Available online at www.sciencedirect.com
SCIENCE DIRECT
An International Journal
computers & mathematics with applications
ELSEVIER Computers and Mathematics with Applications 48 (2004) 885–895
www.elsevier.com/locate/camwa

Preinvex Functions and Weak Efficient Solutions for Some Vectorial Optimization Problem in Banach Spaces

L. BATISTA DOS SANTOS*
IMECC-UNICAMP
CP 6065, 13081-970, Campinas-SP, Brazil
lucelina@ime.unicamp.br

R. OSUNA-GÓMEZ† ‡
Departamento de Estadística e Investigación Operativa Facultad de Matemáticas
Universidad de Sevilla, Sevilla, 41012, Spain
rafaela@us.es

M. A. ROJAS-MEDAR# ‡
IMECC-UNICAMP
CP 6065, 13081-970, Campinas-SP, Brazil
marko@ime.unicamp.br

A. RUFIÁN-LIZANA‡
Departamento de Estadística e Investigación Operativa Facultad de Matemáticas
Universidad de Sevilla, Sevilla, 41012, Spain
rufian@us.es

(Received November 2002; revised and accepted May 2003)

Abstract—In this work, we introduce the notion of preinvex function for functions between Banach spaces. By using these functions, we obtain necessary and sufficient conditions of optimality for vectorial problems with restrictions of inequalities. Moreover, we will show that this class of problems has the property that each local optimal solution is in fact global. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Multiobjective optimization, Preinvexity, Optimality conditions.

*This author is supported by CNPq-Brasil.
† This author is partially supported by MCYT-Spain, Grant MTM2004-01433.
‡ The authors are partially supported by MCYT-Spain Grant BFM2003-06579.
# This author is partially supported by CNPq-Brazil Grant 301354/03-0, and FAPESP-Brazil Grant 01/07557-3.
The authors would like to thank very much the anonymous referees whose remarks helped to improve the first version of this paper in some important points.

0898-1221/04/$ - see front matter Typeset by AMS-TEX
doi:10.1016/j.camwa.2003.05.013

1. INTRODUCTION AND FORMULATION OF THE PROBLEM

Throughout this paper, E, F, and G will be real Banach spaces. We shall consider the following problem of optimization:

$$\begin{aligned} &\text{Minimize} \quad f(x), \\ &\text{subject to} \\ &\qquad -g(x) \in K, \\ &\qquad x \in S \subset E, \end{aligned} \tag{P}$$

where f and g are mappings from E into F and G, respectively, and where S and K are two subsets of E and G. We assume that the spaces F and G are ordered by cones $Q \subset F$, $K \subset G$ and that these cones are closed, convex, and with nonempty interior.

We denote by $\mathcal{F} = \{x \in S : -g(x) \in K\}$ the feasible set of (P).

We can consider the following partial order in F:

$$y, z \in F, \qquad y \preceq_F z \Leftrightarrow z - y \in Q$$

(analogously for G).

Also, we can consider the following relation:

$$y, z \in F, \qquad y \prec_G z \Leftrightarrow z - y \in \operatorname{int} Q$$

(where $\operatorname{int} Q$ is the interior of Q).

Then, we have two concepts of solution for (P).

DEFINITION 1.1. *We say that $x_0 \in \mathcal{F}$ is an efficient solution for (P) if $x \in \mathcal{F}$, $f(x) \preceq_F f(x_0) \Rightarrow f(x) = f(x_0)$.*

DEFINITION 1.2. *We say that $x_0 \in \mathcal{F}$ is a weak efficient solution for (P) if there is not $x \in \mathcal{F}$, such that $f(x) \prec_F f(x_0)$.*

This class of problems has been investigated extensively in recent years. For example, when in (P) we take $E = \mathbb{R}^n$, $F = \mathbb{R}^p$, and $G = \mathbb{R}^m$; $Q = \mathbb{R}^p_+$ and $K = \mathbb{R}^m_+$) was studied by Clarke [1], Craven [2], Minami [3], Mishra and Mukherjee [4], Osuna-Gómez [5] (see also [6–8]), among others with relation to the optimality conditions (under various differentiability hypotheses). The infinite-dimensional case was considered by El Abdouni and Thibault [9], Coladas, Li and Wang [10], and Brandão, Rojas-Medar and Silva [11]. In [10], problem (P) was studied in the absence of the constraint $-g(x) \in K$ and some regularity on f and S was assumed.

El Abdouni and Thibault studied a more general vectorial mathematical programming problem than (P). They considered not only inequality constraints, but also equality constraints and proved necessary conditions of optimality of the Fritz John kind in the nonsmooth case. The approach which they used is that devised by Clarke [1] for nonsmooth optimization problems, which is based on the Ekeland variational principle. However, no Karush-Kuhn-Tucker type conditions nor sufficient conditions of optimality have been derived for problem (P). In [11], these last results were proved by using the invexity notion between Banach spaces (introduced here), also they introduced a Mond-Weir like dual for problem (P) and established duality relations. Our purpose in this work is to extend those results given in [5] (see also [6–8]), for arbitrary Banach spaces.

The notion of convexity is very important in optimization theory. The following results are well known: if $\theta : S \subset \mathbb{R}^n \to \mathbb{R}$ is a convex function defined on S, where S is a nonempty, convex subset of $\mathbb{R}^n$, then

(1) if $\bar{x} \in S$ is a local minimum of θ on S, then $\bar{x}$ is a global minimum of θ (on S);
(2) if θ is differentiable on S and S is open set, $\nabla\theta(x_1)(x_2 - x_1) \le \theta(x_2) - \theta(x_1)$, $\forall\, x_1, x_2 \in S$ (and, in particular, if $\bar{x} \in S$, $\nabla\theta(\bar{x}) = 0$, then $\bar{x}$ is a global minimum of S).

These two properties of convex functions are very important in optimization theory.

In fact, there exist other class of functions that are not convex and that have analogous properties: they are the generalized convex functions ([12–15].)

We suggest the following definition of preinvex functions between Banach spaces, analogously as was done in [11], for the case of invexity. We note that this definition generalizes the notion given previously by Hanson and Mond [16], for the scalar case. We will prove that this class of functions satisfies properties analogous to (1) and (2) and this will be useful in obtaining some optimality conditions for problem (P) (in the sense of weak efficiency). We would like to say that the vectorial problems between Banach spaces has many applications in mathematical economies and engineering. In fact, when we study the multiobjective control problems where the dynamics are given by partial differential equations and we impose conditions of positivity of the solutions in some Sobolev space, we can write this restriction as the restriction g in problem (P). Furthermore, we can consider as objective functional f, for example $f = (f_1, f_2, \dots, f_n)$ or $f = f_\lambda$, where $\lambda \in J$, in the first case are finite objectives, and in the second case are infinite objectives, that can represent perturbations in some components of the solution.

The paper is organized as follows. In Section 2, we give the definition of preinvexity and we prove some results. In Section 3, we study the optimality conditions. In Section 4, we prove the global results for weak efficiency.

2. PREINVEX FUNCTIONS

In this section, we define the preinvexity for functions between Banach spaces and we study some properties. Also, we stress the alternative theorem of Gordan type for preinvex functions. These results will be crucial to obtain the optimality conditions for problem (P).

DEFINITION 2.1. *(See [16].) Let E be a Banach space. The function $\theta : \Omega \subset E \to \mathbb{R}$ is called preinvex with respect to η on $S \subset \Omega$, if for all $x_1, x_2 \in S$ and for each $\lambda \in (0,1)$, there exists a vector function $\eta : S \times S \to E$, such that*

$$\theta (x_2 + \lambda \eta (x_1, x_2)) \leq \lambda \theta (x_1) + (1 - \lambda)\, \theta (x_2).$$

Moreover, if the set $S \subset E$ has the following property

$$x_2 + \lambda \eta (x_1, x_2) \in S, \qquad \forall\, x_1, x_2 \in S, \quad \forall\, \lambda \in (0,1),$$

we will say that S is invex with respect to the vectorial function η.

Let $Q^* := \{\omega^* \in F^* : \langle \omega^*, x\rangle \geq 0, \forall\, x \in Q\}$, the *dual cone* of Q and F^* the topological dual of F. We denote $\langle \cdot, \cdot \rangle$ the *canonical duality* in the $F^* \times F$ (that is, $\langle \omega^*, x\rangle = w^*(x)$, $\forall\, w^* \in F^*$, $\forall\, x \in F$).

We generalize Definition 2.1 to the functions between Banach spaces in the following way.

DEFINITION 2.2. *The function $f : \Omega \subset E \to F$ is called preinvex with respect to η on $S \subset \Omega$ if for each $\omega^* \in Q^*$, the composition function $\omega^* \circ f$ is preinvex with respect to η, in the sense of Definition 2.1.*

LEMMA 2.3. *Definition 2.2 is equivalent to: for all $x_1, x_2 \in S$ and each $\lambda \in (0,1)$, there exists a vector $\eta : S \times S \to E$, such that*

$$f (x_2 + \lambda \eta (x_1, x_2)) \preceq_F \lambda f (x_1) + (1 - \lambda)\, f (x_2). \tag{1}$$

To prove this result, we will need recall the following lemma (see [17, p. 215]).

LEMMA 2.4. *Let F be a Banach space ordered by the cone $Q \subset F$, with Q convex and closed. If there exists $y \in F$, such that $\langle y^*, y\rangle \geq 0$, $\forall\, y^* \in Q^*$, then $y \in Q$.*

The inverse affirmation is clearly true.

PROOF OF LEMMA 2.3. Let $x_1, x_2 \in S$ and $\lambda \in (0,1)$. The following equivalences are true

$$\begin{aligned} & f(x_2 + \lambda\eta(x_1,x_2)) \preceq_F \lambda f(x_1) + (1-\lambda) f(x_2) \\ & \iff \lambda f(x_1) + (1-\lambda) f(x_2) - f(x_2 + \lambda\eta(x_1,x_2)) \in Q \\ & \iff \omega^*(\lambda f(x_1) + (1-\lambda) f(x_2) - f(x_2 + \lambda\eta(x_1,x_2))) \geq 0, \qquad \forall\,\omega^* \in Q^*, \\ & \iff \omega^* \circ f(x_2 + \lambda\eta(x_1,x_2)) \leq \lambda\omega^* \circ f(x_1) + (1-\lambda)\,\omega^* \circ f(x_2), \qquad \forall\,\omega^* \in Q^*, \end{aligned}$$

where the first equivalence follows from the definition of $\preceq_F$, the second from Lemma 2.4 and the third from the linearity of ω^*. ∎

DEFINITION 2.5. *Let $f : \Omega \subset E \to F$. We say that f is directionally differentiable at x_0 in the direction d, denote $f'(x_0, d)$, if the following limit exists*

$$\lim_{\lambda \to 0^+} \frac{f(x_0 + \lambda d) - f(x_0)}{\lambda}.$$

The following property of the directionally differentiable preinvex functions will be extensively used in the rest of the paper.

LEMMA 2.6. *Let $f : \Omega \subset E \to F$ be a preinvex function on $S \subset \Omega$, directionally differentiable. Then,*

$$(\omega^* \circ f)'(x, \eta(x,y)) \leq \omega^* \circ f(y) - \omega^* \circ f(x),$$

$\forall\,\omega^* \in Q^*, \forall\, x, y \in S.$

PROOF. Assume that f is preinvex on S. Then, by Definition 2.2 $\omega^* \circ f$ is preinvex on S, for all $\omega^* \in Q^*$ (in the sense of Definition 2.1).

Then, by Definition 2.1,

$$\omega^* \circ f(x + \lambda\eta(x,y)) \leq \omega^*(\lambda f(y) + (1-\lambda) f(x)), \tag{2}$$

$\forall\,\lambda \in (0,1)$.

From (2)

$$\omega^* \circ f(x + \lambda\eta(x,y)) - \omega^* \circ f(x) \leq \lambda\omega^*(f(y) - f(x)). \tag{3}$$

Dividing the inequality (3) by λ, and taking the limit when $\lambda \to 0^+$, we obtain

$$(\omega^* \circ f)'(x, \eta(x,y)) \leq \omega^* \circ f(y) - \omega^* \circ f(x),$$

$\forall\,\omega^* \in Q^*, \forall\, x, y \in S.$ ∎

We recall the following result, see [18, p. 54].

LEMMA 2.7. *If $Q \subset F$ is a convex cone, $\operatorname{int} Q \neq \emptyset$ and $0 \neq p \in Q^*$, then $p(s) > 0$ when $s \in \operatorname{int} Q$.*

Also, we will prove the following alternative theorem of Gordan's type. This result will be useful in the next sections.

THEOREM 2.8. *Let $f : E \to F$ be a preinvex function with respect to η on $\mathcal{F} \subset E$, where $\mathcal{F}$ is an invex set with respect to η. Let $Q \subset F$ be a convex cone with nonempty interior. Then, either*

(i) *there exists $x \in \mathcal{F}$, such that $-f(x) \in \operatorname{int} Q$, or*
(ii) *there exists $p \in Q^* \setminus \{0\}$, such that $(p \circ f)(\mathcal{F}) \subset \mathbb{R}_+$.*

PROOF. First, we assume that Systems (i) and (ii) have solutions $x \in \mathcal{F}$ and $p \in Q^* \setminus \{0\}$.

Then, from Lemma 2.7, we have that $p(f(x)) < 0$, with $x \in \mathcal{F}$, consequently, we obtain an contradiction with (ii).

Now, we assume that System (ii) has no solution. We will prove that System (i) has a solution.

We put

$$A := f(\mathcal{F}) + \operatorname{int} Q.$$

Set A is open: in fact, let $k \in A$. Then, there exist $x \in \mathcal{F}$ and $s \in \operatorname{int} Q$, such that $k = f(x) + s$.

Since $s \in \operatorname{int} Q$, there exist a ball N with center at zero, such that $s + N \subset Q$.

But, $k + N = f(x) + (s + N) \subset A$, and consequently, A is open.

Now, we will prove that A is convex. Let $k_1, k_2 \in A$ and $\tau \in (0,1)$.

Then, $k_1 = f(x_1) + s_1$, $k_2 = f(x_2) + s_2$, with $x_1, x_2 \in \mathcal{F}$ and $s_1, s_2 \in \operatorname{int} S$.

$$(1-\tau)\, k_1 + \tau k_2 = [(1-\tau)\, f(x_1) + \tau f(x_2)] + [(1-\tau)\, s_1 + \tau s_2]. \tag{4}$$

But, since f is preinvex, we have

$$(1-\tau)\, f(x_1) + \tau f(x_2) \in f(x_2 + \tau\eta(x_1, x_2)) + Q \tag{5}$$

and

$$(1-\tau)\, s_1 + \tau s_2 \in \operatorname{int} Q. \tag{6}$$

By hypothesis, $\mathcal{F}$ is invex, that is,

$$x_2 + \tau\eta(x_1, x_2) \in \mathcal{F}, \tag{7}$$

is true.

From (4)–(7), we obtain $(1-\tau)k_1 + \tau k_2 \in A$, that is, the set A is convex.

Since system (1) has no solution, then $0 \notin A$. From Hahn-Banach theorem, there exists $p \in F^* \setminus \{0\}$, such that

$$p(A) \subset \mathbb{R}_+. \tag{8}$$

We fix $s \in \operatorname{int} Q$. We would like to prove: $p(f(x)) \geq 0$, $\forall\, x \in \mathcal{F}$.

Since $s \in \operatorname{int} Q$, we have

$$s + N \subset \operatorname{int} Q, \tag{9}$$

for some ball N.

For $\tau \in \mathbb{R}_+$ sufficiently big, we have $(1/\tau)f(x) \in N$ and from (9) we have $s - (1/\tau)f(x) \in \operatorname{int} Q$ and recalling that $\operatorname{int} Q$, is a cone, we obtain $\tau s - f(x) \in \operatorname{int} Q$, that is $\tau s \in f(x) + \operatorname{int} Q \subset A$, and therefore, by (8) we have

$$p(s) \geq 0, \qquad \forall\, s \in \operatorname{int} Q. \tag{10}$$

But, for each $\varepsilon > 0$ sufficiently small, such that $k = f(x) + \varepsilon s \in A$, and therefore,

$$(p \circ f)(x) = p(k) - \varepsilon p(s) \geq -\varepsilon p(s) \to 0,$$

as $\varepsilon \to 0^+$, consequently

$$(p \circ f)(x) \geq 0, \qquad \forall\, x \in \mathcal{F}. \tag{11}$$

For each $s_0 \in Q$, $p(s_0) = (1/\tau)p(\tau s_0)$ and for $\tau > 0$ small, $\tau s_0 \in \operatorname{int} Q$, therefore of (10) we have $p(s_0) \geq 0\ \forall\, s_0 \in \operatorname{int} Q$, that is,

$$p \in Q^* \backslash \{0\}, \tag{12}$$

and (11) and (12) imply that p is a solution of System (ii). ∎

3. CONDITIONS OF OPTIMALITY

This section is divided in three parts. In Section 3.1, we study the scalar optimization problem, i.e., when the objective function is real-valued; in Section 3.2, we establish and prove the scalarization theorem. This theorem will be important because of the relationship the optimal solution of the scalar problem with the vectorial problem, and in Section 3.3, we use the above result to obtain optimality conditions for the vectorial problem (P).

3.1. Conditions of Optimality for Scalar Problems

Now, we consider the following scalar optimization problem:

$$\begin{aligned} &\text{Minimize} \quad \theta(x), \\ &\text{subject to} \\ &\qquad -g(x) \in K, \\ &\qquad x \in S \subset E, \end{aligned} \tag{PM}$$

where E,G are Banach spaces, G is ordered by the closed convex cone with nonempty interior K, $\theta : E \to \mathbb{R}$, $g : E \to G$ are directionally differentiable and S is a nonempty open subset of E.

THEOREM 3.1. *We assume that the functions in problem (PM), θ and g are preinvex functions with respect to the same η and are directionally differentiable. Let $\bar{x}$ be a solution of (PM). Then, there exist $\lambda^* \geq 0$ and $\mu^* \in K^*$, not simultaneously zero such that*

$$(\lambda^*\theta)'(\bar{x}, \eta(\bar{x}, y)) + (\mu^* \circ g)'(\bar{x}, \eta(\bar{x}, y)) \geq 0, \qquad \forall\, y \in S,$$

and

$$\langle \mu^*, g(\bar{x}) \rangle = 0.$$

PROOF. From the hypotheses make, we have that the feasible set $\mathcal{F} := \{x \in S : -g(x) \in K\}$ is invex with respect to η.

Let $\bar{x}$ be the solution of (PM). In this case, the system

$$-\begin{bmatrix} \theta(x) - \theta(\bar{x}) \\ g(x) \end{bmatrix} \in \operatorname{int}(\mathbb{R}_+ \times K)$$

has no solution $x \in \mathcal{F}$.

From Theorem 2.8, we have that there exists $p = (\tau, v^*) \in (\mathbb{R}_+ \times K^*)\backslash\{(0,0)\}$, such that

$$\tau[\theta(x) - \theta(\bar{x})] + v^* \circ g(x) \geq 0, \qquad \forall\, x \in \mathcal{F}, \tag{13}$$

consequently,

$$v^* \circ g(\bar{x}) = 0. \tag{14}$$

We observe that for each $\lambda > 0$ sufficiently small, we have $\bar{x} + \lambda\eta(\bar{x}, y) \in F$, $\forall\, y \in S$ since $\mathcal{F}$ is invex with respect to η.

From (13) and (14), we obtain

$$\begin{aligned} &\lim_{\lambda \to 0+} \frac{\tau\theta(\bar{x} + \lambda\eta(\bar{x}, y)) - \tau\theta(\bar{x}) + v^* \circ g(\bar{x} + \lambda\eta(\bar{x}, y)) - v^* \circ g(\bar{x})}{\lambda} \\ &\qquad = (\tau\theta)'(\bar{x}, \eta(\bar{x}, y)) + (v^* \circ g)'(\bar{x}, \eta(\bar{x}, y)) \geq 0. \end{aligned} \tag{15}$$

Setting in (15) $\tau = \lambda^*$ and $\mu = v^*$, we obtain the desired result. ∎

3.2. A Theorem of Scalarization

We will consider the following optimization problem

$$\begin{aligned} &\text{Minimize} \quad f(x), \\ &\text{subject to} \\ &\qquad x \in \Gamma, \end{aligned} \tag{P1}$$

where $f : E \to F$, $\Gamma \subset E$, E and F are Banach spaces, F is ordered by the closed, convex cone Q with nonempty interior.

The following theorem of scalarization is true for problem (P1).

THEOREM 3.2. *Assume that in (P1) the function f is preinvex with respect to η in the set Γ and that the feasible set Γ is invex with respect to η. If $x^* \in \Gamma$ is a weak efficient solution of (P1), then there exists $\omega^* \in Q^* \setminus \{0\}$, such that*

$$\omega^* \circ f(x^*) \le \omega^* \circ f(x), \qquad \forall x \in \Gamma.$$

PROOF. We consider the following sets

$$U := \{u \in F : 0 \prec_F u\}; \qquad V := \{v \in F : v \preceq_F f(x^*) - f(x), \text{ for some } x \in \Gamma\}.$$

Since x^* is a weak efficient solution of (P1), we obtain $U \cap V = \emptyset$.

In fact, assume the contrary, that is, that there exists $z \in U \cap V$.

In this case, there exists $x \in \Gamma$ such that $0 \prec_F z \preceq_F f(x^*) - f(x)$.

But, $z \in \operatorname{int} Q$, and consequently, there exists a ball N with center at zero such that $z + N \subset Q$.

We have, $z \preceq_F f(x^*) - f(x)$, that is, $f(x^*) - f(x) \in z + Q$, consequently,

$$(f(x^*) - f(x)) + N \subset (z + N) + Q \subset Q + Q \subset Q$$

(because Q is a convex cone). Moreover, $f(x^*) - f(x) \in \operatorname{int} Q$ and we deduce that

$$f(x) \prec_F f(x^*), \qquad x \in \Gamma,$$

this is a contradiction with the fact that x^* is a weak efficient solution of (P1). Then, $U \cap V = \emptyset$.

U is obviously open and convex (because $U = \operatorname{int} Q$ and Q is convex).

In view of the fact that the function f preinvex and the set Γ is invex, we have that V is convex.

In fact, let $v_1, v_2 \in V$ and $\lambda \in (0,1)$. Then, there exist $x_1, x_2 \in \Gamma$, such that

$$v_1 \preceq_F f(x^*) - f(x_1) \qquad \text{and} \qquad v_2 \preceq_F f(x^*) - f(x_2).$$

It is easy to see that

$$\lambda v_1 \preceq_F \lambda f(x^*) - \lambda f(x_1) \qquad \text{and} \qquad (1-\lambda) v_2 \preceq_F (1-\lambda) f(x^*) - (1-\lambda) f(x_2),$$

and we deduce that

$$\begin{aligned} \lambda v_1 + (1-\lambda) v_2 &\preceq_F f(x^*) - [\lambda f(x_1) + (1-\lambda) f(x_2)] \\ &\preceq_F f(x^*) - f(x_2 + \lambda \eta(x_1, x_2)), \end{aligned}$$

where the last inequality is consequence of the preinvexity of f. Since Γ is an invex set with respect to η, we have $x_2 + \lambda \eta(x_1, x_2) \in \Gamma$, and therefore, V is convex.

From Hahn-Banach theorem, there exists $w^* \in F^* \setminus \{0\}$, such that

$$\langle w^*, v \rangle \le 0 \le \langle w^*, u \rangle, \qquad \forall u \in U, \quad \forall v \in V.$$

The second inequality implies $\langle w^*, u \rangle \ge 0$, $\forall u \in \operatorname{int} Q$.

But, Q is convex with nonempty interior, and then is verified that $\overline{\operatorname{int} Q} = Q$ ([19], p. 413) and this implies $w^* \in Q^*$.

We observe that for each $x \in \Gamma$, we have $f(x^*) - f(x) \in V$, moreover

$$\langle w^*, f(x^*) - f(x) \rangle \le 0, \qquad \forall x \in \Gamma.$$

∎

3.3. Conditions of Optimality for Vectorial Problems

In this section, we obtain optimality conditions for problem (P).

Observe that the weak efficient solutions of vectorial preinvex functions are fully characterized by a stationary condition.

THEOREM 3.3. *Let $f : \Omega \subset E \to F$ be a preinvex function on $S \subset \Omega$ with respect to η and are directionally differentiable. Then, $\bar{x}$ is a weak efficient solution of f on the open set S if and only if*

$$(\omega^* \circ f)'(\bar{x}, \eta(\bar{x}, y)) \geq 0, \tag{16}$$

$\forall y \in S, \forall \omega^* \in Q^*$.

PROOF. First, we show the implication ($\Rightarrow$). We assume that $\bar{x}$ is a weak efficient solution and that (16) is not true.

In this case, there exist $y \in S$ and $\omega^* \in Q^*$, such that

$$(\omega^* \circ f)'(\bar{x}, \eta(\bar{x}, y)) < 0. \tag{17}$$

Since S is open and $\bar{x} \in S$, we have that $\bar{x} + \lambda\eta(\bar{x}, y) \in S$, for $\lambda > 0$ sufficiently small.

From (17), we obtain

$$\lim_{\lambda \to 0^+} \frac{\omega^* \circ f(\bar{x} + \lambda\eta(\bar{x}, y)) - \omega^* \circ f(\bar{x})}{\lambda} < 0,$$

and therefore, for $\lambda > 0$ sufficient small, we get

$$\omega^*(f(\bar{x} + \lambda\eta(\bar{x}, y)) - f(\bar{x})) < 0.$$

Since $\omega^* \in Q^*$, $\omega^* \neq 0$, we have

$$f(\bar{x} + \lambda\eta(\bar{x}, y)) \prec_F f(\bar{x}),$$

with $\bar{x} + \lambda\eta(\bar{x}, y) \in S$. This is a contradiction with the fact that $\bar{x}$ is a weak efficient solution.

Now, we prove the reverse implication ($\Leftarrow$).

To do this, we assume that condition (16) is true and that $\bar{x}$ is not a weak efficient solution.

In this case, there exists $y \in S$, such that $f(y) \prec_F f(\bar{x})$.

Let $\omega^* \in Q^* \setminus \{0\}$ (it is possible to show that $Q^* \neq \{0\}$; see [20]) and we obtain

$$\omega^* \circ f(y) - \omega^* \circ f(\bar{x}) < 0. \tag{18}$$

Then,

$$0 \leq (\omega^* \circ f)'(\bar{x}, \eta(\bar{x}, y)) \leq \omega^* \circ f(y) - \omega^* \circ f(\bar{x}) < 0.$$ ∎

Next, we give some optimality conditions (necessary and sufficient conditions) for problem (P).

THEOREM 3.4. NECESSARY CONDITION. *Assume that in problem (P) functions f and g are preinvex with respect to the same η, are directionally differentiable, and the set S is invex with respect to η. If $\bar{x}$ is a weak efficient solution of (P), then there exist $\lambda^* \in Q^*$, $\mu^* \in K^*$, not all zero, such that*

$$(\lambda^* \circ f)'(\bar{x}, \eta(\bar{x}, y)) + (\mu^* \circ g)'(\bar{x}, \eta(\bar{x}, y)) \geq 0, \qquad \forall y \in \mathcal{F},$$

and

$$\langle \mu^*, g(\bar{x}) \rangle = 0.$$

PROOF. From the hypotheses done, we have that the feasible set $\mathcal{F}$ is invex with respect to η. By using Theorem 3.2, there exists $\bar{\lambda}^* \in Q^* \setminus \{0\}$ such that

$$\bar{\lambda}^* \circ f(\bar{x}) \leq \bar{\lambda}^* \circ f(x), \qquad \forall x \in \mathcal{F}.$$

Then, by applying Theorem 3.1 , there exist $\alpha \geq 0$ and $\mu^* \in K^*$, not all zero, such that

$$\alpha\left(\bar{\lambda}^* \circ f\right)'(\bar{x}, \eta(\bar{x}, y)) + (\mu^* \circ g)'(\bar{x}, \eta(\bar{x}, y)) \geq 0, \qquad y \in \mathcal{F},$$

and

$$\langle \mu^*, g(\bar{x}) \rangle = 0.$$

$\lambda^* = \alpha\bar{\lambda}^*$ is a sufficient set and we obtain the desirable result. ∎

THEOREM 3.5. SUFFICIENT CONDITION. *Assume that in problem (P) functions f and g are preinvex with respect to to the same function η, directionally differentiable and that set S is invex with respect to η. If there exist $\bar{x} \in \mathcal{F}$ and $(\lambda^*, \mu^*) \in Q^* \times K^*$, with $\lambda^* \neq 0$, such that*

$$(\lambda^* \circ f)'(\bar{x}, \eta(\bar{x}, x)) + (\mu^* \circ g)'(\bar{x}, \eta(\bar{x}, x)) \geq 0, \qquad \forall y \in \mathcal{F}, \tag{19}$$

and

$$\langle \mu^*, g(\bar{x}) \rangle = 0. \tag{20}$$

Then, $\bar{x}$ is a weak efficient solution of (P).

PROOF. Assume the contrary, that is $\bar{x}$ is not a weak efficient solution of (P). Then, there exist $x \in \mathcal{F}$ such that $f(x) \prec_F f(\bar{x})$ and since $\lambda^* \in Q^*, \lambda^* \neq 0$, by using Lemma 2.7, we have $\lambda^*(f(x) - f(\bar{x})) < 0$ and using Lemma 2.6, we obtain

$$(\lambda^* \circ f)'(\bar{x}, \eta(\bar{x}, x)) < 0. \tag{21}$$

Also, we have

$$(\mu^* \circ g)'(\bar{x}, \eta(\bar{x}, x)) \leq \mu^* \circ g(x) - \mu^* \circ g(\bar{x}) \leq 0,$$

where the first inequality is obtained from Lemma 2.6 and the second by the feasibility of x and of (20). Consequently, we have

$$(\mu^* \circ g)'(\bar{x}, \eta(\bar{x}, x)) \leq 0. \tag{22}$$

Adding the inequalities (21) and (22), we obtain

$$(\lambda^* \circ f)'(\bar{x}, \eta(\bar{x}, x)) + (\mu^* \circ g)'(\bar{x}, \eta(\bar{x}, x)) < 0.$$

This is a contradiction with (19), because $x \in \mathcal{F}$.

Therefore, $\bar{x}$ is a weak efficient solution for (P). ∎

REMARK 3.6. *We observe that λ^* can be 0, in this case the problem is called abnormal. To guarantee $\lambda^* \neq 0$, we must impose some conditions on the data. Usually, we assume the following.*

SLATER REGULARITY CONDITION. $\exists x_0 \in \mathcal{F}$, such that $g(x_0) \prec_F 0$.

COROLLARY 3.7. *On the hypotheses of the Theorem 3.4, if the Slater regularity condition is verified, then $\lambda^* \neq 0$.*

PROOF. In fact, we assume that the hypotheses of the Theorem 3.4 is verified and the Slater regularity conditions is true. If we consider $\lambda^* = 0$, we will prove a contradiction. To do this, we observe that there exists $\mu^* \in K^* \backslash \{0\}$, such that

$$(\mu^* \circ g)'(\bar{x}, \eta(\bar{x}, y)) \geq 0, \qquad \forall y \in \mathcal{F}, \tag{23}$$

and

$$\langle \mu^*, g(\bar{x}) \rangle = 0. \tag{24}$$

But,

$$\begin{aligned} (\mu^* \circ g)'(\bar{x}, \eta(\bar{x}, x_0)) &\leq \mu^* \circ g(x_0) - \mu^* \circ g(\bar{x}) \\ &= \mu^* \circ g(x_0) < 0 \end{aligned} \tag{25}$$

(where the first inequality is consequence of Lemma 2.6, the equality is obtained (24) and the last inequality from $g(x_0) \prec_F 0$ and $\mu^* \neq 0$). This is a contradiction with (23). Consequently, $\lambda^* \neq 0$. ∎

4. GLOBAL WEAK EFFICIENCY

In this section, we will consider the following optimization problem

$$\begin{aligned} &\text{Minimize} \quad f(x), \\ &\text{subject to} \\ &\qquad x \in \Gamma, \end{aligned} \tag{P1}$$

where $f : E \to F$, $\Gamma \subset E$, E and F are Banach spaces, F is ordered by the cone, closed, convex, pointed, and with interior nonempty Q.

We will call $x_0 \in \Gamma$ a *global weak efficient solution* for problem (P1) if it is a weak efficient solution of f on the set Γ, in the sense of Definition 2.

Also, we will say that $x_0 \in \Gamma$ is *local weak efficient solution* for problem (P1) if there exists some neighborhood N of x_0, such that x_0 is a weak efficient solution of f on the set $\Gamma \cap N$.

Now, we will prove that if f is a preinvex function in problem (P1), then local efficiency implies global efficiency. In fact, we have the following theorem.

THEOREM 4.1. *If f is preinvex with respect to η and the set Γ is invex with respect to η, then the solution weakly efficient local to (P1) is one solution weakly efficient global to (P1).*

PROOF. Assume that the function f is preinvex on Γ and that $\bar{x} \in \Gamma$ is a local weak efficient solution of (P1), but that is not global.

Then, there exist $x' \in \Gamma$ such that

$$f(\bar{x}) - f(x') \in \operatorname{int} Q. \tag{26}$$

Since f is preinvex and Γ is invex (with respect to η), there exists a function $\eta : E \times E \to E$ such that $\bar{x} + \alpha\eta(x', \bar{x}) \in \Gamma$, for each $\alpha \in (0, 1)$ and

$$f(\bar{x} + \alpha\eta(x', \bar{x})) \preceq_F \alpha f(x') + (1 - \alpha) f(\bar{x}),$$

or equivalently,

$$\alpha f(x') + (1 - \alpha) f(\bar{x}) - f(\bar{x} + \alpha\eta(x', \bar{x})) \in Q$$

or

$$\alpha (f(x') - f(\bar{x})) + f(\bar{x}) - f(\bar{x} + \alpha\eta(x', \bar{x})) \in Q, \qquad \forall \alpha \in (0, 1). \tag{27}$$

Since Q is a pointed cone, from (26) and (27), we obtain $\eta(x', \bar{x}) \neq 0$.

We observe

$$\begin{aligned} f(\bar{x}) - f(\bar{x} + \alpha\eta(x', \bar{x})) &= [\alpha (f(x') - f(\bar{x})) + f(\bar{x}) - f(\bar{x} + \alpha\eta(x', \bar{x}))] + \alpha (f(\bar{x}) - f(x')) \\ &\in Q + \operatorname{int} Q \subset \operatorname{int} Q, \qquad \forall \alpha \in (0, 1) \end{aligned}$$

this is contradiction with the optimality of the point $\bar{x}$. ∎

REFERENCES

1. F.H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, (1983).
2. B.D. Craven, Nonsmooth multiobjective programming, *Numer. Funct. Anal. Optimiz.* **10**, 49–64, (1989).
3. M. Minami, Weak Pareto-optimal necessary conditions in a nondifferential multiobjective program on a Banach space, *Journal Optim. Theory Appl.* **41**, 451–461, (1983).
4. S.K. Mishra and R.N. Mukherjee, On generalised convex multiobjective nonsmooth programming, *Journal of the Australian Mathematical Society* **Ser B 38**, 140–148, (1996).
5. R. Osuna Gómez, Programación con objetivos múltiples: Dualidad, Doctoral Thesis, Universidad de Sevilla, (1995).
6. R. Osuna-Gómez, A. Rufián-Lizana and P. Ruíz-Canales, Multiobjective fractional programming with generalized convexity, *Sociedad de Estadística e Investigación Operativa Top 8* **1**, 97–110, (2000).

7. R. Osuna-Gómez, A. Beato-Moreno and A. Rufián-Lizana, Generalized convexity in multiobjective programming, *Journal of Mathematical Analysis and Applications* **233**, 205–220, (1999).
8. R. Osuna-Gómez, A. Rufián-Lizana and P. Ruíz-Canales, Invex functions and generalized convexity in multiobjective programming, *Journal of Optimization Theory and Applications* **98**, 651–661, (1998).
9. B. El Abdouni and L. Thibault, Lagrange multipliers for Pareto nonsmooth programming problems in Banach spaces, *Optimization* **26**, 277–285, (1992).
10. L. Coladas, Z. Li and S. Wang, Optimality conditions for multiobjective and nonsmooth minimisation in abstract spaces, *Bulletin of the Australian Mathematical Society* **50**, 205–218, (1994).
11. A.J.V. Brandão, M.A. Rojas-Medar and G.N. Silva, Optimality conditions for Pareto nonsmooth nonconvex programming in Banach spaces, *Journal Optim. Theory Appl.* **103**, 65–73, (1999).
12. O.L. Mangasarian, *Nonlinear Programming*, McGraw-Hill, New York, (1969).
13. K.H. Elster and R. Nehse, *Optimality Conditions for Some Nonconvex Problems*, Springer-Verlag, New York, (1980).
14. M.A. Hanson, On sufficiency of the Kuhn-Tucker conditions, *Journal Mathematical Analysis and Applications* **80**, 545–550, (1981).
15. M.A. Hanson, Invexity and the Kuhn-Tucker theorem, *Journal of Mathematical Analysis and Applications* **236**, 594–604, (1999).
16. M. A. Hanson and B. Mond, Convex transformable programming problems and invexity, *Journal of Information and Optimization Sciences* **8**, 201–207, (1987).
17. D. G. Luenberger, *Optimization by Vector Spaces Methods*, John Willey & Sons, (1969).
18. B. D. Craven, *Control and Optimization*, Chapman & Hall, (1995).
19. N. Dunfort and J. Schwarz, *Linear Operators, Part I*, Interscience, (1957).
20. I. V. Girsanov, *Lectures on Mathematical Theory of Extremum Problems, Lecture Notes in Economics and Mathematical Systems, Volume 67*, (1972).