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Commutators, Lefschetz fibrations and the signatures of surface bundles ☆

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Abstract

We construct examples of Lefschetz fibrations with prescribed singular fibers. By taking differences of pairs of such fibrations with the same singular fibers, we obtain new examples of surface bundles over surfaces with nonzero signature. From these we derive new upper bounds for the minimal genus of a surface representing a given element in the second homology of a mapping class group. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

It is an elementary fact that the Euler characteristic is multiplicative in fiber bundles. According to a classical result of Chern et al. [2] the same holds for the signature, provided that the fundamental group of the base acts trivially on the cohomology of the fiber. Atiyah [1]

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and, independently, Kodaira [9] showed that this assumption on the monodromy is necessary, by exhibiting surface bundles over surfaces with nonzero signature.

In the case of bundles whose fiber is a sphere or torus, it is easy to see that the signature must vanish. Therefore, only the signature of surface bundles of higher genus is interesting. For a closed oriented surface F of genus $h \geqslant 2$, Teichmüller theory implies that the identity component of the group of orientation-preserving diffeomorphisms is contractible. It follows that every oriented bundle with fiber F over a base B is determined by (the conjugacy class of) its monodromy representation

$$\rho : \pi_1(B) \to \Gamma_h,$$

where Γ_h is the mapping class group of F, consisting of isotopy classes of orientation-preserving diffeomorphisms. If the base B is also two-dimensional, then the signature of the total space X is four times the first Chern number of the flat symplectic bundle obtained by composing ρ with the action of Γ_h on the homology of F, see [1,6]. In particular, the signature vanishes if the genus of B is 0 or 1. The signature also vanishes for all bundles with fiber genus 2, because of Igusa's theorem $H_2(\Gamma_2, \mathbb{Q}) = 0$. Thus, we may assume that the fiber genus h is $\geqslant 3$.

Combining the work of Meyer [15] and of Harer [5], one sees that the signature of the total space X is given by the homology class of $\rho_*[B]$ in the homology of Γ_h. More precisely, the second integral homology of the mapping class group is infinite cyclic, generated by the Meyer signature cocycle corresponding to the signature of the total space. This means that determining the maximal signature of a surface bundle with given fiber and base genus is equivalent to calculating the Gromov–Thurston norm in the second homology of the mapping class group. This is essentially Problem 2.18 in Kirby's list [8]. To address this problem, consider the function

$$g_h(n) = \min\{g \mid \exists \text{ a } \Sigma_h\text{-bundle } X \to \Sigma_g \text{ with } \sigma(X) = 4n\}.$$

Using Seiberg–Witten gauge theory, the first nontrivial lower bound for this function was proved in [11]:

$$g_h(n) \geqslant \frac{2|n|}{h-1} + 1. \tag{1}$$

The only systematic upper bound for this function was proved in [3], where it was shown that for every fiber genus $h \geqslant 3$ there is a surface bundle over a surface of genus 111 with signature 4. Pulling back to coverings of the base, one has

$$g_h(n) \leqslant 110|n| + 1. \tag{2}$$

A non-explicit improvement of (2) in some cases was given in [19].

In this paper we obtain new upper bounds for the function $g_h(n)$ by constructing examples of surface bundles in which the base genus is comparatively small. We found these examples by first constructing Lefschetz fibrations with singular fibers corresponding to expressions of products of Dehn twists as products of commutators, and then taking differences of Lefschetz fibrations with the same singular fibers to obtain smooth surface bundles. We have chosen to present the examples in the way we originally found them, although it would have been possible, after the fact, to eliminate the Lefschetz fibrations from the presentation and write down the monodromy representations of the surface bundles directly. We believe that the subtraction of

Lefschetz fibrations presented in Section 2, also used in [19], is of interest in its own right, in addition to being a useful stepping stone in the construction of surface bundles.

Our first main theorem is the following improvement of (2):

Theorem 1. *For every $h \geqslant 3$ there is a surface bundle of genus h over the surface of genus 9 with signature 4. In particular,* $g_h(n) \leqslant 8|n| + 1$.

Notice that all these examples over Σ_9 have the same signature. By considering sections of our fibrations we can construct surface bundles with fiber genus h over Σ_9 for which the signature grows linearly with h. More precisely, we have:

Theorem 2. *For every $h \geqslant 3$ there are surface bundles of fiber genus h over the surface of genus 9 with signature at least* $4(h-2)/3$.

This result allows us to prove upper bounds for $g_h(n)$ which have the same shape as the lower bound (1), in that the fiber genus appears in the denominator. We only formulate these upper bounds in the asymptotic case, when n becomes large. It is easy to see that the limit

$$G_h = \lim_{n\to\infty} \frac{g_h(n)}{n}$$

exists and is finite for all h. The inequality (1) implies $G_h \geqslant 2/(h-1)$. Using our new examples, we will prove:

Theorem 3. *If $h \geqslant 3$ is odd, then $G_h \leqslant 16/(h-1)$. If $h \geqslant 4$ is even, then $G_h \leqslant 16/(h-2)$.*

This paper is organized as follows. In Section 2 we review the basic facts about Lefschetz fibrations and describe the "subtraction operation" for them in detail. Section 3 is devoted to the proof of various identities in the mapping class group expressing certain products of Dehn twists as products of commutators. In Section 4 we calculate the signatures of the corresponding Lefschetz fibrations using the Meyer signature cocycle [15]. In the last section we give the proofs of the theorems stated above.

2. Subtracting Lefschetz fibrations

We begin by recalling the definition and basic properties of Lefschetz fibrations. More details can be found in [4,12]. Let X be a compact oriented 4-manifold, and B a compact oriented surface.

Definition 1. A smooth map $f : X \to B$ is called a Lefschetz fibration if it is surjective and if for each critical point $p \in X$ there are local complex coordinates (z_1, z_2) on X around p and z on B around $f(p)$ compatible with the orientations and such that $f(z_1, z_2) = z_1^2 + z_2^2$.

It follows that a Lefschetz fibration has at most finitely many critical points $p_1, \ldots, p_k$. It is easy to see that by a slight perturbation one can achieve that f is injective on its critical set $C = \{p_1, \ldots, p_k\}$. We will always assume that this additional property holds.

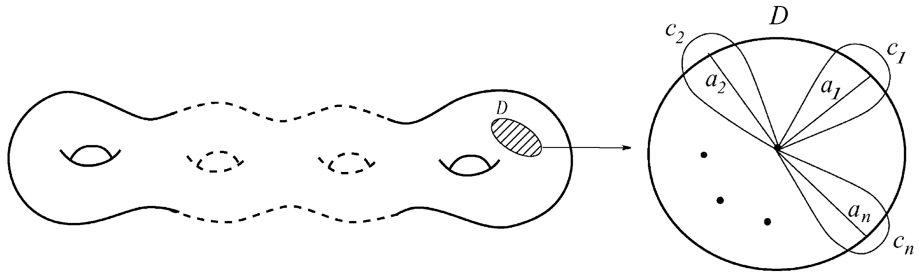


Fig. 1. Choice for loops defining vanishing cycles.

The *genus* of f is defined to be the genus of a regular fiber. If B is connected, the genus is well-defined. Even when B is not connected, we will assume that all regular fibers have the same genus. Fibers of f passing through elements of C are *singular* fibers. Notice that if $v(f(C))$ denotes an open tubular neighborhood of the set of critical values $f(C)$, then the restriction of f to $f^{-1}(B \setminus v(f(C)))$ is a smooth surface bundle over the surface-with-boundary $B \setminus v(f(C))$.

A singular fiber $f^{-1}(q_i)$, where $q_i = f(p_i)$, can be described by its *monodromy*, which is an element in the mapping class group Γ_h. To determine this element, however, we need to fix a base point $\tau \in B \setminus f(C)$, an identification of $f^{-1}(\tau)$ with the closed oriented surface F of genus h, and a loop c_i in B based at τ which has linking number $+1$ with q_i. The restriction of f to the preimage of this loop is an F-bundle over S^1 which can be described by a single element $t_i \in \Gamma_h$. In fact, by performing this procedure for all loops in $B \setminus f(C)$ we get a map $\varphi : \pi_1(B \setminus f(C)) \to \Gamma_h$. It can be shown that t_i is a right-handed Dehn twist along a simple closed curve $v_i \subset f^{-1}(\tau)$ called the *vanishing cycle* corresponding to the singular fiber $f^{-1}(q_i)$. Notice that, even after fixing $\tau \in B$ and the identification $F \approx f^{-1}(\tau)$, both t_i and v_i depend on the chosen loop c_i.

It is convenient to fix the following conventions. Suppose that all q_i lie on the boundary of a disk $D \subset B$ centered at $\tau \in B$. Let a_i denote the radial curve in D connecting τ with q_i and form c_i as the boundary of an appropriate neighborhood of a_i, cf. Fig. 1. By fixing a generating system $\{a_1, b_1, \ldots, a_g, b_g\}$ of $\pi_1(B \setminus D)$, the map φ can be encoded by a sequence $(t_1, \ldots, t_s, \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$, where α_i and $\beta_i \in \Gamma_h$ tell us the monodromy of the fibration along a_i and b_i. It is easy to see that these elements satisfy the relation $\prod_{j=1}^{g} [\alpha_i, \beta_i] \cdot \prod_{i=1}^{s} t_i = 1$ in the mapping class group. Conversely, for $h \geqslant 2$ a word of the form $\prod_{j=1}^{g} [\alpha_i, \beta_i] \cdot \prod_{i=1}^{s} t_i$ representing 1 in Γ_h (with t_i being right-handed Dehn twists) gives rise to a Lefschetz fibration of genus h over a surface B of genus g.

As we noted already, the vanishing cycles and the corresponding Dehn twists depend on the chosen loops c_i. It is easy to see that a cyclic permutation of the indices can be compensated by changing the identification $F \approx f^{-1}(\tau)$, so the resulting Lefschetz fibration remains the same. One can also change the word by *elementary transformations* without changing the Lefschetz fibration, i.e. the path c_i can be changed as indicated by Fig. 2. By applying an elementary transformation as shown by the figure, we replace t_i and t_{i+1} by t_{i+1} and $t_{i+1}^{-1} t_i t_{i+1}$. (Notice that this change has no effect on the product of these elements.) The new vanishing cycles are easy to determine since for any mapping class g the conjugate $g^{-1} t_v g$ of the Dehn twist t_v is

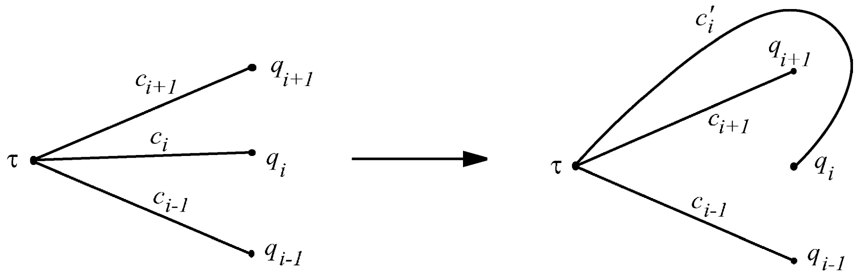


Fig. 2. Elementary transformation.

simply the Dehn twist $t_{g(v)}$. It is not hard to prove that if two words give rise to equivalent fibrations then the words can be transformed into each other by applying combinations of the two operations just described.

A singular fiber $f^{-1}(q_i)$ is *nonseparating* if the corresponding vanishing cycle v_i is nonseparating, equivalently its homology class is nonzero in $H_1(f^{-1}(\tau);\mathbb{Z})$. If v_i is a separating curve, equivalently its homology class is zero, then $f^{-1}(q_i)$ is called *separating*. A vanishing cycle v_i and the corresponding singular fiber are of *type* 0 if v_i is nonseparating; they are of *type* $j \in \{1,\dots,[h/2]\}$ if the vanishing cycle separates the surface of genus h into two components with genera j and $h-j$. Although the vanishing cycle depends on the chosen path c_i, its type is independent of this choice. From the classification of surfaces, one can prove that for two simple closed curves of the same type there exists a diffeomorphism of the ambient surface mapping one into the other. This implies that two singular fibers of the same type have fiber- and orientation-preservingly diffeomorphic tubular neighborhoods.

The combinatorial data of a Lefschetz fibration can be encoded as follows:

Definition 2. The vector $\mu_{\text{comb}}(X) = (\mu_0,\dots,\mu_{[h/2]}) \in \mathbb{Z}^{[h/2]+1}$ associated to the Lefschetz fibration $f:X \to B$ is constructed by taking μ_j to be the number of singular fibers of type j $(j = 0,1,\dots,[h/2])$. Following [18] we say that two fibrations $f_i:X_i \to B_i$ $(i = 1,2)$ are combinatorially equivalently if $\mu_{\text{comb}}(X_1) = \mu_{\text{comb}}(X_2)$.

The construction we use to produce new examples of surface bundles is a procedure for taking the difference of two combinatorially equivalent Lefschetz fibrations. If X_1 and X_2 are combinatorially equivalent as in Definition 2, with critical values $\{q_i^1\}_{i=1}^s$ and $\{q_i^2\}_{i=1}^s$, respectively, then a surface bundle $X_1 - X_2 \to B_1 - B_2$ can be constructed in the following way: order the q_i^1's and q_j^2's so that singular fibers with coinciding lower index have the same type. Fix an orientation- and fiber-preserving diffeomorphism ϕ_i between the boundaries of tubular neighborhoods of fibers with lower index i $(i = 1,\dots,s)$. The union of these maps will be denoted by ϕ. Now glue $X_1 \setminus (\bigcup_{i=1}^s \nu(f_1(q_i^1)))$ to $\overline{X_2 \setminus (\bigcup_{i=1}^s \nu(f_2(q_i^2)))}$ using ϕ. Notice that by reversing the orientation on X_2, the map ϕ becomes orientation-reversing, hence the resulting manifold $Y = X_1 - X_2$ inherits a natural orientation. Since ϕ is fiber-preserving, Y admits a smooth fibration with fibers of genus h over a compact surface B which we will denote by $B_1 - B_2$.

Lemma 4. *If $X_1 \to B_1$ and $X_2 \to B_2$ are combinatorially equivalent Lefschetz fibrations with s singular fibers, then $Y = X_1 - X_2$ is a smooth surface bundle with signature $\sigma(Y) = \sigma(X_1) - \sigma(X_2)$ over the surface $B = B_1 - B_2$ with Euler characteristic $\chi(B) = \chi(B_1) + \chi(B_2) - 2s$.*

Proof. By construction, Y is an oriented smooth surface bundle over a surface B. The claim about the Euler characteristic of the base is obvious. The claim about the signature is an instance of Novikov additivity. □

Note that we did not assume X_1 and X_2 to be connected. This means that basepoints have to be chosen in each component of B_i, and the vectors of combinatorial data have to be summed over all components to determine combinatorial equivalence. If X_1 or X_2 happens to be connected, then so is $X_1 - X_2$.

The main property we used in the above construction is that the manifolds $X_1 \setminus (\bigcup_{i=1}^{s} \nu(f_1(q_i^1)))$ and $X_2 \setminus (\bigcup_{i=1}^{s} \nu(f_2(q_i^2)))$ have diffeomorphic boundaries and, after reversing the orientation of one of them, this diffeomorphism can be chosen to be fiber-preserving and orientation-reversing. A variation of this construction goes as follows: Suppose that partitions of the critical values $\{q_i^1\}_{i=1}^{s}$ and $\{q_i^2\}_{i=1}^{s}$ are given together with a system of disjoint disks $D_j^k \subset B_k$ ($k = 1, 2$ and $j = 1, \ldots, m$) such that each disk contains exactly one equivalence class of the partitions. Suppose furthermore that we can pair up these disks in a way that the surface bundles $X_1|_{D_j^1}$ are isomorphic to $X_2|_{D_j^2}$ for all $j = 1, \ldots, m$. Then X_2 can be subtracted from X_1 along the disks D_j^k, i.e. the manifold

$$Y = \left(X_1 \Bigg\backslash \left(\bigcup_{j=1}^{m} f_1^{-1}(\operatorname{int} D_j^1)\right)\right) \cup \overline{\left(X_2 \Bigg\backslash \left(\bigcup_{j=1}^{m} f_2^{-1}(\operatorname{int} D_j^2)\right)\right)}$$

admits the structure of a surface bundle. The signature $\sigma(Y)$ is again given by $\sigma(X_1) - \sigma(X_2)$, while the Euler characteristic of the base is equal to $\chi(B_1) + \chi(B_2) - 2m$.

Remark 1. The definition of $X_1 - X_2$ is a special case of this latter construction, corresponding to the situation when each equivalence class of the partition consists of a unique critical value. By considering a partition with larger equivalence classes we get smaller m which results in a smaller genus for the base. Notice that in the special case of $X_1 - X_2$ the assumption $X_1|_{D_j^1} \approx X_2|_{D_j^2}$ can be easily checked by determining the type of the singular fibers over the disks. In general, however, the types of the singular fibers over the disks do not specify the diffeomorphism type of the above fibration, since fibers of the same type can be glued together in many different ways resulting in various fibrations over D_j^k.

Remark 2. There is a generalization of Lefschetz fibrations, called achiral Lefschetz fibrations, where one allows singular fibers whose monodromies are left-handed Dehn twists, cf. [4]. Keeping track of the chirality of the singular fibers, it is clear that the subtraction operation described above generalizes to the category of achiral Lefschetz fibrations.

We conclude this section by discussing the relation between the word specifying a Lefschetz fibration and sections of the fibration. Suppose that $f:X\to B$ is a given Lefschetz fibration. A map $\sigma:B\to X$ is called a *section* if $f\circ\sigma=\mathrm{id}_B$. The *self-intersection* (or *square*) of the section σ is simply the self-intersection number of the homology class $[\sigma(B)]\in H_2(X;\mathbb{Z})$. In the following $\Gamma_{h,1}$ denotes the mapping class group of the closed oriented surface of genus h with one marked point and Γ_h^1 denotes the mapping class group with respect to one boundary component (fixed pointwise). Note that by collapsing the boundary circle to a point we get a natural surjection $\varphi:\Gamma_h^1\to\Gamma_{h,1}$ with kernel the subgroup generated by the Dehn twist Δ_∂ along a curve isotopic to the boundary circle (cf. [20], for example). Moreover, by forgetting the marked point we have an obvious map $\Gamma_{h,1}\to\Gamma_h$.

The following two well-known facts show how the existence of a section (and its square) is reflected in the monodromy representation of a Lefschetz fibration. Suppose that the monodromy representation of $f:X\to B$ is given by the relator $\prod_{j=1}^{g}[a_i,b_i]\cdot\prod_{i=1}^{s}t_i$ representing 1 in Γ_h.

Proposition 5. *The fibration admits a section if and only if t_i and $a_j,b_j\in\Gamma_h$ admit lifts $\tilde t_i,\tilde a_j,\tilde b_j\in\Gamma_{h,1}$ such that $\prod_{j=1}^{g}[\tilde a_j,\tilde b_j]\cdot\prod_{i=1}^{s}\tilde t_i$ represents 1 in $\Gamma_{h,1}$. A section of $f:X\to B$ is given once such a lift is fixed.*

Suppose now that a fibration $f:X\to B$ with a section is given, so a lift $\prod_{j=1}^{g}[\tilde a_j,\tilde b_j]\cdot\prod_{i=1}^{s}\tilde t_i$ of $\prod_{j=1}^{g}[a_j,b_j]\cdot\prod_{i=1}^{s}t_i$ is fixed. Take a lift t_i' of $\tilde t_i$ (and a_j',b_j' of $\tilde a_j,\tilde b_j$) in Γ_h^1 and consider $\prod_{j=1}^{g}[a_j',b_j']\cdot\prod_{i=1}^{s}t_i'\in\Gamma_h^1$. From the discussion above, this product is in $\ker\varphi$, hence it is equal to Δ_∂^n for some $n\in\mathbb{Z}$.

Proposition 6 (cf. Smith [18]). *The self-intersection number of the section given by the above lift is equal to $-n$.*

Next we would like to show that after subtracting Lefschetz fibrations with sections, under favorable circumstances the resulting fibration admits a section whose self-intersection number is equal to the difference of the self-intersection numbers of the sections of the individual fibrations. For this, suppose that two fibrations $f_i:X_i\to B_i$ $(i=1,2)$ are given by their monodromy representations $\Pi[a_i,b_i]\cdot\Pi t_i$ and $\Pi[c_j,d_j]\cdot\Pi s_j$, respectively. Suppose furthermore that the disks $D_i\subset B_i$ along which the subtraction operation will be performed contain the singular fibers corresponding to the Dehn twists $t_{i_1}\dots t_{i_k}$ (and $s_{i_1}\dots s_{i_k}$ resp.).

Proposition 7. *If the lifts $\tilde t_{i_n}$ giving rise to the sections coincide with $\tilde s_{i_n}$ $(n=1,\dots,k)$ in $\Gamma_{h,1}$, then the difference of the two fibrations admits a section. The self-intersection of this section is given by the difference of the self-intersection of the individual pieces.*

Proof. The assumption shows that there is a diffeomorphism $f_1^{-1}(D_1)\to f_2^{-1}(D_2)$ mapping the sections into each other. Now the statement is obvious—notice that the self-intersection is the difference of the two self-intersections since in the subtracting operation we change the orientation of X_2. $\square$

Remark 3. The assumption on coinciding lifts cannot be relaxed, as the following example shows. Take two copies of the trivial bundle $\Sigma \times S^2 \to S^2$, fix two sections in each and blow up one section in each copy. In this way we get two Lefschetz fibrations (each with a single singular fiber) for which the subtraction operation (along the singular fibers) applies and gives $\Sigma_h \times S^2 \to S^2$ back. The section blown up, however, can be glued only to the section in the other copy also blown up, because otherwise we would find a homology class in $\Sigma_h \times S^2$ with odd square, which is clearly impossible.

Surface bundles with sections of self-intersection zero can be summed along their sections by performing a fiberwise connected sum. This is an instance of Gompf's symplectic sum operation, but for our purposes the symplectic aspect is irrelevant.

Lemma 8. *If $X_i \to B$ with $i = 1,2$ are two surface bundles with fiber genera h_i over the same base surface and both fibrations admit sections with self-intersection zero, then there is a surface bundle over B with fiber genus $h_1 + h_2$ and signature $\sigma(X_1) + \sigma(X_2)$.*

Proof. The signature is additive when summing along embedded surfaces of self-intersection zero. □

3. Commutators in mapping class groups

Let $F^r_{h,s}$ be an oriented surface of genus h with s marked points and r boundary components. The *mapping class group* $\Gamma^r_{h,s}$ of F consists of the isotopy classes of orientation-preserving diffeomorphisms of F which are the identity on each boundary component and preserve the set of marked points. The isotopies are not allowed to permute marked points or to rotate boundary components. The groups $\Gamma^0_{h,s}$, $\Gamma^r_{h,0}$ and $\Gamma^0_{h,0}$ will be denoted by $\Gamma_{h,s}$, Γ^r_h and Γ_h, respectively.

We say that two simple closed curves a and b on F are *topologically equivalent* if there exists a diffeomorphism of F mapping a to b. For a group G and $x, y \in G$, the commutator $[x, y]$ denotes the element $xyx^{-1}y^{-1}$ and x^y denotes the conjugate yxy^{-1}.

It follows easily from the definition of a Dehn twist that if a is a simple closed curve on F and f is an orientation-preserving diffeomorphism of F, then $f t_a f^{-1} = t_{f(a)}$ in $\Gamma^r_{h,s}$.

If a and b are two topologically equivalent simple closed curves on F, then $t_a t_b^{-1}$ is a commutator. More precisely, if $f(a) = b$ then $t_a t_b^{-1} = [t_a, f]$.

Let a and b be two simple closed curves on F. If a is disjoint from b, then the supports of the Dehn twists t_a and t_b can be chosen to be disjoint. Hence, t_a commutes with t_b. If a intersects b transversely at one point, then it is easy to see that $t_a t_b(a) = b$. It follows that t_a and t_b satisfy the *braid relation* $t_a t_b t_a = t_b t_a t_b$.

The following two relations in the mapping class group are also well-known. The first one is the *lantern relation* (cf. [7]). Let S be a sphere with four boundary components d_1, d_2, d_3 and d_4. Suppose that S is embedded in F. Then there are three simple closed curves α, β, γ on S, as illustrated in Fig. 3(i), which satisfy the lantern relation

$$t_{d_1} t_{d_2} t_{d_3} t_{d_4} = t_\alpha t_\beta t_\gamma.$$

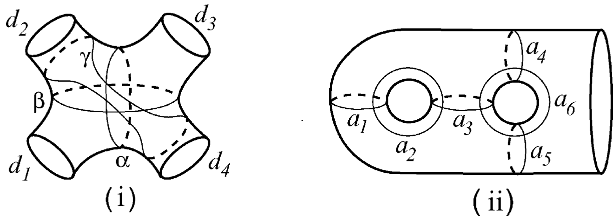


Fig. 3. Curves of the lantern and two-holed torus relation.

The second relation is the *two-holed torus relation* or *chain relation*. Let a_1, a_2, a_3 be three nonseparating simple closed curves on F such that a_2 intersects a_1 and a_3 transversely only once, a_1 is disjoint from a_3 and $a_1 \cup a_3$ does not disconnect F. A regular neighborhood of $a_1 \cup a_2 \cup a_3$ is a torus with two nonseparating boundary components, say a_4 and a_5 (cf. Fig. 3(ii)). Clearly, a_4 and a_5 are disjoint from a_1, a_2, a_3 and from each other. By using the braid relation and the fact that t_{a_1} commutes with t_{a_3}, the relation given by Proposition 3 in [13] is easily shown to be equivalent to the two-holed torus relation

$$t_{a_4}t_{a_5} = (t_{a_1}t_{a_2}t_{a_3})^4.$$

Now we describe various commutator relations in mapping class groups. These relations will be used in the next section to construct the Lefschetz fibrations used in the course of the proofs of the theorems stated in Section 1.

Lemma 9. *Let a, b, c and d be four simple closed curves on F such that a is disjoint from b, c is disjoint from d, and the complements of $a \cup b$ and $c \cup d$ in F are connected. Then $t_a t_b^{-1} t_c t_d^{-1}$ is a commutator.*

Proof. By the classification of surfaces, there exists a diffeomorphism g of F such that $g(a) = d$ and $g(b) = c$. Then

$$t_a t_b^{-1} t_c t_d^{-1} = t_a t_b^{-1} t_{g(b)} t_{g(a)}^{-1} = t_a t_b^{-1} g t_b t_a^{-1} g^{-1} = [t_a t_b^{-1}, g]. \qquad \square$$

Proposition 10. *Let $h \geqslant 3$ and let a be a simple closed curve on F. In the mapping class group $\Gamma_{h,s}^r$ of F*

(a) *t_a^2 can be written as a product of two commutators,*
(b) *if a is nonseparating, then t_a^4 can be written as a product of three commutators.*

Proof. Suppose that the surface of genus 3 with two holes in Fig. 4 is embedded in F. Consider the curves on F given in the figure. The sphere S with four holes of the lantern relation, see Fig. 3, can be embedded in F so that the curves $d_1, d_2, d_3, d_4, \alpha, \beta, \gamma$ become, respectively $a_2, a_1, a_4, a_5, x, a_3, b_1$. This gives us the relation

$$t_{a_1}t_{a_2}t_{a_4}t_{a_5} = t_x t_{a_3} t_{b_1}. \tag{3}$$

Similarly, two other embeddings of S give the relations

$$t_x t_{a_4} t_{a_6} t_{a_8} = t_{a_5} t_{b_2} t_{b_3} \tag{4}$$

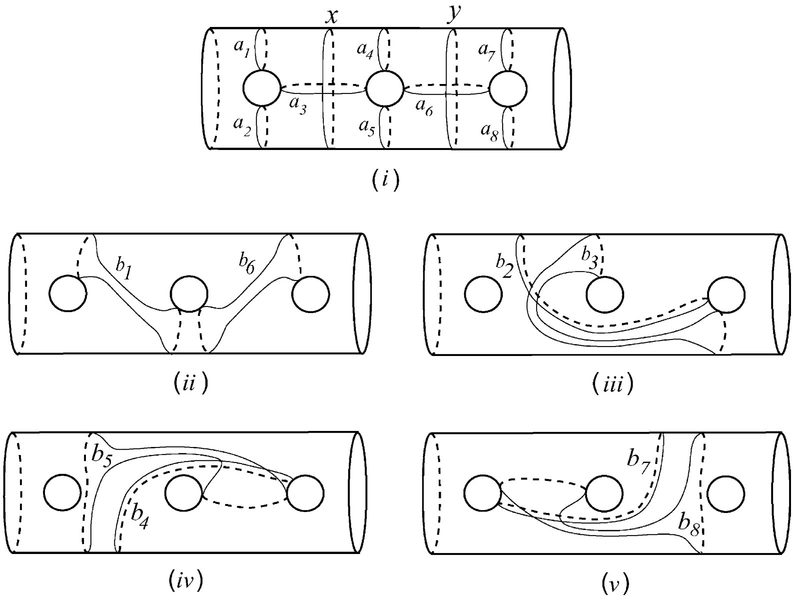


Fig. 4.

and

$$t_x t_{a_5} t_{a_6} t_{a_7} = t_{a_4} t_{b_4} t_{b_5}. \tag{5}$$

If we multiply both sides of (3) by $t_{b_2} t_{b_3}$, use (4) and cancel t_x, we obtain

$$t_{a_4}^2 t_{a_1} t_{a_2} t_{a_6} t_{a_8} = t_{a_3} t_{b_1} t_{b_2} t_{b_3}, \tag{6}$$

or, equivalently,

$$t_{a_4}^2 = t_{a_3} t_{a_6}^{-1} t_{b_1} t_{a_8}^{-1} t_{b_2} t_{a_1}^{-1} t_{b_3} t_{a_2}^{-1}. \tag{7}$$

Similarly, equalities (4) and (5) yield the equality

$$t_x^2 = t_{b_2} t_{a_6}^{-1} t_{b_3} t_{a_7}^{-1} t_{b_4} t_{a_6}^{-1} t_{b_5} t_{a_8}^{-1}. \tag{8}$$

Applying Lemma 9 to (7) and (8) proves that $t_{a_4}^2$ and t_x^2 are products of two commutators. Any nonseparating simple closed curve is topologically equivalent to a_4. If a is a separating simple closed curve on F, then the surface of genus 2 on the right hand side of x can be embedded in F so that x is topologically equivalent to a. Now, the proof of (a) follows from the fact that a conjugate of a commutator is again a commutator.

Similarly, two more embeddings of the lantern give the relations

$$t_{a_4} t_{a_5} t_{a_7} t_{a_8} = t_{b_6} t_{a_6} t_y, \tag{9}$$

$$t_y t_{a_2} t_{a_3} t_{a_4} = t_{a_5} t_{b_7} t_{b_8}. \tag{10}$$

Multiplying (9) by $t_{b_7} t_{b_8}$ from the left and using (10) gives

$$t_{a_4}^2 t_{a_2} t_{a_3} t_{a_7} t_{a_8} = t_{b_7} t_{b_8} t_{b_6} t_{a_6}. \tag{11}$$

By combining (6) and (11), we get

$$t_{a_4}^2 t_{a_1} t_{a_2} t_{a_6} t_{a_8} t_{a_4}^2 t_{a_2} t_{a_3} t_{a_7} t_{a_8} = t_{a_3} t_{b_1} t_{b_2} t_{b_3} t_{b_7} t_{b_8} t_{b_6} t_{a_6}.$$

Cancelling t_{a_3} and t_{a_6} yields

$$t_{a_4}^4 t_{a_1} t_{a_2}^2 t_{a_7} t_{a_8}^2 = t_{b_1} t_{b_2} t_{b_3} t_{b_7} t_{b_8} t_{b_6}.$$

Any simple closed curve on the left hand side is disjoint from each closed curve on the right hand side. Note also that the complements of $a_1 \cup b_1$, $a_2 \cup b_2$, $a_2 \cup b_3$, $a_7 \cup b_7$, $a_7 \cup b_8$ and of $a_8 \cup b_6$ are all connected. Lemma 9 now implies that $t_{a_4}^4$ is a product of three commutators, implying (b). □

Proposition 11. *Let $h \geqslant 2$ and let a and b be two simple closed curves intersecting each other transversely at one point on F. Then $t_a^4 t_b^4$ is a product of three commutators.*

Proof. Suppose that the two-holed torus of Fig. 3(ii) is embedded in F in such a way that a_4 and a_5 are nonseparating on F. The curve a_2 intersects $t_{a_1}(a_2)$ transversely at one point. Since a intersects b transversely at one point also and since any two such pairs are topologically equivalent, we can assume that $a = a_2$ and $b = t_{a_1}(a_2)$. By the two-holed torus relation, we have $t_{a_4} t_{a_5} = (t_{a_1} t_{a_2} t_{a_3})^4$. Let us denote t_{a_i} by t_i. Then, we obtain

$$\begin{aligned}
t_4 t_5 &= (t_1 t_2 t_3 t_1 t_2 t_3)(t_1 t_2 t_3 t_1 t_2 t_3) \\
&= (t_1 t_2 t_1 t_3 t_2 t_3)(t_1 t_2 t_1 t_3 t_2 t_3) \\
&= (t_2 t_1 t_2 t_2 t_3 t_2)(t_2 t_1 t_2 t_2 t_3 t_2) \\
&= t_2 t_2 t_3 t_2 t_2 t_1 t_2 t_2 t_3 t_2 t_2 t_1 \\
&= (t_2 t_2 t_3 t_2^{-1} t_2^{-1}) t_2 t_2 t_2 t_2 (t_1 t_2 t_2 t_2 t_2 t_1^{-1}) t_1 t_1 (t_1^{-1} t_2^{-1} t_2^{-1} t_3 t_2 t_2 t_1).
\end{aligned}$$

If $v = t_{a_2}^2(a_3)$ and $w = t_{a_1}^{-1} t_{a_2}^{-2}(a_3)$, we have the equality

$$(t_{a_4} t_v^{-1} t_{a_5} t_w^{-1}) = t_a^4 t_b^4 t_{a_1}^2.$$

Now, $t_{a_4} t_v^{-1} t_{a_5} t_w^{-1}$ is a commutator and $t_{a_1}^2$ is a product of two commutators. This observation completes the proof of Proposition 11. □

4. Signature computations

The relations expressing certain products of Dehn twists as products of commutators proved in Section 3 allow us to construct corresponding Lefschetz fibrations. These fibrations, and their signatures, depend on the choices we make for the diffeomorphisms occurring in the commutator relations.

In this section the genus of the fiber F is $h \geqslant 3$. The base B of genus g will be denoted by Σ_g if it is closed, and by Σ_g^r if it has r boundary components. For a smooth surface bundle $X \to$

Σ_g^r, the signature is completely determined by the corresponding monodromy representation. We shall pass back and forth between surface bundles over bases with boundary and Lefschetz fibrations over closed bases using the following well-known fact, see [4,14,17]:

Proposition 12. *The signature of a fibered neighborhood of a nonseparating, respectively separating, singular fiber in a Lefschetz fibration is equal to* 0, *respectively to* -1.

Now fix a symplectic basis for $H_1(F,\mathbb{Z})$, so that the monodromy representation

$$\rho : \pi_1(\Sigma_g^r) \to \Gamma_h$$

of X composed with the action of the mapping class group on homology

$$\phi : \Gamma_h \to Sp(2h,\mathbb{Z})$$

yields a symplectic representation χ of the fundamental group of the base. The following result of Meyer [15] allows us to calculate the signature:

Theorem 13. *Let* $f : X \to \Sigma_g^r$ *be an oriented surface bundle with monodromy representation* $\rho : \pi_1(\Sigma_g^r) \to \Gamma_h$. *Fix a standard presentation of* $\pi_1(\Sigma_g^r)$ *as follows*:

$$\pi_1(\Sigma_g^r) = \left\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_r \;\middle|\; \prod_{i=1}^{g}[a_i, b_i]\prod_{j=1}^{r} c_j = 1 \right\rangle,$$

and let $\tau_h : Sp(2h,\mathbb{Z}) \times Sp(2h,\mathbb{Z}) \to \mathbb{Z}$ *by the cocycle defined in* [15].

Then the signature of X *is given by the formula*

$$\sigma(X) = \sum_{i=1}^{g}\tau_h(\kappa_i, \beta_i) - \sum_{i=2}^{g}\tau_h(\kappa_1 \ldots \kappa_{i-1}, \kappa_i) - \sum_{j=1}^{r-1}\tau_h(\kappa_1 \ldots \kappa_g \gamma_1 \ldots \gamma_{j-1}, \gamma_j),$$

where $\alpha_i = \chi(a_i)$, $\beta_i = \chi(b_i)$, $\gamma_i = \chi(c_i)$ *and* $\kappa_i = [\alpha_i, \beta_i]$.

Here is a first application of this formula:

Proposition 14. *There is a Lefschetz fibration* $X \to \Sigma_2$ *with a unique singular fiber and with signature* -1, *whether the vanishing cycle is separating or not*.

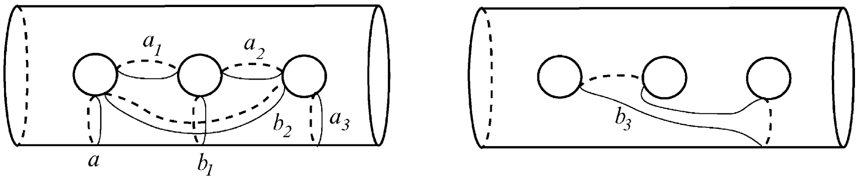
Proof. It is well known that a Dehn twist can be written as a product of two commutators, see [10]. We need to make explicit choices for these commutators. To this end we consider curves $a, a_1, a_2, a_3, b_1, b_2$ and b_3 on a genus 3 subsurface of F as in Fig. 5. Further, we add curves according to Fig. 6. If we choose the genus 3 subsurface suitably, the vanishing cycle v is topologically equivalent to the curve a.

Define diffeomorphisms ϕ_1 and ϕ_2 of F as follows. If a is nonseparating, set

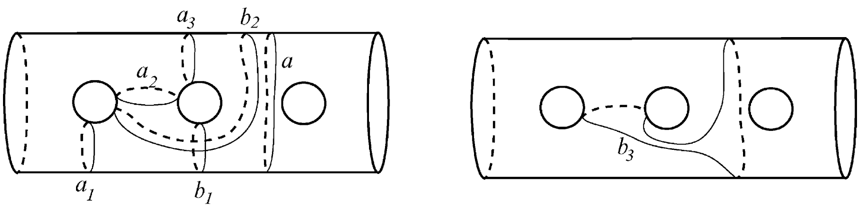
$$\phi_1 = t_{c_1} t_{b_2} t_{c_2} t_{a_2} t_{b_1} t_{c_2} t_{a_1} t_{c_1},$$

and

$$\phi_2 = t_{c_3} t_{b_3} t_{a_3} t_{c_3}.$$



(i) *if a is nonseparating*



(ii) *if a is separating*

Fig. 5.

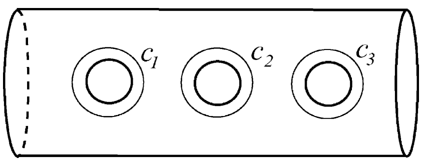


Fig. 6.

If a is separating, set

$$\phi_1 = t_{c_2} t_{a_2} t_{c_1} t_{b_2} t_{a_1} t_{c_1} t_{b_1} t_{c_2}$$

and

$$\phi_2 = t_{c_2} t_{a_3} t_{b_3} t_{c_2}.$$

One can check that $\phi_1(a_1) = b_2$, $\phi_1(b_1) = a_2$ and $\phi_2(a_3) = b_3$.

The lantern relation as in Theorem 2 of [10] implies that

$$t_{a_3} t_{b_3}^{-1} t_{a_2} t_{b_2}^{-1} t_{a_1} t_{b_1}^{-1} t_a = 1.$$

The monodromy representation of the complement of the singular fiber is given by mapping the standard generators of $\pi_1(\Sigma_2^1)$ to $t_{a_3}, \phi_2, \phi_1, t_{a_1}^{-1} t_{b_1}$ and t_a, respectively, as

$$[t_{a_3}, \phi_2][\phi_1, t_{a_1}^{-1} t_{b_1}] t_a = 1.$$

Evaluating the signature cocycle, Theorem 13 shows that the complement of the singular fiber has signature -1 if v is nonseparating, and has signature 0 if v is separating. Now Proposition 12 and Novikov additivity complete the proof. □

Mutatis mutandis, this calculation generalizes to prove the next three Propositions:

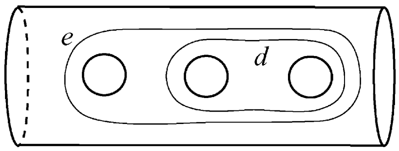


Fig. 7.

Proposition 15. *There is a Lefschetz fibration $X \to \Sigma_2$ with two singular fibers whose monodromies are Dehn twists with the same nonseparating vanishing cycle and signature equal to -2.*

Proposition 16. *There is a Lefschetz fibration $X \to \Sigma_3$ with four singular fibers whose monodromies are Dehn twists with the same nonseparating vanishing cycle and signature equal to -4.*

Proposition 17. *Let a and b be two nonseparating simple closed curves on F which intersect transversely and precisely at one point. There is a Lefschetz fibration $X \to \Sigma_3$ with signature -4 which has eight singular fibers, four of which have monodromy a Dehn twist along a and four of which have monodromy a Dehn twist along b.*

Proof of Propositions 15–17. In all these proofs the signature of the Lefschetz fibration is the same as that of the complement of the singular fibers, because all the vanishing cycles are nonseparating, cf. Proposition 12.

We take curves $x, y, a_1, \dots, a_8$, $b_1, \dots, b_8$ on a genus 3 subsurface of F as in Fig. 4. We also add curves c_1, c_2, c_3 as in Fig. 6, and d and e as in Fig. 7.

For each of the Propositions, the vanishing cycles a and b are topologically equivalent to certain curves a_0 and b_0. We fix the latter explicitly and construct some diffeomorphisms as required by the proofs in Section 3, so that we can write the monodromy representation of the complement of the singular fibers as a relator in the mapping class group. Then the calculation is done by implementing the formula in Theorem 13 with the following data.

For Proposition 15 the base is Σ_2^2 and a_0 is taken to be a_4. The relator giving the monodromy representation is

$$[t_{b_3}^{-1} t_{a_1}, \phi_1][t_{b_1}^{-1} t_{a_2}, \phi_2] t_{a_4}^2 = 1,$$

with

$$\phi_1 = t_e t_{b_2} t_{a_1} t_e t_{c_2} t_{a_6} t_{b_3} t_{c_2},$$

and

$$\phi_2 = t_{c_1} t_{a_3} t_{a_2} t_{c_1} t_d t_{a_8} t_{b_1} t_d.$$

For Proposition 16 the base is Σ_3^4 and a_0 is taken to be a_4 again. The relator giving the monodromy representation is

$$[t_{b_6}^{-1} t_{a_2}, \phi_1][t_{b_7}^{-1} t_{a_8}, \phi_2][t_{b_2}^{-1} t_{a_1}, \phi_3] t_{a_4}^4 = 1,$$

with

$$\phi_1 = t_{c_1} t_{b_8} t_{a_2} t_{c_1} t_{c_3} t_{a_8} t_{b_6} t_{c_3},$$

$$\phi_2 = t_{c_3} t_{b_3} t_{a_8} t_{c_3} t_{c_1} t_{a_2} t_{b_7} t_{c_1},$$

and

$$\phi_3 = t_{c_1} t_{b_1} t_{a_1} t_{c_1} t_{c_3} t_{a_7} t_{b_2} t_{c_3}.$$

For Proposition 17 consider the curves $a_1, \dots, a_6$ on a genus 2 subsurface of the fiber as in Fig. 3(ii). The base surface is Σ_3^8 and the curve a_0 is taken to be a_2 and b_0 is taken to be $t_{a_1}(a_2)$. We first compute the signature corresponding to the relator

$$[t_{a_1}^{-1} t_{a_2}^{-2} t_{a_3} t_{a_2}^{2} t_{a_1} t_{a_5}^{-1}, \phi] t_{a_2}^{4} (t_{a_1} t_{a_2} t_{a_1}^{-1})^4 t_{a_1}^{2} = 1,$$

with

$$\phi = t_{a_2}^{3} t_{a_3} t_{a_6} t_{a_4} t_{a_5} t_{a_6} t_{a_3} t_{a_2} t_{a_6} t_{a_5} t_{a_3} t_{a_6} t_{a_2}^{2} t_{a_1},$$

which is the monodromy of a fibration $X' \to T^2$ with 10 singular fibers. The signature of X' is equal to -6. Subtracting off the fibration of Proposition 15 from X' gives the claim. □

As the expression of a given element in Γ_h as a product of commutators is not unique, it is conceivable that the signature of the corresponding Lefschetz fibrations might be different for different choices of commutators. Then a surface bundle of nonzero signature could be constructed by subtracting the Lefschetz fibrations corresponding to different choices from each other.

5. Bounds on the genus function $g_h(n)$

We now prove the theorems about the minimal genus function $g_h(n)$ stated in the Introduction.

Proof of Theorem 1. We apply the subtraction operation to the Lefschetz fibrations $X_1 \to \Sigma_3$ and $X_2 \to \Sigma_3$ as in Propositions 17 and 16, respectively. In X_1 we group the singular fibers into two groups each containing four singular fibers with coinciding vanishing cycles; in X_2 the singular fibers form one group. Now subtracting two copies of X_2 according to the above pattern we get a surface bundle $Y_h \to \Sigma_9$ of fiber genus h with $\sigma(Y_h) = \sigma(X_1) - 2\sigma(X_2) = -4 - 2(-4) = 4$ (cf. Propositions 16 and 17). Thus $g_h(1) \leqslant 9$, and the claim now follows by pulling Y_h back to unramified coverings of Σ_9 of degree $|n|$. □

Surface bundles over Σ_9 with a higher signature can be constructed as follows.

Proof of Theorem 2. Notice that the relators defining the fibrations we used in the proof of Theorem 1 represent 1 in the mapping class group Γ_h^1 of a surface with one boundary component. According to Proposition 6, this fact shows that the fibrations given by the relators $\prod_{i=1}^{3} [a_i, b_i] t_a^4 t_b^4$ and $\prod_{i=1}^{3} [c_i, d_i] t_a^4$ admit sections with vanishing self-intersection. Since the lifts of the various Dehn twists are chosen to be Dehn twists in Γ_h^1, Proposition 7 implies that

$Y_h \to \Sigma_9$ also admits a section with zero self-intersection for all h. Now write h as $3k+l$ where $l \in \{0,1,2\}$, and apply Lemma 8 to k copies of Y_3 together with the product $\Sigma_l \times \Sigma_9 \to \Sigma_9$. The resulting surface bundle $S_h \to \Sigma_9$ of fiber genus h has $\sigma(S_h) = 4k = 4(h-l)/3 \geqslant 4(h-2)/3$. □

Now we turn to the study of the asymptotic behavior of the genus function.

Proof of Theorem 3. First notice that the proof of Theorem 2 immediately yields $G_h = \lim_{n\to\infty} g_h(n)/n \leqslant 24/(h-l)$ for all h. (As before, $l \in \{0,1,2\}$ is the mod 3 residue of h.)

Now every surface of odd genus is a covering of a genus 3 surface. It was shown in Lemma 4.1 of [16], that after replacing a given surface bundle by a pullback to some covering of the base, the resulting surface bundle admits fiberwise coverings of any given degree. From the multiplicativity of the signature in coverings and the multiplicativity of the Euler characteristic of the fiber in fiberwise coverings, for odd h we obtain

$$\lim_{n\to\infty} \frac{g_h(n)}{n} \leqslant \frac{2}{h-1} \lim_{n\to\infty} \frac{g_3(n)}{n} \leqslant \frac{16}{h-1}. \tag{12}$$

For even h consider the fibration $Z \to \Sigma_{8n+1}$ of fiber genus $h-1$ with signature $2n(h-2)$ we got by taking fiberwise coverings. It is easy to see that since $Y_3 \to \Sigma_9$ admits a section of zero self-intersection, so does $Z \to \Sigma_{8n+1}$. Summing Z and the product fibration $\Sigma_1 \times \Sigma_{8n+1} \to \Sigma_{8n+1}$ along their sections (as in Lemma 8), we get a fibration over Σ_{8n+1} with fiber genus h and signature $2n(h-2)$. These examples yield the bound $G_h \leqslant 16/(h-2)$ once h is even. Consequently the proof of Theorem 3 is complete. □

Remark 4. For certain values of h, the examples of Kodaira [9] give a better upper bound, namely $G_h \leqslant 44/5(h-1)$. Our construction has the advantage of covering all possible values of h (and n).

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