JOURNAL OF ALGEBRA 29, 504–509 (1974)

Rationally Represented Characters and Permutation Characters of Nilpotent Groups

JOHN R. RASMUSSEN

Department of Mathematics, Bowdoin College, Brunswick, Maine 04011

Communicated by W. Feit

Received July 13, 1972

1. INTRODUCTION

A theorem of Artin states that every rationally valued character of a finite group G is a rational sum of permutation characters of G. A natural question to ask is the following: When is every rationally represented character of G an integral sum of permutation characters of G? We call this property P. This paper classifies all nilpotent groups with this property. The main classification is Theorem 5.2.

In Part 3 we prove that all p-groups have property P. This result has been proven by W. Feit, J. Ritter, G. Segal, and others and is included for completeness.

Parts 4 and 5 are the main part of the paper. In Part 4 we characterize the nilpotent groups having property P in terms of the Schur indices of irreducible characters. Part 5 translates this condition into an equivalent statement about the splitting fields of certain algebras, and then we use algebraic number theory to complete the classification.

2. PRELIMINARIES

G will always denote a finite group; F a finite field extension of Q (the rational numbers), and χ an irreducible (absolutely irreducible) character of G. All notation not explained or referred to is as in [5].

The following lemmas on Schur indices are all wellknown, and have been published in more general form in several sources.

LEMMA 2.1. *Let $H \subseteq G$ with ϕ and χ irreducible characters of H and G respectively. Suppose $\phi^* = \chi$ and $F(\chi) = F(\phi)$, then $m_F(\phi) = m_F(\chi)$.* [2].

LEMMA 2.2. *Let $G = G_1 \times G_2$ be the direct product of two groups with irreducible characters χ_1, χ_2 respectively. If $\chi = \chi_1\chi_2$, then $m_F(\chi) \leqslant m_F(\chi_1)m_F(\chi_2)$* [4].

LEMMA 2.3. *Let $G = G_1 \times G_2$ with $(|G_1|, |G_2|) = 1$. Let χ be an irreducible character of G; $\chi = \chi_1\chi_2$. If $m_F(\chi) = 1$, then $m_{F(\chi_1,\chi_2)}(\chi_2) = 1$* [4].

3. PROPERTY P FOR p-GROUPS

The key to the verification of P for p-groups is the following theorem which is probably due to P. Roquette. Roquette uses the theorem, although it is not explicitly stated in [8].

THEOREM 3.1 (Roquette, Solomon [5, 8]). *Let G be a p-group. If χ is a faithful nonlinear irreducible character of G, then either*

(i) *there exists a subgroup $H \subseteq G$ and a character ϕ of H such that $|G : H| = p$; $\chi = \phi^*$ and $Q(\chi) = Q(\phi)$ or*

(ii) *$p = 2$ and G contains a cyclic subgroup of index 2.*

THEOREM 3.2. *Let G be a p-group. Then every rationally represented character of G is an integral sum of permutation characters of G.*

Proof. We proceed by induction on the order of G. It is known [5, p. 62] that

$$O(\chi) = m_Q(\chi) \sum_{\delta} \chi^{\delta} \qquad \delta \in \mathrm{Gal}[Q(\chi) : Q]$$

(χ an irreducible character of G) is a Z-base for the integral ring of rationally represented characters of G. So it suffices to show that each $O(\chi)$ is an integral sum of permutation characters of G. If χ is a nonfaithful irreducible character of G, then we view χ as a character on $G/\ker \chi$ and apply our induction hypothesis. Thus $O(\chi)$ is an integral sum of permutation characters of $G/\ker \chi$. These can also be viewed as permutation characters of G. So we can assume that χ is faithful.

We note that for p-groups, p odd, that $m_Q(\chi) = 1$ for all irreducible characters of G and if p is even, $m_Q(\chi) = 1$ or 2. [5, p. 77]. If χ is a faithful linear character, then G is cyclic; $G = \langle a \rangle$. A calculation shows that $O(\chi) = 1^*_{\langle 1 \rangle} - 1^*_{\langle a^{p^{n-1}} \rangle}$ where $|\langle a \rangle| = p^n$ and χ is any faithful irreducible character of G.

If χ is a faithful nonlinear character we apply Theorem 3.1. Case (i) presents no problem as we restrict $O(\chi)$ to H, and use Lemma 2.1 to obtain

$O(\chi)|_H = \sum_{G/H} O(\phi)^{gH}$ where the gH's are coset representatives of G/H. We apply our induction hypothesis to $O(\phi)$ and induce back up to G. Cancelling p from both sides of the equation gives the desired result.

Case (ii) is broken into two parts. By [6, p. 187] the groups satisfying condition (ii) are as follows:

(1) Dihedral, semidihedral, ordinary nonabelian groups of order 2^{n+1}.

(2) Generalized quaternion.

For (1), let the cyclic subgroup be $\langle a\rangle$, $|\langle a\rangle| = 2^n$ and let b be an involution in $G \sim \langle a\rangle$. By [5, p. 63] $\chi = \lambda^*$ where λ is a faithful linear character of $\langle a\rangle$. A calculation shows that $O(\chi) = 1^*_{\langle b\rangle} - 1^*_{\langle a^{2^{n-1}}, b\rangle}$. For (2), again $\chi = \lambda^*$ where λ is a faithful irreducible character of $\langle a\rangle$. Take $\sum_\delta \chi^\delta|_{\langle a\rangle} = O(\lambda) = 1^*_{\langle 1\rangle} - 1^*_{\langle a^{2^{n-1}}\rangle}$ in $\langle a\rangle$. Inducing back to G, we have $2\sum_\delta \chi^\delta = 1^*_{\langle 1\rangle} - 1^*_{\langle a^{2^{n-1}}\rangle}$. But $m_Q(\chi) = 2$ [5, p. 64], and the result follows.

4. Characterization of Nilpotent Groups With Property P by Means of Schur Indices

The following wellknown result will be used in our characterization.

Lemma 4.1. *Let G be a p-group, and χ an irreducible character of G. If $\chi \neq 1_G$ (the principal character), then there exists a $g \in G$ such that $\chi(g) = \chi(1)\, e^{2\pi i/p}$.*

We now characterize all nilpotent groups which satisfy property P. We introduce the following notation. Let G_2 be the Sylow 2-subgroup of G (if $|G|$ is odd, we let $G_2 = \langle 1\rangle$). Since G is nilpotent $G \cong G_2 \times G/G_2$ and any irreducible character χ of G is of the form $\chi = \chi_2\chi_2'$ where χ_2, χ_2' are irreducible characters of G_2 and G/G_2 respectively.

Theorem 4.2. *Let G be a nilpotent group. Let χ be an irreducible character of G. Set $O(\chi) = m_Q(\chi)\sum_\delta \chi^\delta$ $\delta \in \mathrm{Gal}[Q(\chi) : Q]$, then $O(\chi)$ is an integral sum of permutation characters of G iff $m_Q(\chi) = m_Q(\chi_2)\, m_Q(\chi_2')$.*

Proof.

$$\Leftarrow \quad O(\chi) = m_Q(\chi)\sum_\delta \chi^\delta \qquad \delta \in \mathrm{Gal}[Q(\chi) : Q]$$

$$= m_Q(\chi_2)\, m_Q(\chi_2')\sum_\delta \chi^\delta,$$

$$Q(\chi) = Q(\chi_2\chi_2') = Q(\chi_2, \chi_2'),$$

and

$$\mathrm{Gal}[Q(\chi):Q] = \mathrm{Gal}[Q(\chi_2):Q] \times \mathrm{Gal}[Q(\chi_2'):Q].$$

So $O(\chi) = O(\chi_2)\, O(\chi_2')$.

Now $\chi_2' = \chi_{p_1} \cdots \chi_{p_n}$ where χ_{p_i} is an irreducible character of the Sylow p_i-subgroup of G/G_2. Using the fact that the irreducible character of p-groups have Schur indices 1 when p is odd, we obtain $O(\chi_2') = O(\chi_{p_1})O(\chi_{p_2}) \cdots O(\chi_{p_n})$. The result follows from Theorem 3.2 on p-groups.

$\Rightarrow$ Suppose $m_Q(\chi) \neq m_Q(\chi_2)\, m_Q(\chi_2')$. By [5, p. 77], $m_Q(\chi_2') = 1$ and $m_Q(\chi_2) \leqslant 2$. Thus by Lemma 2.2 $m_Q(\chi) = 1$ and $m_Q(\chi_2) = 2$. Thus $\chi_2' \neq 1_{G/G_2}$ and by repeated application of Lemma 4.1 we know that there exists a $g \in G/G_2$ such that $x_2'(g) = \chi_2'(1)\, e^{2\pi i/n}$ where n divides the product of all primes dividing $|\, G/G_2 \,|$ and $n > 1$. Now

$$\begin{aligned} O(\chi) &= m_Q(\chi) \sum_{\delta} \chi^{\delta} \qquad \delta \in \mathrm{Gal}[Q(\chi):Q] \\ &= \sum_{\delta} \chi^{\delta} \\ &= O(\chi_2') \sum_{\delta} \chi_2^{\delta} \qquad \delta \in \mathrm{Gal}[Q(\chi_2):Q]. \end{aligned}$$

A calculation shows that $O(\chi_2')(g) = (\pm 1)m$ with $(m, 2) = 1$.

Assume that $O(\chi)$ is an integral sum of permutation characters of G. If we evaluate at gg_2 for any $g_2 \in G_2$, we obtain: $O(\chi)(gg_2) = (\pm m) \sum_{\delta} \chi_2^{\delta}(g_2)$ on one side of the equation, and on the other side we obtain an integral sum of permutation characters of G_2. Thus $m \sum_{\delta} \chi_2^{\delta}$ $\delta \in \mathrm{Gal}[Q(\chi_2):Q]$ is an integral sum of permutation characters of G_2. Hence $m_Q(\chi_2) \mid m$; also $m_Q(\chi_2) \mid \deg \chi_2$ which is a power of 2. So $m_Q(\chi_2) = 1$, a contradiction. The proof is complete.

5. A Characterization Using Number Theory

We now complete the characterization of nilpotent groups which have property P.

Given an irreducible character χ of a group G, one associates a division algebra $D(\chi)$ in the following manner: Form the group algebra of G over the field $Q(\chi)$. Let $\mathfrak{A}$ be the simple component of the group algebra on which χ is nonzero. Set $D(\chi)$ to be the Wedderburn component of $\mathfrak{A}$. It is known that $D(\chi)$ is central over $Q(\chi)$, and $[D(\chi):Q(\chi)] = M_Q(\chi)^2$. [7, pp. 542–545].

Let D be the ordinary quaternion algebra over Q (i.e., with basis $1, i, j, k$). Then the following proposition is well known. [8].

PROPOSITION 5.1. *Let G be a nilpotent group and χ an irreducible character of G. If $m_Q(\chi) = 2$, then $D(\chi) = D \otimes_Q Q(\chi)$.* [8].

We now return to our condition on the Schur Indices. In particular we determine when $m_Q(\chi) < m_Q(\chi_2)\, m_Q(\chi_2')$, i.e., $m_Q(\chi) = 1$ and $m_Q(\chi_2) = 2$. By Lemma 2.3 $m_{Q(\chi_1, \chi_2')}(\chi_2) = 1$, but this is the same as saying that $Q(\chi_2, \chi_2')$ splits the division algebra $D(\chi_2)$. Since $D(\chi_2) = D \otimes_Q Q(\chi_2)$ we obtain that $Q(\chi_2, \chi_2')$ splits χ_2 iff $Q(\chi_2, \chi_2')$ splits D.

We now determine when $Q(\chi_2, \chi_2')$ splits D by the use of local indices of D. All facts used about local indices may be found in [1].

It is known that D has nonzero invariants at the primes 2 and ∞. Thus L splits D iff $| L_p : Q_p |$ is even for $p = 2$ and ∞. In our situation we have $L = Q(\chi_2, \chi_2')$ and we must have that $e^{2\pi i/q} \in Q(\chi_2')$ for some odd prime $q \mid | G/G_2 |$. Thus $L_\infty = C$ (the complex numbers) and L_∞ splits D. We are left with the prime 2.

There are two cases to consider:

Case (i). $Q(\chi_2) \neq Q$.

Case (ii). $Q(\chi_2) = Q$.

Case (i) is easily handled by noting that $\sqrt{2} \in Q(\chi_2)$. Now $x^2 + 2$ is irreducible over Q_2, thus $| Q_2(\sqrt{2}) : Q_2 | = 2$. Hence $| L_2 : Q_2 |$ is even and L_2 splits D.

Case (ii). We need to have $| Q_2(\chi_2') : Q_2 |$ even. This is completely discussed in [3].

All of the above is summarized in the following theorem.

THEOREM 5.2. *A nilpotent group G has property P unless the order of G is not a power of 2 and the Sylow 2-subgroup of G has an irreducible character χ with $m_Q(\chi) = 2$ and either*

(i) $Q(\chi) \neq Q$

or

(ii) $Q(\chi) = Q$ *and there exists an odd prime $q \mid | G |$ such that the order of* 2(mod q) *is even.*

Using the following wellknown results:

If $q \equiv \pm 3 \pmod 8$, then the order of 2 (mod q) is even
if $q \equiv -1 \pmod 8$, then the order of 2 (mod q) is odd

we obtain the corollary.

COROLLARY 5.3. *A nilpotent group G has property P unless the order of G* *is not a power of* 2 *and the Sylow* 2*-subgroup of* G *has an irreducible character* χ *with* $m_Q(\chi) = 2$ *and either*

(i) $Q(\chi) \neq Q$

or

(ii) $Q(\chi) = Q$ *and there exists an odd prime* $q \mid |G|$ *with* $q \equiv \pm 3 \pmod 8$, *or* $q \equiv 1 \pmod 8$ *and the order of* $2 \pmod q$ *is even.*

References

1. A. A. Albert, "Structure of Algebras," American Mathematical Society, New York, 1939.
2. R. Brauer, On the algebraic structure of group rings, *J. Math. Soc. Japan* 3 (1951), 237–251.
3. B. Fein, B. Gordon, and J. H. Smith, On the representation of −1 as a sum of two squares in an algebraic number field, *J. Number Theory* 3 (1971), 310–315.
4. B. Fein, Representations of direct products of finite groups, *Pacific J. Math.* **20** (1967), 45–58.
5. W. Feit, "Characters of Finite Groups," Benjamin, New York, 1967.
6. M. Hall, "The Theory of Finite Groups," MacMillan, New York, 1959.
7. B. Huppert, "Endliche Gruppen I," Springer-Verlag, Berlin, 1967.
8. P. Roquette, Realisierung von Darstellungen endlicher nilpotenter Gruppen, *Arch. Math.* **IX** (1958), 241–250.