eds.), Academic Press, New York, 1974], and A. C. Murray and M. M. Protter, [*Indiana U. Math. J.* **24** (1974), 115–130], on a more restricted class of equations.

1. INTRODUCTION

Let D be a bounded domain in $\mathbb{R}^m$, $m \geqslant 2$, and let Γ denote the exterior of the unit ball in $\mathbb{R}^n$, $n \geqslant 2$. Use $r = |y|$ to denote the length of a vector y in $\mathbb{R}^n$. For $R \geqslant 1$, the sets $S(R)$ and $\Gamma(R)$ are defined by

$$S(R) = \{y \in \mathbb{R}^n : r = |y| = R\},$$

$$\Gamma(R) = \{y \in \mathbb{R}^n : 1 < |y| < R\}.$$

Let L be an ultrahyperbolic operator defined in $D \times \Gamma$ by

$$Lu \equiv Au - Bu, \tag{1.1}$$

where

$$Au \equiv (a_{ij}(x, y)\, u_{x_i})_{x_j} \qquad \text{and} \qquad Bu \equiv b_{k\ell}(x, y)\, u_{y_k y_\ell}\,.$$

Repeated indices i, j are to be summed from 1 to m, while repeated indices k, ℓ (and later K, L) are to be summed from 1 to n.

The coefficient matrices $[a_{ij}]$ and $[b_{k\ell}]$ are assumed to be positive definite and symmetric with C^1 entries defined for $(x, y) \in D \times \Gamma$. Further, A is assumed

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to be uniformly elliptic, thus there exist positive constants $\underline{a}$ and $\bar{a}$ such that

$$\underline{a} \mid \xi \mid^2 \leqslant a_{ij}(x, y)\ \xi_i \xi_j \leqslant \bar{a} \mid \xi \mid^2$$

for all $(x, y) \in D \times \Gamma$ and all $\xi \in \mathbb{R}^m$. Also, the coefficients of A are subject to the condition

$$|(a_{ij})_{y_k}| \leqslant \mathscr{M} r^{-1}$$

for some small constant $\mathscr{M}$.

The matrix $[b_{k\ell}]$ is assumed to be close to the identity $[\delta_{k\ell}]$ in the sense that

$$b_{k\ell}(x, y) = \delta_{k\ell} + c_{k\ell}(x, y),$$

where the $c_{k\ell}$ are small, slowly varying functions. Specifically, we assume that there are constants $\mathscr{C}$, $\mathscr{K}$, $\mathscr{L}$, such that

$$\sum_{k,\ell=1}^{n} | c_{k\ell}(x, y)|^2 \leqslant \mathscr{C}^2,$$

$$|(c_{k\ell})_{y_k}| \leqslant \mathscr{K} r^{-1}, \qquad |(c_{k\ell})_{x_i}| \leqslant \mathscr{L} r^{-1},$$

throughout $D \times \Gamma$.

We shall consider solutions of the equation

$$Lu = f(x, y, u, \nabla_x u, \nabla_y u) \tag{1.2}$$

in the region $D \times \Gamma$, where f is subject to a consistency condition

$$f(x, y, 0, 0, 0) = 0$$

and a Lipschitz condition

$$|f(x, y, u, p, q) - f(x, y, u', p', q')| \leqslant \phi_0 \mid u - u' \mid + \phi_1 \mid p - p' \mid + \phi_2 \mid q - q' \mid,$$

where the ϕ_i, $0 \leqslant i \leqslant 2$, are functions of y. Thus we can consider not only solutions of (1.2) but, more generally, solutions of the differential inequality

$$| Lu | \leqslant \phi_0(y) \mid u \mid + \phi_1(y) \mid \nabla_x u \mid + \phi_2(y) \mid \nabla_y u \mid. \tag{1.3}$$

Broadly put, our results say that if a nonzero solution of (1.3) vanishes on $\partial D \times \Gamma$, then it cannot decay arbitrarily fast as $| y | \to \infty$. The precise results can be stated as follows for a solution u of (1.3) which is C^2 on $D \times \Gamma$ and vanishes on $\partial D \times \Gamma$. Assume that L satisfies *Condition C*, a technical hypothesis (spelled out in Section 2) saying that the constants $\mathscr{C}$, $\mathscr{K}$, $\mathscr{L}$, and $\mathscr{M}$ are "small enough." Introduce the "energy"

$$E(u, R) = R^{1-n} \int_{S(R)} \int_D \{| u |^2 + | \nabla_y u |^2 + u_{x_i} a_{ij} u_{x_j}\} \, dx \, d\sigma.$$

If the ϕ_i are bounded in Γ, then

(i) u cannot have bounded support in $D \times \Gamma$, unless $u \equiv 0$;

(ii) $E(u, R)$ cannot decay faster than $\exp(-\rho R^2)$ for all ρ, unless $u \equiv 0$;

and

(iii) there is an explicit lower bound of the form

$$E(u, R) \geq K \exp(-\rho R^2) \, E(u, 1).$$

More generally, suppose that $\phi_i = O(r^\beta)$ for $-\frac{1}{2} < \beta < \infty$. Then the results (i), (ii), and (iii) remain valid when the function $\exp(-\rho R^2)$ is replaced by $\exp(-\rho R^{2+2\beta})$. As a consequence, we have uniqueness for the mixed boundary-value problem

$$\begin{array}{ll} Lu = f(x, y, u, \nabla_x u, \nabla_y u) \text{ in} & D \times \Gamma(R) \\ \text{with Dirichlet data given on} & \partial D \times \Gamma(R) \\ \text{and Cauchy data given on} & D \times S(R). \end{array} \tag{1.4}$$

Results of this type were obtained previously by Murray and Protter [3] and Protter [5] in the special case where $[b_{k\ell}]$ is the identity matrix $[\delta_{k\ell}]$. The burden of this paper is to extend the estimate procedure of [3] to the case of variable $b_{k\ell}$.

Other authors have considered the uniqueness question for certain boundary-value problems for the special ultrahyperbolic equations

$$\sum_{i=1}^{m} u_{x_i x_i} - \sum_{k=1}^{n} u_{y_k y_k} = 0 \tag{1.5}$$

or

$$\sum_{i,j=1}^{m} (a_{ij} u_{x_i})_{x_j} - \sum_{k=1}^{n} u_{y_k y_k} = cu. \tag{1.6}$$

In [4], Owens gives examples of bounded domains V such that a solution of (1.5) in V is determined by giving both its value on all of ∂V and its normal derivative on an appropriate part of ∂V. The domain considered in our problem (1.4) is not among those Owens discusses, nor is our boundary condition quite as severe as his.

In [1], Diaz and Young consider the Dirichlet and Neumann problems for (1.6) in a region $D \times P$ where D is a bounded domain in x-space and P is a bounded parallelepiped. Their conditions for uniqueness relate the dimensions of P to the eigenvalues of a related problem in D.

In [2], Levine considered the abstract Cauchy problem for certain ordinary differential equations in Hilbert space. Our Eq. (1.2) can be interpreted in the terminology of [2] by taking $r = |y|$ for an independent variable. In this framework, Levine's results do not apply to the problem (1.4). However, they do apply to the analogous problem for $x \in D$, $r \geqslant R$.

Paper [3] contains a discussion of related work on the question of asymptotic behavior.

The main results of this paper are established in Section 3 by means of a weighted energy inequality. This inequality is stated and proved as Theorem 1 in Section 2. The proof of this theorem is quite technical, and one may prefer to omit it on first reading.

2. A Weighted Energy Estimate

We consider an operator L defined by (1.1) and having the properties described above. We assume that L satisfies

Condition C.

$$\tfrac{1}{2} - \mathscr{C}[4n + 2 + \mathscr{C}] - 6\mathscr{K} n^{3/2}(1 + \mathscr{C}) \geqslant 0, \tag{C_1}$$

$$1 - \mathscr{C}(n + n^{1/2} - 1) - \mathscr{K} n^{3/2} \geqslant \tfrac{3}{4}, \tag{C_2a}$$

$$\tfrac{3}{4}\,\underline{a} - \mathscr{L} m \bar{a}^2 - \mathscr{M} n^{1/2} m(1 + \mathscr{C}) \geqslant \tfrac{1}{2}\,\underline{a}, \tag{C_2b}$$

$$2n^2 \mathscr{L} < \tfrac{1}{2}, \tag{C_3}$$

$$1 - \mathscr{C} - (1 + \mathscr{C})\{\mathscr{C}\,|\,2\alpha - 3\,| + 2\mathscr{C} + n^{3/2}\mathscr{K}\} \geqslant \tfrac{4}{5}, \tag{C_4a}$$

$$\tfrac{1}{5} - (1 + \mathscr{C})\{n^{1/2}\mathscr{C} + n\mathscr{C} + n^{3/2}\mathscr{K}\} \geqslant 0. \tag{C_4b}$$

These inequalities are not chosen to be "best possible," but rather to fit naturally into the estimates that will arise. These hypotheses are expressed in terms of the dimensions m and n, the moduli of ellipticity of A, and a parameter α which will permit us to handle a variety of growth properties for the ϕ_i in (1.3).

For convenient reference, let $\mathscr{U}$ denote the class of functions $u = u(x, y)$ which are C^2 in $D \times \Gamma$, C^1 on the closure of $D \times \Gamma$, and zero on $(\partial D) \times \Gamma$.

Theorem 1. *Suppose $u \in \mathscr{U}$, L satisfies Condition* C, *and $\alpha > 1$. Then there* *are computable positive constants k_i, $0 \leqslant i \leqslant 3$, such that for all sufficiently large λ*

$$\int_{\Gamma(R)} r^{3-\alpha-n} e^{2\lambda r^2} \| Lu \|^2 \, dy + k_0(\lambda\alpha)^3 R^{2\alpha-1} e^{2\lambda R^\alpha} E(u, R)$$
$$\geqslant k_1(\lambda\alpha)^3 \int_{\Gamma(R)} r^{2\alpha-n-1} e^{2\lambda r^\alpha} \| u \|^2 \, dy$$
$$+ k_2\lambda\alpha \int_{\Gamma(R)} r^{1-n} e^{2\lambda r^\alpha} \{\| \nabla_x u \|^2 + \| \nabla_y u \|^2\} \, dy$$
$$+ k_3 e^{2\lambda} E(u, 1). \tag{2.1}$$

The k_i are independent of u; the necessary size of λ depends on the behavior of u on $D \times S(1)$.

The rest of this section is devoted to the proof of the weighted energy estimate (2.1) for a function u in $\mathscr{U}$. For parameters $\lambda > n$ and $\alpha > 1$, we introduce the auxiliary function

$$w(x, y) = u(x, y) \exp(\lambda r^\alpha).$$

Then computation shows that

$$e^{\lambda r^\alpha} Lu = Aw - Bw + 2\lambda\alpha r^{\alpha-2} y_k b_{k\ell} w_{y_\ell}$$
$$- \lambda\alpha r^{\alpha-4}\{(\lambda\alpha r^\alpha - \alpha + 2)\, y_k b_{k\ell} y_\ell - \delta_{k\ell} b_{k\ell} r^2\} w. \tag{2.2}$$

For brevity let q denote the quantity

$$q = (\lambda\alpha r^\alpha - \alpha + 2)\, y_k b_{k\ell} y_\ell - \delta_{k\ell} b_{k\ell} r^2.$$

By squarring (2.2) and dropping a positive term on the right, we can obtain the initial inequality

$$e^{2\lambda r^\alpha} | Lu |^2 \geqslant 4(\lambda\alpha)^2 r^{2\alpha-4} (y_k b_{k\ell} w_{y_\ell})^2$$
$$+ 2(2\lambda\alpha r^{\alpha-2} y_k b_{k\ell} w_{y_\ell})(Aw - Bw - \lambda\alpha r^{\alpha-4} q w). \tag{2.3}$$

Once multiplied through by $r^{3-\alpha-n}$, (2.3) yields

$$r^{3-\alpha-n} e^{2\lambda r^\alpha} | Lu |^2 \geqslant 4(\lambda\alpha)^2 r^{\alpha-1-n} (y_k b_{k\ell} w_{y_\ell})^2$$
$$- 4\lambda\alpha r^{1-n} y_k b_{k\ell} w_{y_\ell} Bw$$
$$+ 4\lambda\alpha r^{1-n} y_k b_{k\ell} w_{y_\ell} Aw$$
$$- 4(\lambda\alpha)^2 r^{\alpha-3-n} q y_k b_{k\ell} w_{y_\ell} w.$$

We integrate this over $D \times \Gamma(R)$ and let T_i denote the ith term on the right side of the result. Thus

$$\int_{\Gamma(R)} r^{3-\alpha-n} e^{2\lambda r^\alpha} \int_D | Lu |^2 \, dx \, dy \geqslant T_1 + T_2 + T_3 + T_4 , \tag{2.4}$$

where

$$T_1 = 4(\lambda\alpha)^2 \int_{\Gamma(R)} r^{\alpha-1-n} \int_D | y_k b_{k\ell} w_{y_\ell} |^2 \, dx \, dy,$$

$$T_2 = -2\lambda\alpha \int_D 2 \int_{\Gamma(R)} r^{1-n} y_k b_{k\ell} w_{y_\ell} b_{\mathrm{LK}} w_{y_\mathrm{K} y_\mathrm{L}} \, dy \, dx,$$

$$T_3 = 2\lambda\alpha \int_D 2 \int_{\Gamma(R)} r^{1-n} y_k b_{k\ell} w_{y_\ell} (a_{ij} w_{x_j})_{x_i} \, dy \, dx,$$

and

$$T_4 = -2(\lambda\alpha)^2 \int_D 2 \int_{\Gamma(R)} r^{\alpha-3-n} q y_k b_{k\ell} w w_{y_\ell} \, dy \, dx.$$

The next task is to obtain useful estimates of the T_i by careful exploitation of the hypotheses on $[a_{ij}]$ and $[b_{k\ell}]$. These estimates and their proofs appear in the next three lemma. For any smooth $v = v(x, y)$ it will be convenient to use $\| v \|^2$ to denote the integral of $| v(x, y)|^2$ over domain D in x-space. Let $\nu = (\nu_1 ,..., \nu_n)$ denote the outer unit normal on the boundary of $\Gamma(R)$. The expression $d\sigma$ refers to the usual $(n - 1)$ measure on hypersurfaces in $\mathbb{R}^n$.

LEMMA 1.

$$T_1 + T_2 \geqslant -2\lambda\alpha \int_D \int_{\partial\Gamma(R)} r^{1-n}\{2(y_k b_{k\ell} w_{y_\ell})\, w_{y_\mathrm{K}} - (w_{y_k} b_{k\ell} w_{y_\ell})\, y_\mathrm{K}\}\, b_{\mathrm{KL}} \nu_\mathrm{L} \, d\sigma \, dx$$
$$+ \lambda\alpha \int_{\Gamma(R)} r^{1-n} \| \nabla_y w \|^2 \, dy. \tag{2.5}$$

Proof. We first study T_2 alone. Its integrand is

$$\mathscr{I}_2(x, y) = -2r^{1-n} y_k b_{k\ell} w_{y_\ell} b_{\mathrm{KL}} w_{y_\mathrm{K} y_\mathrm{L}} .$$

In order to integrate $\mathscr{I}_2(x, y)$ over $D \times \Gamma(R)$ by means of the divergence theorem, we use the identity

$$\begin{aligned} \mathscr{I}_2(x, y) = &-[r^{1-n}\{2(y_k b_{k\ell} w_{y_\ell})\, w_{y_\mathrm{K}} - (w_{y_k} b_{k\ell} w_{y_\ell})\, y_\mathrm{K}\}\, b_{\mathrm{KL}}]_{y_\mathrm{L}} \\ &- 2(n-1)\, r^{-1-n} (y_k b_{k\ell} w_{y_\ell})^2 \\ &+ r^{1-n}\{(n-1)\, r^{-2} (y_k b_{k\ell} y_\ell) - \delta_{k\ell} b_{k\ell}\}(w_{y_\mathrm{K}} b_{\mathrm{KL}} w_{y_\mathrm{L}}) \\ &+ 2r^{1-n} \sum_{k=1}^{n} (b_{k\ell} w_{y_\ell})^2 \\ &+ r^{1-n} y_k \{2 w_{y_\ell} w_{y_\mathrm{K}} (b_{k\ell} b_{\mathrm{KL}})_{y_\mathrm{L}} - (b_{k\ell} b_{\mathrm{KL}})_{y_\ell} w_{y_\mathrm{K}} w_{y_\mathrm{L}}\}. \end{aligned}$$

The first term is a divergence: call it $\mathcal{T}$. The second term is negative, but it can be dominated by T_1. The next two terms can be estimated fairly directly since

$$\{(n-1)\, r^{-2} y_k b_{k\ell} y_\ell - \delta_{k\ell} b_{k\ell}\} \geqslant (n-1)(1-\mathscr{C}) - n - n^{1/2}\mathscr{C},$$

$$(w_{y_K} b_{KL} w_{y_L}) \geqslant (1-\mathscr{C}) |\nabla_y w|^2,$$

and

$$\sum_k (b_{k\ell} w_{y_\ell})^2 = |\nabla_y w|^2 + 2 w_{y_k} c_{k\ell} w_{y_\ell} + \sum_k (c_{k\ell} w_{y_\ell})^2 \geqslant (1-2\mathscr{C}) |\nabla_y w|^2.$$

In the last term of $\mathscr{I}_2$, Cauchy–Schwarz estimates yield

$$|y_k\{2 w_{y_\ell} w_{y_K} (b_{KL} b_{k\ell})_{y_L} - (b_{KL} b_{k\ell})_{y_\ell} w_{y_K} w_{y_L}\}| \leqslant 6 n^{3/2} \mathscr{K}(1+\mathscr{C}) |\nabla_y w|^2.$$

Thus we find that

$$\mathscr{I}_2(x,y) \geqslant \mathcal{T} - 2(n-1)\, r^{-1-n} (y_k b_{k\ell} w_{y_\ell})^2$$
$$+ r^{1-n}\{1 - \mathscr{C}(4n + 2 + \mathscr{C}) - \mathscr{K} 6 n^{3/2}(1+\mathscr{C})\} |\nabla_y w|^2.$$

By applying (C_1), we get

$$\mathscr{I}_2(x,y) > \mathcal{T} + \tfrac{1}{2} r^{1-n} |\nabla_y w|^2 - 2(n-1)\, r^{-1-n} (y_k b_{k\ell} w_{y_\ell}). \qquad (2.6)$$

By integrating (2.6) over $D \times \Gamma(R)$ and then applying the divergence theorem, we get

$$T_2 \geqslant -2\lambda\alpha \int_{\partial\Gamma(R)} \int_D r^{1-n}\{2(y_k b_{k\ell} w_{y_\ell})\, w_{y_K} - (w_{y_k} b_{k\ell} w_{y_\ell})\, y_K\}\, b_{KL} \nu_L \, dx\, d\sigma$$
$$+ \lambda\alpha \int_{\Gamma(R)} \int_D r^{1-n} |\nabla_y w|^2 \, dx\, dy$$
$$- 4(n-1)\, \lambda\alpha \int_{\Gamma(R)} \int_D r^{-1-n} (y_k b_{k\ell} w_{y_\ell})^2 \, dx\, dy.$$

To finish the proof of Lemma 1 from this point it suffices to recall that $\lambda > n$ and $\alpha > 1$, so

$$T_1 - 4(n-1)\, \lambda\alpha \int_{\Gamma(R)} r^{-1-n} \| y_k b_{k\ell} w_{y_\ell} \|^2 \, dy \geqslant 0.$$

Lemma 2.

$$T_3 \geqslant -2\lambda\alpha \int_D \int_{\partial\Gamma(R)} r^{1-n} y_k b_{k\ell} \nu_\ell w_{x_i} a_{ij} w_{x_j} \, d\sigma\, dx$$
$$+ \underline{a}\lambda\alpha \int_{\Gamma(R)} r^{1-n} \| \nabla_x w \|\, 2 \, dy$$
$$- 2\lambda\alpha\mathscr{L} n^2 \int_{\Gamma(R)} r^{1-n} \| \nabla_y w \|^2 \, dy. \qquad (2.7)$$

The last term on the right side of (2.7) can be dominated by the last term on the right of (2.5) by invoking the condition

$$2n^2\mathscr{L} < \tfrac{1}{2}. \tag{C_3}$$

It should also be remarked that if the $b_{k\ell}$ are independent of x, then $\mathscr{L} = 0$ and the conditions are needed only on $\mathscr{C}$, $\mathscr{K}$, $\mathscr{M}$.

Proof. We consider the integrand in T_3, namely

$$\mathscr{I}_3(x, y) = 2r^{1-n}y_k b_{k\ell} w_{y_\ell} Aw.$$

Preparing to use the divergence theorem, we find that

$$\begin{aligned}\mathscr{I}_3(x, y) = {} & (2r^{1-n}y_k b_{k\ell} w_{y_\ell} a_{ij} w_{x_j})_{x_i} \\ & - (r^{1-n}y_k b_{k\ell} w_{x_i} a_{ij} w_{x_j})_{y_\ell} \\ & + (r^{1-n}y_k b_{k\ell})_{y_\ell} w_{x_i} a_{ij} w_{x_j} \\ & + r^{1-n}y_k b_{k\ell} w_{x_i} (a_{ij})_{y_\ell} w_{x_j} \\ & - 2r^{1-n}y_k (c_{k\ell})_{x_i} w_{y_\ell} a_{ij} w_{x_j}.\end{aligned}$$

Before integrating, notice that $w_{y_\ell} = 0$ on $(\partial D) \times \mathbb{R}^n$, since $w = u \exp(\lambda r^\alpha) = 0$ for $x \in \partial D$. After integrating and making the natural estimates, we get

$$\begin{aligned}T_3 = {} & 2\lambda\alpha \int_{\Gamma(R)} \int_D \mathscr{I}_3(x, y)\, dx\, dy \\ \geqslant {} & -2\lambda\alpha \int_{\partial\Gamma(R)} \int_D r^{1-n} y_k b_{k\ell} \nu_\ell w_{x_i} a_{ij} w_{x_j}\, dx\, d\sigma \\ & + 2\lambda\alpha\mathscr{S} \int_{\Gamma(R)} r^{1-n} \| \nabla_x w \|^2\, dy \\ & - 2\lambda\alpha\mathscr{L} n^2 \int_{\Gamma(R)} r^{1-n} \| \nabla_y w \|^2\, dy,\end{aligned} \tag{2.8}$$

where $\mathscr{S}$ stands for the quantity

$$\mathscr{S} = \{1 - \mathscr{C}(n + n^{1/2} - 1) - \mathscr{K} n^{3/2}\}\underline{a} - \mathscr{L} m\bar{a}^2 - \mathscr{M} n^{1/2} m(1 + \mathscr{C}).$$

Clearly the hypotheses (C_2a) and (C_2b) are chosen to give the result $\mathscr{S} \geqslant \frac{1}{2}\underline{a}$. So the estimate (2.7) follows from (2.8).

LEMMA 3. *There is a constant $\lambda_0 = \lambda_0(\alpha, n)$, independent of $\mathscr{C}$ and $\mathscr{K}$, such that if $\lambda > \lambda_0$, then*

$$\begin{aligned}T_4 \geqslant {} & -2(\lambda\alpha)^2 \int_{\partial\Gamma(R)} r^{\alpha-3-n} q y_k b_{k\ell} \nu_\ell \| w \|^2\, d\sigma \\ & + \tfrac{1}{5}(\lambda\alpha)^3 \int_{\Gamma(R)} r^{2\alpha-1-n} \| w \|^2\, dy.\end{aligned} \tag{2.9}$$

Proof. After integrating by parts, one can re-express T_4 as

$$T_4 = -2(\lambda\alpha)^2 \int_{\partial\Gamma(R)} r^{\alpha-3-n} q y_k b_{k\ell} \nu_\ell \| w \|^2 \, d\sigma$$
$$+ 2(\lambda\alpha)^2 \int_{\Gamma(R)} \| w \|^2 \{r^{-n}(r^{\alpha-3}q)\, y_k b_{k\ell}\}_{y_\ell} \, dy. \tag{2.10}$$

The derivation of (2.9) from (2.10) requires a very delicate estimate of the divergence term

$$J \equiv \{r^{-n}(r^{\alpha-3}q)\, y_k b_{k\ell}\}_{y_\ell} \,.$$

Computation yields

$$J = r^{\alpha-n-3} q[\delta_{k\ell} c_{k\ell} + y_k (c_{k\ell})_{y_\ell} - n r^{-2} y_k c_{k\ell} y_\ell] + r^{-n} y_k b_{k\ell} (r^{\alpha-3} q)_{y_\ell} \,.$$

By (C_1) we have $\mathscr{C} < 1$. Since $\lambda > n$ and $\alpha > 1$, it follows that in $D \times \Gamma$

$$q \geqslant r^2\{(\lambda\alpha r^\alpha - \alpha + 2)(1 - \mathscr{C}) - (n + n^{1/2}\mathscr{C})\} \geqslant 1 - \mathscr{C}(n + n^{1/2} + 1).$$

Thus (C_1) is sufficient to keep $q > 0$. After expanding the quantity $y_k b_{k\ell}(r^{\alpha-3}q)_{y_\ell}$, and grouping its terms according to the powers of r, one can obtain an estimate of the form

$$y_k b_{k\ell}(r^{\alpha-3}q)_{y_\ell} \geqslant \lambda\alpha r^{2\alpha-1} Q_1 - r^{\alpha-1} Q_2 \,,$$

where Q_1 and Q_2 are algebraic expressions in α, n, $\mathscr{C}$, and $\mathscr{K}$. Assumption (C_4a) makes $Q_1 \geqslant \frac{4}{5}$, and thus

$$y_k b_{k\ell}(r^{\alpha-3}q)_{y_\ell} \geqslant \tfrac{4}{5}\, \lambda\alpha r^{2\alpha-1} - r^{\alpha-1} Q_2 \,.$$

Either (C_1) or (C_4a) allows Q_2 to be bounded above in terms of n and α alone. Take λ_0 so large that $\frac{1}{5}\lambda_0 > Q_2$. Then for $\lambda > \lambda_0$

$$y_k b_{k\ell}(r^{\alpha-3}q)_{y_\ell} \geqslant \tfrac{3}{5}\, \lambda\alpha r^{2\alpha-1},$$

and

$$J \geqslant \tfrac{3}{5}\, \lambda\alpha r^{2\alpha-n-1} - r^{\alpha-n-3} q[n^{1/2}\mathscr{C} + n^{3/2}\mathscr{K} + n\mathscr{C}].$$

Since $q \leqslant (\lambda\alpha r^\alpha - \alpha + 2)\, r^2(1 + \mathscr{C})$, one now sees that

$$J \geqslant r^{2\alpha-n-1}\lambda\alpha[\tfrac{3}{5} - (1 + \mathscr{C})\{n^{1/2}\mathscr{C} + n\mathscr{C} + n^{3/2}\mathscr{K}\}]$$
$$- r^{\alpha-n-1}(2 - \alpha)(1 + \mathscr{C})\{n^{1/2}\mathscr{C} + n^{3/2}\mathscr{K} + n\mathscr{C}\}.$$

Using (C_4b), we get

$$J \geqslant \tfrac{1}{5}\, r^{\alpha-n-1}\{2\lambda\alpha r^\alpha - | 2 - \alpha |\}.$$

Take λ_0 also larger than $|2 - \alpha|$. Then for $\lambda > \lambda_0$ we arrive at the result

$$J \geqslant \tfrac{1}{5} \lambda\alpha r^{\alpha-n-1}.$$

Putting this into (2.10) we finish the proof of Lemma 3.

Returning to the main line of the proof, we will assume that $\lambda > \max\{n, \lambda_0\}$ so the result of Lemma 3 will be valid. Applying the three lemmas to estimate the right side of (2.4) one derives the following inequality, in which $\mathscr{I}(R)$ and $\mathscr{I}(1)$ denote certain boundary integrals to be detailed presently:

$$\begin{aligned}\int_{\Gamma(R)} r^{3-\alpha-n} e^{2\lambda r^\alpha} \| Lu \|^2 \, dy + 2\lambda\alpha \mathscr{I}(R) & \\ \geqslant 2\lambda\alpha\mathscr{I}(1) + \tfrac{1}{5}(\lambda\alpha)^3 \int_{\Gamma(R)} r^{2\alpha-n-1} e^{2\lambda r^\alpha} \| u \|^2 \, dy & \\ + \lambda\alpha \int_{\Gamma(R)} r^{1-n} \{ \tfrac{1}{2} \| \nabla_y w \|^2 + \underline{a} e^{2\lambda r^\alpha} \| \nabla_x u \|^2 \} \, dy. & \qquad (2.11)\end{aligned}$$

Notice that $\partial\Gamma(R)$ is composed of the two spheres $S(R)$ and $S(1)$. The outer unit normal ν from $\partial\Gamma(R)$ is therefore given by $\nu = R^{-1}y$ on $S(R)$ and by $\nu = -y$ on $S(1)$.

The terms $\mathscr{I}(\rho)$ for $\rho = R$ and $\rho = 1$ have the form

$$\begin{aligned}\mathscr{I}(\rho) = & \int_{S(\rho)} \int_D r^{-n} \{ 2(y_k b_{k\ell} w_{y_\ell})^2 - (w_{y_k} b_{k\ell} w_{y_\ell})(y_K b_{KL} y_L) \} \, dx \, d\sigma \\ & + \int_{S(\rho)} \int_D r^{-n} (y_k b_{k\ell} y_\ell)(w_{x_i} a_{ij} w_{x_j}) \, dx \, d\sigma \\ & + (\lambda\alpha) \int_{S(\rho)} r^{\alpha-4-n} q (y_k b_{k\ell} y_\ell) \| w \|^2 \, d\sigma.\end{aligned}$$

The next objectives are an upper bound for $\mathscr{I}(R)$ and a lower bound for $\mathscr{I}(1)$.

Since $[b_{ij}]$ is symmetric and positive definite

$$(y_k b_{k\ell} w_{y_\ell})^2 \leqslant (y_K b_{KL} y_L)(w_{y_k} b_{k\ell} w_{y_\ell}).$$

Standard methods lead to

$$w_{y_k} b_{k\ell} w_{y_\ell} \leqslant (1 + \mathscr{C})\, 2e^{2\lambda r^\alpha} \{ (\lambda\alpha)^2 r^{2\alpha-2} u^2 + | \nabla_y u |^2 \}.$$

Because of (C_1) one can verify that in $D \times \Gamma$

$$0 < q < \lambda\alpha r^{\alpha+2}(1 + \mathscr{C}).$$

Using these remarks one concludes that

$$\mathscr{I}(R) \leqslant 3(\lambda\alpha)^2(1+\mathscr{C})^2 \int_{S(R)} r^{2\alpha-n} e^{2\lambda r^\alpha} \| u \|^2 \, d\sigma$$
$$+ 2(1+\mathscr{C})^2 \int_{S(R)} r^{2-n} e^{2\lambda r^\alpha} \| \nabla_y u \|^2 \, d\sigma$$
$$+ (1+\mathscr{C}) \int_{S(R)} r^{2-n} e^{2\lambda r^\alpha} \int_D u_{x_i} a_{ij} u_{x_j} \, dx \, d\sigma.$$

Recalling the definition of the "energy" $E(u, R)$, we see that

$$\mathscr{I}(R) \leqslant 3(\lambda\alpha)^2(1+\mathscr{C})^2 R^{2\alpha-1} e^{2\lambda R^\alpha} E(u, R). \tag{2.12}$$

The argument leading to a lower bound for $\mathscr{I}(1)$ in terms of $E(1, u)$ is contained in the proof of the final lemma.

LEMMA 4. *There is a λ_1 such that if $\lambda > \max\{2n, \lambda_1\}$, then*

$$\mathscr{I}(1) \geqslant \tfrac{1}{2}(1-\mathscr{C})^2 e^{2\lambda} E(u, 1). \tag{2.13}$$

The value of λ_1 depends only on the behavior of u and $\nabla_y u$ on $D \times S(1)$.

Proof. By expressing $\mathscr{I}(1)$ almost entirely in terms of u, one may obtain the inequality

$$\mathscr{I}(1) \geqslant \lambda\alpha e^{2\lambda}(1-\mathscr{C}) \int_{S(1)} q \| u \|^2 \, d\sigma$$
$$+ e^{2\lambda}(1-\mathscr{C}) \int_{S(1)} \int_D u_{x_i} a_{ij} u_{x_j} \, dx \, d\sigma$$
$$+ \int_{S(1)} \int_D (\nu_k b_{k\ell} w_{y_\ell})^2 \, dx \, d\sigma$$
$$- e^{2\lambda} \int_{S(1)} \int_D \{(u_{y_k} b_{k\ell} u_{y_\ell})(\nu_K b_{KL} \nu_L) - (\nu_k b_{k\ell} u_{y_\ell})^2\} \, dx \, d\sigma. \tag{2.14}$$

Under (C_1) and with $\lambda > 2n$, $r = 1$, one gets

$$q \geqslant (\lambda\alpha - \alpha + 2)(1-\mathscr{C}) - n(1+\mathscr{C}) \geqslant \tfrac{3}{4}\lambda\alpha(1-\mathscr{C}).$$

The proofs now proceeds by separate arguments depending on the behavior of u on $D \times S(1)$.

Case 1. Assume that the integral of $\| u \|^2$ over $S(1)$ is positive. Inequality (2.14) can be weakened to the form

$$\mathscr{I}(1) \geqslant \tfrac{3}{4}(\lambda\alpha)^2 e^{2\lambda}(1-\mathscr{C})^2 \int_{S(1)} \| u \|^2 \, d\sigma$$
$$+ e^{2\lambda}(1-\mathscr{C}) \int_{S(1)} \int_D u_{x_i} a_{ij} u_{x_j} \, dx \, d\sigma$$
$$- e^{2\lambda}(1+\mathscr{C})^2 \int_{S(1)} \| \nabla_y u \|^2 \, d\sigma. \tag{2.15}$$

Under (C_1) one can show that if

$$\lambda > \lambda_1 \equiv \left[10 \int_{S(1)} \| \nabla_y u \|^2 \, d\sigma\right]^{1/2} \left[\int_{S(1)} \| u \|^2 \, d\sigma\right]^{-1/2}$$

then

$$\tfrac{1}{4} \lambda^2 (1 - \mathscr{C})^2 \int_{S(1)} \| u \|^2 \, d\sigma - (1 + \mathscr{C})^2 \int_{S(1)} \| \nabla_y u \|^2 \, d\sigma$$
$$\geqslant \tfrac{1}{2} (1 - \mathscr{C})^2 \int_{S(1)} \| \nabla_y u \|^2. \tag{2.16}$$

From (2.15) and (2.16) it follows that

$$\mathscr{I}(1) \geqslant \tfrac{1}{2} e^{2\lambda} (1 - \mathscr{C})^2 \int_{S(1)} \{\| u \|^2 + u_{x_i} a_{ij} u_{x_j} + \| \nabla_y u \|^2\} \, d\sigma.$$

This is exactly the required bound (2.13).

Case 2. Assume that $\| u \|^2$ vanishes identically on $S(1)$; so $u(x, y) = 0$ for all $x \in D$, $y \in S(1)$. Considering u as a function of y for a fixed $x \in D$, we now have $\nabla_y u = \pm \mid \nabla_y u \mid \nu$, since ν is the outer unit normal from $\Gamma(R)$ on $S(1)$. Thus the inequality (2.14) takes the form

$$\mathscr{I}(1) \geqslant \int_{S(1)} \int_D (\nu_k b_{k\ell} w_{y_\ell})^2 \, dx \, d\sigma.$$

But in this case, one also finds that

$$\nu_k b_{k\ell} w_{y_\ell} = e^{\lambda} \nu_k b_{k\ell} u_{y_\ell} = \pm e^{\lambda} \mid \nabla_y u \mid \nu_k b_{k\ell} \nu_\ell$$

on $D \times S(1)$. Thus

$$\mathscr{I}(1) \geqslant e^{2\lambda} (1 - \mathscr{C})^2 \int_{S(1)} \| \nabla_y u \|^2 \, d\sigma. \tag{2.17}$$

Because u and $\nabla_x u$ vanish in $D \times S(1)$ in this case, (2.17) is equivalent to

$$\mathscr{I}(1) \geqslant e^{2\lambda} (1 - \mathscr{C})^2 \int_{S(1)} \{\| \nabla_y u \|^2 + \| u \|^2 + u_{x_i} a_{ij} u_{x_j}\} \, d\sigma,$$

which leads to (2.13) without further conditions on λ.

Having completed the proof of Lemma 4, we return to the proof of Theorem 1. We now require that $\lambda \geqslant \max\{2n, \lambda_0, \lambda_1\}$ in order to assure the validity of (2.11), (2.12), and (2.13).

Combining these three inequalities we find that

$$\int_{\Gamma(R)} r^{3-\alpha-n} e^{2\lambda r^\alpha} \| Lu \|^2 \, dy + 6(1+\mathscr{C})^2 (\lambda\alpha)^3 R^{2\alpha+1} e^{2\lambda R^\alpha} E(u, R)$$

$$\geqslant \tfrac{1}{5} (\lambda\alpha)^3 \int_{\Gamma(R)} r^{2\alpha-n-1} e^{2\lambda r^\alpha} \| u \|^2 \, dy$$

$$+ \lambda\alpha \int_{\Gamma(R)} r^{1-n} \{ \tfrac{1}{2} \| \nabla_y w \|^2 + \underline{a} \| \nabla_x w \|^2 \} \, dy + (1-\mathscr{C})^2 \lambda\alpha e^{2\lambda} E(u, 1). \quad (2.18)$$

It remains only to estimate the two terms referring to w instead of u. Since $w = u \exp(\lambda r^\alpha)$, it follows that

$$| \nabla_x w |^2 = e^{2\lambda r^\alpha} | \nabla_x u |^2$$

and

$$| \nabla_y w |^2 = e^{2\lambda r^\alpha} \sum_{k=1}^{n} \{ \lambda\alpha r^{\alpha-2} y_k u + u_{y_k} \}^2.$$

But the bound

$$| 2(\lambda\alpha r^{\alpha-2} y_k u) \, u_{y_k} | \leqslant \tfrac{6}{5} (\lambda\alpha r^{\alpha-1} u)^2 + \tfrac{5}{6} | \nabla_y u |^2$$

leads to

$$\| \nabla_y w \|^2 \geqslant e^{2\lambda r^\alpha} \{ -\tfrac{1}{5} (\lambda\alpha r^{\alpha-1})^2 \| u \|^2 + \tfrac{1}{6} \| \nabla_y u \|^2 \}.$$

Thus (2.18) will yield the desired inequality (2.1) once we set

$$k_0 \geqslant 6(1+\mathscr{C})^2, \qquad k_1 = \tfrac{1}{10},$$

$$k_2 = \min \{ \tfrac{1}{2}, \underline{a} \} \qquad k_3 \leqslant (1-\mathscr{C})^2.$$

The proof of Theorem 1 is finally complete.

3. The Main Results

We now apply the weighted energy inequality of Theorem 1 to the study of solutions of ultrahyperbolic equations.

THEOREM 2. *Suppose that u belongs to $\mathscr{U}$ and satisfies*

$$| Lu | \leqslant \phi_0 | u | + \phi_1 | \nabla_x u | + \phi_2 | \nabla_y u |. \quad (1.3)$$

If ϕ_0, ϕ_1, and ϕ_2 are bounded in Γ, then for some positive constants k, K, and for all sufficiently large λ

$$K\lambda^3 R^3 e^{2\lambda R^2} E(u, R) \geqslant kE(u, 1) + \int_{\Gamma(R)} r^{1-n} e^{2\lambda r^2} \| u \|^2 \, dy. \qquad (3.1)$$

Proof. We invoke (2.1) with $\alpha = 2$ and λ sufficiently large. Because of the assumption on the ϕ_i, we have

$$\| Lu \|^2 \leqslant \Phi\{\| u \|^2 + \| \nabla_x u \|^2 + \| \nabla_y u \|^2\}$$

for some Φ. Putting this bound on $\| Lu \|^2$ into (2.1), we can obtain the inequality

$$\begin{aligned} &k_0 8\lambda^3 R^3 e^{2\lambda R^2} E(u, R) \\ &\quad \geqslant \int_{\Gamma(R)} \{8k_1\lambda^3 r^2 - \Phi\} \, r^{1-n} e^{2\lambda r^2} \| u \|^2 \, dy \\ &\qquad + \int_{\Gamma(R)} \{2k_2\lambda - \Phi\} \, r^{1-n} e^{2\lambda r^2} \{\| \nabla_x u \|^2 + \| \nabla_y u \|^2\} \, dy + k_3 E(u, 1). \end{aligned}$$

If λ is taken not only so large that (2.1) holds, but also so large that

$$8k_1\lambda^3 - \Phi \geqslant 1, \qquad \text{and} \qquad 2k_2\lambda - \Phi \geqslant 0,$$

then we get the inequality (3.1) claimed by Theorem 2.

The crucial observation about (3.1) is that the right side is a nonnegative increasing function of R. From (3.1) one is led to the following results.

THEOREM 3. *Suppose $u \in \mathcal{U}$ and u satisfies* (1.3). *Assume L satisfies Condition C and the ϕ_i are bounded in Γ. Then*

(i) *if u has bounded support, then $u \equiv 0$;*

(ii) *$E(u, R)$ cannot decay arbitrarily fast, unless $u \equiv 0$;*

(iii) *there are positive constants K and ρ, such that for $R > 1$*

$$E(u, R) \geqslant Ke^{-\rho R^2} E(u, 1).$$

Proof. (i) Suppose that the support of u is contained in some $D \times \Gamma(R)$. Then $E(u, R)$ will be zero. From (3.1) it follows that $\| u \| = 0$ in $\Gamma(R)$, and thus that the support of u is empty.

(ii) Suppose $E(u, R)$ is $o(e^{-\rho R^2})$ for all $\rho > 0$. Then (3.1) forces $\| u \|$ to vanish in all $\Gamma(R)$.

(iii) This follows immediately from (3.1).

In the proof of Theorem 2, one should notice that the validity of (3.1) for any given value of R requires only that the ϕ_i be bounded in $D \times \Gamma(R)$ and that u

solve (1.3) in $D \times \Gamma(R)$ and vanish on $(\partial D) \times \Gamma(R)$. This remark allows us to treat the question of uniqueness for a mixed boundary value problem.

THEOREM 4. *Suppose L satisfies Condition* C, *and the* ϕ_i *are bounded in* $\Gamma(R)$. *Then there is at most one solution of the problem*

$$
\begin{aligned}
&Lu = f(x, y, u, \nabla_x u, \nabla_y u) \qquad \text{in } D \times \Gamma(R);\\
&u \text{ is specified on } (\partial D) \times \Gamma;\\
&u \ \text{ and } \ u_n = r^{-1} y_k u_{y_k} \text{ are specified on } D \times S(R).
\end{aligned} \tag{3.2}
$$

Proof. Suppose u and v are both solutions. Set $U = u - v$. Then the Lipschitz condition on f forces U to satisfy the inequality

$$| LU | \leqslant \phi_0 \,| U | + \phi_1 \,| \nabla_x U | + \phi_2 \,| \nabla_y U |.$$

Clearly $U = 0$ on $(\partial D) \times \Gamma$. Also, it is easy to verify that $E(U, R) = 0$. Thus by Theorem 2, it follows that $U \equiv 0$ in $D \times \Gamma(R)$.

These results can be extended and sharpened by considering various possible growth conditions on the $\phi_i(y)$. We discuss rather informally the case

$$| \phi_0(y)| \leqslant \Phi r^{\beta}, \qquad | \phi_i(y)| \leqslant \Psi r^{\gamma}, \qquad i = 1, 2,$$

where $-\frac{1}{2} < \beta, \gamma < \infty$. Now if u solves (1.3), we can conclude that

$$\| Lu \|^2 \leqslant 3\Phi^2 r^{2\beta} \| u \|^2 + 3\Psi^2 r^{2\gamma} \{\| \nabla_x u \|^2 + \| \nabla_y u \|^2\}.$$

If u both solves (1.3) and belongs to class $\mathscr{U}$, then we can invoke Theorem 1 to get

$$
\begin{aligned}
&k_0(\lambda\alpha)^3 \, R^{2\alpha-1} e^{2\lambda R^{\alpha}} E(u, r)\\
&\quad \geqslant \int_{\Gamma(R)} \{k_1(\lambda\alpha)^3 - 3\Phi^2 r^{4+2\beta-3\alpha}\} \, r^{2\alpha-n-1} e^{2\lambda r^{\alpha}} \| u \|^2 \, dy\\
&\qquad + \int_{\Gamma(R)} \{k_2 \lambda\alpha - 3\Psi^2 r^{2+2\gamma-\alpha}\} \, r^{1-n} e^{2\lambda r^{\alpha}} \| \nabla u \|^2 \, dy + k_3 E(u, 1)
\end{aligned} \tag{3.3}
$$

for $\alpha > 1$ and λ sufficiently large. Now we pick $\alpha = \max\{\frac{1}{3}(4 + 2\beta), 2 + 2\gamma\} > 1$. The effect is to make the powers of r in the curly brackets, $\{\cdots\}$, in (3.3) non-positive. Thus we can pick λ so large that (3.3) holds and that

$$k_1 \lambda^3 - 3\Phi^2 \geqslant 1, \qquad \text{and} \qquad k_2 \lambda - 3\Psi^2 \geqslant 0.$$

For all such large λ, (3.3) yields

$$k_0(\lambda\alpha)^3 \, R^{2\alpha-1} e^{2\lambda R^{\alpha}} E(u, R) \geqslant k_3 E(u, 1) + \int_{\Gamma(R)} r^{2\alpha-1-n} e^{2\lambda r^{\alpha}} \| u \|^2 \, dy.$$

This is the analog of Theorem 2 and the results analogous to those in Theorem 3 can be easily recognized.

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