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Partial differential equations

On uniqueness for a rough transport–diffusion equation

*Sur l'unicité pour une équation de transport–diffusion irrégulière*

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ABSTRACT

In this Note, we study a transport–diffusion equation with rough coefficients, and we prove that solutions are unique in a low-regularity class.

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R É S U M É

Dans cette Note, nous étudions une équation de transport–diffusion à coefficients irréguliers, et nous prouvons l'unicité de sa solution dans une classe de fonctions peu régulières.

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1. Introduction

In this note, we address the problem of uniqueness for a transport–diffusion equation with rough coefficients. Our primary interest and motivation is a uniqueness result for an equation obeyed by the vorticity of a Leray-type solution of the Navier–Stokes equation in the full, three-dimensional space [5]. The main theorem of this note is the following.

Theorem 1.1. *Let v be a divergence free vector field in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ and a be a function in $L^2(\mathbb{R}_+ \times \mathbb{R}^3)$. Assume that a is a distributional solution of the Cauchy problem*

$$(C) \begin{cases} \partial_t a + \nabla \cdot (av) - \Delta a = 0 \\ a(0) = 0, \end{cases} \quad (1)$$

where the initial condition is understood in the distributional sense. Then a is identically zero on $\mathbb{R}_+ \times \mathbb{R}^3$.

As a preliminary remark, the assumptions on both v and a entail that $\partial_t a$ belongs to $L^1_{loc}(\mathbb{R}_+, H^{-2}(\mathbb{R}^3))$ and thus, in particular, a is also in $\mathcal{C}(\mathbb{R}_+, \mathcal{D}'(\mathbb{R}^3))$. In [Theorem 1.1](#), a is to be thought of as a scalar component of the vorticity of v , which is in the original problem a Leray solution of the Navier–Stokes equation. In particular, we only know that a belongs to $L^2(\mathbb{R}_+ \times \mathbb{R}^3)$ and $L^\infty(\mathbb{R}_+, \dot{H}^{-1}(\mathbb{R}^3))$, though we will not use the second assumption. The reader accustomed to

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three-dimensional fluid mechanics will notice that, comparing the above equation with the actual vorticity equations in 3D, a term of the type $a\partial_t v$ is missing. In the original problem, where this Theorem first appeared, we actually rely on a double application of [Theorem 1.1](#). For some technical reasons, only the second application of [Theorem 1.1](#) takes in account the above-mentioned term.

As opposed to the standard DiPerna–Lions theory, we cannot assume that a is in $L^\infty(\mathbb{R}_+, L^p(\mathbb{R}^3))$ for some $p \geq 1$. However, our proof does bear a resemblance to the work of DiPerna and Lions; our result may thus be viewed as a generalization of their techniques, see [\[1\]](#), [\[2\]](#), [\[3\]](#) and [\[4\]](#). Because of the low regularity of both the vector field v and the scalar field a , the use of energy-type estimates seems difficult. This is the main reason why we rely instead on a duality argument, embodied by the following theorem.

Theorem 1.2. *Given v a divergence free vector field in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ and a smooth φ_0 in $\mathcal{D}(\mathbb{R}^3)$, there exists a distributional solution of the Cauchy problem*

$$(C') \begin{cases} \partial_t \varphi - v \cdot \nabla \varphi - \Delta \varphi = 0 \\ \varphi(0) = \varphi_0 \end{cases} \tag{2}$$

with the bounds

$$\|\varphi(t)\|_{L^\infty(\mathbb{R}^3)} \leq \|\varphi_0\|_{L^\infty(\mathbb{R}^3)} \tag{3}$$

and

$$\|\partial_j \varphi(t)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \|\nabla \partial_j \varphi(s)\|_{L^2(\mathbb{R}^3)}^2 ds \leq \|\partial_j \varphi_0\|_{L^2(\mathbb{R}^3)}^2 + \|\varphi_0\|_{L^\infty(\mathbb{R}^3)}^2 \|\partial_j v\|_{L^2(\mathbb{R}_+ \times \mathbb{R}^3)}^2 \tag{4}$$

for $j = 1, 2, 3$ and any positive time t .

By reversing the arrow of time, this amounts to build, for any strictly positive T , a solution on $[0, T] \times \mathbb{R}^3$ of the Cauchy problem

$$(-C') \begin{cases} -\partial_t \varphi - v \cdot \nabla \varphi - \Delta \varphi = 0 \\ \varphi(T) = \varphi_T, \end{cases} \tag{5}$$

where we have set $\varphi_T := \varphi_0$ for the reader's convenience.

2. Proofs

We begin with the dual existence result.

Proof (of [Theorem 1.2](#)). Let us choose some mollifying kernel $\rho = \rho(t, x)$ and denote $v^\delta := \rho_\delta * v$, where $\rho_\delta(t, x) := \delta^{-4} \rho(\frac{t}{\delta}, \frac{x}{\delta})$. Let (C'_δ) be the Cauchy problem (C') , where we replaced v by v^δ . The existence of a (smooth) solution φ^δ to (C'_δ) is then easily obtained thanks to, for instance, a Friedrichs method combined with heat kernel estimates. We now turn to estimates uniform in the regularization parameter δ . The first one is a sequence of energy estimates done in L^p with $p \geq 2$, which yields the maximum principle in the limit. Multiplying the equation on φ^δ by $\varphi^\delta |\varphi^\delta|^{p-2}$ and integrating in space and time, we get

$$\frac{1}{p} \|\varphi^\delta(t)\|_{L^p(\mathbb{R}^3)}^p + (p-1) \int_0^t \|\nabla \varphi^\delta(s) |\varphi^\delta(s)|^{\frac{p-2}{2}}\|_{L^2(\mathbb{R}^3)}^2 ds = \frac{1}{p} \|\varphi_0\|_{L^p(\mathbb{R}^3)}^p. \tag{6}$$

Discarding the gradient term, taking the p -th root on both sides and letting p go to infinity gives

$$\|\varphi^\delta(t)\|_{L^\infty(\mathbb{R}^3)} \leq \|\varphi_0\|_{L^\infty(\mathbb{R}^3)}. \tag{7}$$

To obtain the last estimate, let us derive for $1 \leq j \leq 3$ the equation satisfied by $\partial_j \varphi^\delta$. We have

$$\partial_t \partial_j \varphi^\delta - v^\delta \cdot \nabla \partial_j \varphi^\delta - \Delta \partial_j \varphi^\delta = \partial_j v^\delta \cdot \nabla \varphi^\delta. \tag{8}$$

Multiplying this new equation by $\partial_j \varphi^\delta$ and integrating in space and time gives

$$\frac{1}{2} \|\partial_j \varphi^\delta(t)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \|\nabla \partial_j \varphi^\delta(s)\|_{L^2(\mathbb{R}^3)}^2 ds = \frac{1}{2} \|\partial_j \varphi_0\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \int_{\mathbb{R}^3} \partial_j \varphi^\delta(s, x) \partial_j v^\delta(s, x) \cdot \nabla \varphi^\delta(s, x) dx ds. \tag{9}$$

Since v is divergence free, the gradient term in the left-hand side does not contribute to Equation (9). Denote by $I(t)$ the last integral written above. Integrating by parts and recalling that v is divergence free, we have

$$\begin{aligned} I(t) &= - \int_0^t \int_{\mathbb{R}^3} \varphi^\delta(s, x) \partial_j v^\delta(s, x) \cdot \nabla \partial_j \varphi^\delta(s, x) \, dx \, ds \\ &\leq \|\varphi_0\|_{L^\infty(\mathbb{R}^3)} \int_0^t \|\partial_j v^\delta(s)\|_{L^2(\mathbb{R}^3)} \|\nabla \partial_j \varphi^\delta(s)\|_{L^2(\mathbb{R}^3)} \, ds \\ &\leq \frac{1}{2} \int_0^t \|\nabla \partial_j \varphi^\delta(s)\|_{L^2(\mathbb{R}^3)}^2 \, ds + \frac{1}{2} \|\varphi_0\|_{L^\infty(\mathbb{R}^3)}^2 \int_0^t \|\partial_j v^\delta(s)\|_{L^2(\mathbb{R}^3)}^2 \, ds. \end{aligned}$$

And finally, the energy estimate on $\partial_j \varphi^\delta$ reads

$$\|\partial_j \varphi^\delta(t)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \|\nabla \partial_j \varphi^\delta(s)\|_{L^2(\mathbb{R}^3)}^2 \, ds \leq \|\partial_j \varphi_0\|_{L^2(\mathbb{R}^3)}^2 + \|\varphi_0\|_{L^\infty(\mathbb{R}^3)}^2 \|\partial_j v\|_{L^2(\mathbb{R}_+ \times \mathbb{R}^3)}^2. \tag{10}$$

Thus, the family $(\varphi^\delta)_\delta$ is bounded in $L^\infty(\mathbb{R}_+, H^1(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^2(\mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$. Up to some extraction, we have the weak convergence of $(\varphi^\delta)_\delta$ in $L^2(\mathbb{R}_+, \dot{H}^2(\mathbb{R}^3))$ and its weak-* convergence in $L^\infty(\mathbb{R}_+, H^1(\mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$ to some function φ .

By interpolation, we also have $\nabla \varphi^\delta \rightharpoonup \nabla \varphi$ weakly in $L^4(\mathbb{R}_+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$ as $\delta \rightarrow 0$. As a consequence, because $v^\delta \rightarrow v$ strongly in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ as $\delta \rightarrow 0$, the following convergences hold:

$$\begin{aligned} \Delta \varphi^\delta &\rightharpoonup \Delta \varphi \text{ in } L^2(\mathbb{R}_+ \times \mathbb{R}^3); \\ v^\delta \cdot \nabla \varphi^\delta, \partial_t \varphi^\delta &\rightharpoonup v \cdot \nabla \varphi, \partial_t \varphi \text{ in } L^{\frac{4}{3}}(\mathbb{R}_+, L^2(\mathbb{R}^3)). \end{aligned}$$

In particular, such a φ is a distributional solution of (C') with the desired regularity. \square

We now state a Lemma that will be useful in the final proof.

Lemma 2.1. *Let v be a fixed, divergence-free vector field in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$. Let $(\varphi^\delta)_\delta$ be a bounded family in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$. Let $\rho = \rho(x)$ be some smooth function supported inside the unit ball of \mathbb{R}^3 and define $\rho_\varepsilon := \varepsilon^{-3} \rho(\frac{\cdot}{\varepsilon})$. Define the commutator $C^{\varepsilon, \delta}$ by*

$$C^{\varepsilon, \delta}(s, x) := v(s, x) \cdot (\nabla \rho_\varepsilon * \varphi^\delta(s))(x) - (\nabla \rho_\varepsilon * (v(s) \varphi^\delta(s)))(x).$$

Then

$$\|C^{\varepsilon, \delta}\|_{L^2(\mathbb{R}_+ \times \mathbb{R}^3)} \leq \|\nabla \rho\|_{L^1(\mathbb{R}^3)} \|\nabla v\|_{L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))} \|\varphi^\delta\|_{L^\infty(\mathbb{R}_+, H^1(\mathbb{R}^3))}. \tag{11}$$

This type of lemma is absolutely not new. Actually, it is strongly reminiscent of Lemma II.1 in [2] and serves the same purpose. We are now in position to prove the main theorem of this note.

Proof (of Theorem 1.1). Let $\rho = \rho(x)$ be a radial mollifying kernel and define $\rho_\varepsilon(x) := \varepsilon^{-3} \rho(\frac{x}{\varepsilon})$. Convolving the equation on a by ρ_ε gives, denoting $a_\varepsilon := \rho_\varepsilon * a$,

$$(C_\varepsilon) \quad \partial_t a_\varepsilon + \nabla \cdot (a_\varepsilon v) - \Delta a_\varepsilon = \nabla \cdot (a_\varepsilon v) - \rho_\varepsilon * \nabla \cdot (av). \tag{12}$$

Notice that even without any smoothing in time, $a_\varepsilon, \partial_t a_\varepsilon$ are in $L^\infty(\mathbb{R}_+, C^\infty(\mathbb{R}^3))$ and $L^1(\mathbb{R}_+, C^\infty(\mathbb{R}^3))$ respectively, which is enough to make the upcoming computations rigorous. In what follows, we let φ^δ be a solution of the Cauchy problem $(-C'_\delta)$, with $(-C'_\delta)$ being $(-C')$ with v replaced by v^δ . Let us now multiply, for $\delta, \varepsilon > 0$ the equation (C_ε) by φ^δ and integrate in space and time. After integrating by parts (which is justified by the high regularity of the terms we have written), we get

$$\int_0^T \int_{\mathbb{R}^3} \partial_t a_\varepsilon(s, x) \varphi^\delta(s, x) \, dx \, ds = \langle a_\varepsilon(T), \varphi_T \rangle_{\mathcal{D}'(\mathbb{R}^3), \mathcal{D}(\mathbb{R}^3)} - \int_0^T \int_{\mathbb{R}^3} a_\varepsilon(s, x) \partial_t \varphi^\delta(s, x) \, dx \, ds$$

and

$$\int_0^T \int_{\mathbb{R}^3} [\nabla \cdot (v(s, x)a_\varepsilon(s, x)) - \rho_\varepsilon(x) * \nabla \cdot (v(s, x)a(s, x))] \varphi^\delta(s, x) \, dx \, ds = \int_0^T \int_{\mathbb{R}^3} a(s, x) C^{\varepsilon, \delta}(s, x) \, dx \, ds,$$

where the commutator $C^{\varepsilon, \delta}$ has been defined in the Lemma. From these two identities, it follows that

$$\begin{aligned} \langle a_\varepsilon(T), \varphi_T \rangle_{\mathcal{D}'(\mathbb{R}^3), \mathcal{D}(\mathbb{R}^3)} &= \int_0^T \int_{\mathbb{R}^3} a(s, x) C^{\varepsilon, \delta}(s, x) \, dx \, ds \\ &\quad - \int_0^T \int_{\mathbb{R}^3} a_\varepsilon(s, x) (-\partial_t \varphi^\delta(s, x) - v(s, x) \cdot \nabla \varphi^\delta(s, x) - \Delta \varphi^\delta(s, x)) \, dx \, ds. \end{aligned}$$

From the Lemma, we know that $(C^{\varepsilon, \delta})_{\varepsilon, \delta}$ is bounded in $L^2(\mathbb{R}^+ \times \mathbb{R}^3)$. Because $v \cdot \nabla \varphi^\delta \rightarrow v \cdot \nabla \varphi$ in $L^{\frac{4}{3}}(\mathbb{R}^+, L^2(\mathbb{R}^3))$ as $\delta \rightarrow 0$, the only weak limit point in $L^2(\mathbb{R}^+ \times \mathbb{R}^3)$ of the family $(C^{\varepsilon, \delta})_{\varepsilon, \delta}$ as $\delta \rightarrow 0$ is $C^{\varepsilon, 0}$. Thanks to the smoothness of a_ε for each fixed ε , we can take the limit $\delta \rightarrow 0$ in the last equation, which leads to

$$\langle a_\varepsilon(T), \varphi_T \rangle_{\mathcal{D}'(\mathbb{R}^3), \mathcal{D}(\mathbb{R}^3)} = \int_0^T \int_{\mathbb{R}^3} a(s, x) C^{\varepsilon, 0}(s, x) \, dx \, ds. \tag{13}$$

Again, the family $(C^{\varepsilon, 0})_\varepsilon$ is bounded in $L^2(\mathbb{R}^+ \times \mathbb{R}^3)$ and its only limit point as $\varepsilon \rightarrow 0$ is 0, simply because $v \cdot \nabla \varphi_\varepsilon - \rho_\varepsilon * (v \cdot \nabla \varphi) \rightarrow 0$ in $L^{\frac{4}{3}}(\mathbb{R}^+, L^2(\mathbb{R}^3))$. Taking the limit $\varepsilon \rightarrow 0$, we finally obtain

$$\langle a(T), \varphi_T \rangle_{\mathcal{D}'(\mathbb{R}^3), \mathcal{D}(\mathbb{R}^3)} = 0. \tag{14}$$

This being true for any test function φ_T , $a(T)$ is the zero distribution and finally $a \equiv 0$. \square

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