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Pseudo-abelian integrals: Unfolding generic exponential case ☆

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ABSTRACT

We consider functions of the form $H_0 = P_1^{a_1} \cdots P_k^{a_k} e^{R/Q}$, with P_i, R, and $Q \in \mathbb{R}[x, y]$, which are (generalized Darboux) first integrals of the polynomial system $M d \log H_0 = 0$. We assume that H_0 defines a family $\gamma(h) \subset H_0^{-1}(h)$ of real cycles in a region bounded by a polycycle.

To each polynomial form η one can associate the pseudo-abelian integrals $I(h)$ of $M^{-1}\eta$ along $\gamma(h)$, which is the first order term of the displacement function of the orbits of $M dH_0 + \delta\eta = 0$.

We consider Darboux first integrals unfolding H_0 (and its saddle-nodes) and pseudo-abelian integrals associated to these unfoldings. Under genericity assumptions we show the existence of a uniform local bound for the number of zeros of these pseudo-abelian integrals.

The result is a part of a program to extend Varchenko–Khovanskii's theorem from abelian integrals to pseudo-abelian integrals and prove the existence of a bound for the number of their zeros in function of the degree of the polynomial system only.

1. Introduction and main results

This paper is a part of a program for generalizing the results of Varchenko and Khovanskii [8,14] giving the boundedness of the number of zeros $A(n)$ of abelian integrals corresponding to polynomial

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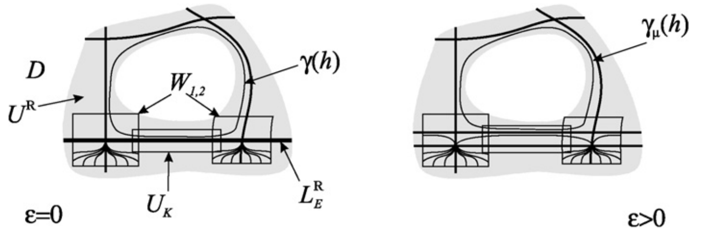

Fig. 1. Unfolding of the polycycle D.

deformations of degree n of Hamiltonian vector fields. We want to generalize this result to deformations of polynomial Darboux integrable systems. The general strategy as in [8,14] is to prove local boundedness and use compactness of the product of the parameter space by the limit periodic sets (see also Roussarie [13]). In previous papers [9], [2] we proved local boundedness of the number of zeros of pseudo-abelian integrals under generic hypothesis. We prove here an analogous result in one of the first non-generic cases where an exponential factor appears in the first integral. Generically, in the unfolding two invariant algebraic curves bifurcate from the exponential factor in saddle-node bifurcations. Other non-generic cases have been studied in [1] and [10].

Consider a real rational closed meromorphic one-form θ_0 having a generalized Darboux first integral of the form

$$H_0 = P_1^{a_1} \cdots P_k^{a_k} e^{R/Q}, \qquad \theta_0 = d(\log H_0). \tag{1}$$

Choose a limit periodic set, i.e. bounded component of $\mathbb{R}^2 \setminus \{Q \prod P_i = 0\}$ filled by cycles $\gamma(h) \subset \{H_0 = h\}$, $h \in (0, b)$. Denote by $D \subset H^{-1}(0)$ the polycycle which is in the boundary of this limit periodic set. The other component of the boundary of the limit periodic set belongs to $H^{-1}(b)$.

Let $U^{\mathbb{R}}$ be a neighborhood of D in $\mathbb{R}^2$, and let U be a neighborhood of D in $\mathbb{C}^2$.

We assume that $Q^{-1}(0)$ contains one or more edges of D. If the curve $Q^{-1}(0)$ does not cut the polycycle D, then the first integral has a form $H = f^* \prod P_i^{a_i}$, where f^* is a non-vanishing holomorphic function in a neighborhood of the polycycle and the proof in [9] or [2] goes through without any modification. Note that the assumption that the curve $Q^{-1}(0)$ cuts the polycycle D implies that $R^{-1}(0) \cap Q^{-1}(0) = \emptyset$. Indeed, in a neighborhood of any (transversal) intersection point $p \in R^{-1}(0) \cap Q^{-1}(0)$ the first integral function reads $H = e^{x/y}$ and so the point $(0, 0)$ does not belong to the closure of a bounded region filled with closed orbits $\gamma(h)$.

Denote the union of the edges of D lying in $Q^{-1}(0)$ by L_E. Each of the vertices of D lying on L_E is a saddle-node and L_E lies in the strong variety of these saddle-nodes. (See Fig. 1.)

We assume that the form θ_0 is generic:

Definition 1. Denote $L_i^{\mathbb{R}} = P_i^{-1}(0)$, $L_E^{\mathbb{R}} = Q^{-1}(0)$ and $L_i^{\mathbb{C}}$ and $L_E^{\mathbb{C}}$ their complexification. We assume that the following properties are satisfied by θ_0 in the neighborhood U of the polycycle D:

(1) the curves $P_j^{-1}(0)$, $Q^{-1}(0)$ are smooth and reduced,
(2) $P_i^{-1}(0)$ and $P_j^{-1}(0)$, as well as $Q^{-1}(0)$ and $P_j^{-1}(0)$ intersect transversally.

Consider an unfolding $\theta_{\varepsilon,\alpha}$ of the form θ_0, where $\theta_{\varepsilon,\alpha}$ are real rational closed one-forms with the Darboux first integral

$$H_{\varepsilon,\alpha} = P_1^{a_1} \cdots P_k^{a_k} Q^{\alpha - 1/\varepsilon} (Q + \varepsilon R)^{1/\varepsilon}, \qquad \theta_{\varepsilon,\alpha} = d(\log H_{\varepsilon,\alpha}). \tag{2}$$

The foliation defined by $\theta_{\varepsilon,\alpha}$ has a maximal nest of cycles $\gamma_{\varepsilon,\alpha}(h) \subset \{H_{\varepsilon,\alpha} = h\}$, $h \in (0, b(\varepsilon,\alpha))$, filling a connected component of $\mathbb{R}^2 \setminus \{Q(Q+\varepsilon R)\prod P_i = 0\}$ whose boundary is a polycycle $D_{\varepsilon,\alpha}$ close to D. Consider pseudo-abelian integrals of the form

$$I_{\varepsilon,\alpha}(h) = \int\limits_{\gamma_{\varepsilon,\alpha}(h)} M^{-1}\eta, \quad \text{where } M = Q(Q+\varepsilon R)\prod_{i=1}^{k} P_i \tag{3}$$

and η is a polynomial one-form of degree at most n.

This integral appears as the linear term with respect to δ of the displacement function of a polynomial deformation

$$M\theta_{\varepsilon,\alpha} + \delta\eta = 0 \tag{4}$$

of the Darboux integrable polynomial vector field with the first integral $H_{\varepsilon,\alpha}$, see [2] and [9].

Theorem 1. *Under the genericity assumptions of Definition* 1 *we have that the number of isolated zeros of pseudo-abelian integrals $I_{\varepsilon,\alpha}$ in their maximal interval of definition* $(0, b(\varepsilon,\alpha))$ *is locally uniformly bounded.*

More precisely, for any n there exist an $\varepsilon_0 > 0$ and an upper bound N, depending on θ_0 and n only, such that for any $|\varepsilon|, |\alpha| < \varepsilon_0$ and any η, $\deg\eta \leqslant n$, the number of isolated zeros of pseudo-abelian integral (3) *in* $(0, b(\varepsilon,\alpha))$ *is at most N.*

In fact, by Varchenko–Khovanskii's theorem [8,14] the number of zeros of $I(h)$ in any interval $[r, b(\varepsilon,\alpha))$ is locally uniformly bounded for any $r > 0$. That is the only point that has to be proved is the local boundedness of the number of zeros of pseudo-abelian integrals in some interval $(0, r)$, for $r > 0$ sufficiently small, i.e. for values corresponding to a neighborhood of the polycycle D.

Following long tradition of [3,5], we completely abandon polynomial settings for analytic ones, and prove more general Theorem 2 below. Theorem 2 deals with unfoldings of a real analytic integrable foliation defined in a neighborhood of the polycycle D and claims that, assuming local analytic analogues of conditions of Theorem 1, the number of zeros of corresponding pseudo-abelian integrals is locally uniformly bounded. Theorem 1 follows from this as indicated above.

Let θ_0 be a closed meromorphic one-form defined in a topological annulus $U^{\mathbb{R}} \subset \mathbb{R}^2$ and satisfying the following conditions:

- $\theta_0 = d(\frac{R}{S}) + \sum a_i \frac{dP_i}{P_i} + \theta'$, where R, S and P_i are analytic in $U^{\mathbb{R}}$, and θ' is a closed one-form analytic in $U^{\mathbb{R}}$;
- $P^{-1}(0), S^{-1}(0)$ are smooth, reduced and intersect transversally.

We assume that the foliation defined by θ_0 in $U^{\mathbb{R}}$ has a nest of cycles accumulating to a polycycle $D \subset U^{\mathbb{R}}$ lying in a polar locus of θ_0, and let $U^{\mathbb{R}}$ be a sufficiently small neighborhood of D. This in particular implies that $\theta' = df^*$ for some analytic in $U^{\mathbb{R}}$ function f^*, which can be further assumed to be equal to zero (by changing P_1 to $P_1 \exp(f^*/a_1)$). We assume that some edges of D lie on $\{Q = 0\}$, as the other case was considered before [2,9].

Consider a finite-dimensional analytic (with topology of uniform convergence on compact sets) family Θ of pairs (θ_μ, η_μ) of one-forms defined in a complex neighborhood U of the polycycle D, $\mu \in \mathbb{R}^m$. We assume that θ_μ is a real meromorphic closed one-form and η_μ is real holomorphic one-form in U.

Assume that the polar locus D_μ of θ_μ is a union of deformations of components of D: this means that the forms $Q_{1,\mu}Q_{2,\mu}P_{1,\mu}\cdots P_{k,\mu}\theta_\mu$ are holomorphic one-forms on U, where $Q_{1,\mu}$, $Q_{2,\mu}$ and $P_{i,\mu}$ are analytic in μ families of real holomorphic functions defined in U, with $Q_{1,0} = Q_{2,0} = Q$ and $P_{i,0} = P_i$. The function $M_\mu = Q_{1,\mu}Q_{2,\mu}P_{1,\mu}\cdots P_{k,\mu}$ will be called the *integrating factor* of θ_μ.

Assume moreover that the real foliations defined by θ_μ have nests of cycles $\gamma_\mu(h) \subset \{H_\mu = h\}$ accumulating to D_μ, where H_μ is the first integral of the foliation defined by θ_μ, namely $H_\mu = \exp(\int \theta_\mu)$.

Theorem 2. *There exists $r > 0$ such that the number of zeros of the pseudo-abelian integral*

$$I_\mu(h) = \int\limits_{\gamma_\mu(h)} M^{-1}\eta_\mu$$

in $(0, r)$ is uniformly bounded over all μ in a sufficiently small neighborhood of 0 in $\mathbb{R}^m$.

Example 1. The family (2) satisfies conditions of Theorem 2: in this case $\mu = (\varepsilon, \alpha)$, $Q_{1,\mu}$ and $P_{i,\mu}$ do not depend on μ and $Q_{2,\mu} = Q + \varepsilon R$.

2. Plan of the proof

2.1. Analytic continuation of pseudo-abelian integral

The first step is to show that the integral $I_\mu(h)$ can be analytically continued to the universal cover of the punctured disc $\{0 < |h| < r\}$ for some sufficiently small r. As in [2], this is obtained by transporting the cycle of integration to nearby leaves. More precisely, in a complex neighborhood of the polycycle D we construct two linearly independent real vector fields preserving the foliation and transversal to it. This allows to define lifting of vector fields from a punctured neighborhood of zero in $\mathbb{C}_h$ to the neighborhood U of D as linear combinations of these vector fields, see Section 3. We transport the real cycles $\gamma_\mu(h)$ using flows of these liftings.

Remark 1. Our construction of local transport of cycles differs from the one used in [11]. Both constructions start from local vector fields (so-called “Clemens symmetries”), and then use partition of unity to get a transport defined in a neighborhood of D. However, we glue together the vector fields themselves, and not their flows as in [11].

2.2. Variation relation

The form θ_μ has a first order pole on $P^{-1}_{j,\mu}(0)$, so from closedness of θ_μ it follows that the residue of θ_μ on $P^{-1}_{j,\mu}(0)$ is well defined. We will denote it by $a_{j,\mu}$.

The main feature of the constructed transport is that the lifting of $ih\partial_h$ is $2\pi a_{j,\mu}$-periodic in a neighborhood of separatrics lying on $\{P_{j,\mu} = 0\}$. This implies that the cycle $\gamma_\mu(h) \subset \{H_\mu = h\}$ and its transport to $\gamma_\mu(he^{2\pi i a_{j,\mu}}) \subset \{H_\mu = he^{2\pi i a_{j,\mu}}\}$ coincide in this neighborhood, so the difference $\gamma_\mu(he^{\pi i a_{j,\mu}}) - \gamma_\mu(he^{-\pi i a_{j,\mu}})$ does not intersect a neighborhood of $\{P_{j,\mu} = 0\}$.

For pseudo-abelian integrals this geometric observation translates into the following construction. Define the variation operator Var_a as the difference between counterclockwise and clockwise continuation of $I_\mu(h)$:

$$Var_a(I_\mu)(h) = I_\mu\big(he^{ia\pi}\big) - I_\mu\big(he^{-ia\pi}\big), \tag{5}$$

and denote by $Var_{a_1,\ldots,a_k}$ the composition $Var_{a_{1,\mu}} \circ \cdots \circ Var_{a_{k,\mu}}$.

The key of the proof [2,9] of the local boundedness of the number of zeros of a generic Darboux integrals on $H = P_1^{a_1} \cdot \ldots \cdot P_k^{a_k} P_k^{a_{k+1}}$ was a lemma stating that $Var_{a_1,\ldots,a_k,a_{k+1}} I(h) \equiv 0$. The main result was then deduced from this by induction observing (via a generalization of Petrov’s trick) that the operators Var_a reduce the number of isolated zeros of pseudo-abelian integrals by a constant locally bounded for any analytic family Θ. Here Proposition 5 provides a suitable form of Petrov's trick. The vanishing of the iterated variation permitted to start the induction using Gabrielov's theorem.

In our present situation we do not know how to associate a variation to the edge corresponding to the exponential factor in the first integral ($Q = 0$ in Theorem 1 or $S = 0$ in Theorem 2). We consider only iterated variation $Var_{a_1,\ldots,a_k}$ associated to all other edges. The operator $Var_{a_1,\ldots,a_k}$ does not annihilate completely the pseudo-abelian integral, but produces a univalued function in a transverse parameter, see Theorem 3. This transverse parameter is shown to be $-1/\omega$, where ω is a Pfaffian function generalizing the classical Ecalle–Roussarie compensator.

More precisely, we define a compensator $\omega(h, \varepsilon, \alpha)$ by the following relation

$$\widetilde{H}\left(-\frac{1}{\omega(h, \varepsilon, \alpha)}, \varepsilon, \alpha\right) = h, \tag{6}$$

where

$$\widetilde{H}(x, \varepsilon, \alpha) = \begin{cases} x^\alpha (\frac{x-\varepsilon}{x})^{1/\varepsilon}, & \text{for } \varepsilon \neq 0, \\ x^\alpha e^{-1/x}, & \text{for } \varepsilon = 0. \end{cases}$$

$\omega(h, \varepsilon, \alpha)$ is a Pfaffian function of h:

$$\frac{\alpha(-1 - \varepsilon\omega) + \omega}{\omega(1 + \varepsilon\omega)} d\omega = \frac{dh}{h}. \tag{7}$$

In Section 7 we prove existence of this function and investigate its analytic properties. Note that $\omega(h, \varepsilon, 0)$ is the usual Roussarie–Ecalle compensator, i.e. $\omega(h, \varepsilon, 0) = \frac{h^\varepsilon - 1}{\varepsilon}$, for $\varepsilon \neq 0$.

Theorem 3. *For a pseudo-abelian integral $I_\mu(h)$ corresponding to the family Θ there exist several pairs of real analytic functions $(\varepsilon_i(\mu), \alpha_i(\mu))$, $\varepsilon_i(0) = \alpha_i(0) = 0$, such that*

$$Var_{a_1,\ldots,a_k}(I_\mu)(h) = \sum_{i=1}^{N} f_i\left(-\frac{1}{\omega(h, \varepsilon_i(\mu), \alpha_i(\mu))}, \varepsilon_i(\mu), \alpha_i(\mu), \mu\right), \tag{8}$$

where $f_i(u, \varepsilon, \alpha, \mu)$ are meromorphic in u in some small disc and depend analytically on $\varepsilon, \alpha\mu$ varying in some small bidisc near the origin in $\mathbb{R}_{(\varepsilon,\alpha)} \times \mathbb{R}_\mu$.

Example 2. It will follow from the proofs that the number N of such pairs is at most the number of arcs of D lying on $\{Q = 0\}$. However, for the family (2) there is only one pair of parameters ε_i, α_i in (8) coinciding with the parameters ε, α of the family.

2.3. End of the proof: Application of Petrov trick

Fewnomials theory of Khovanskii enables us to start the proof by induction. It gives that the number of zeros of the right-hand side of (8) on any interval $0 \leqslant u \leqslant r$ is uniformly bounded for all μ sufficiently small. Theorem 2 (and therefore Theorem 1) follow next by Petrov's argument, which allows to estimate the number of real zeros of J in terms of the number of zeros of $Var_a J$, see Lemma 5. The key technical difficulty is to prove existence of a suitable asymptotic series for $Var_{a_1,\ldots,a_k}(I_\mu)(h)$, see Proposition 6, which allows to translate a priori estimates on the growth of the pseudo-abelian integral $I_\mu(h)$ to estimates on variation of its argument along small arcs.

3. Transport of cycles near the polycycle

In this section we construct a pair $v^\mu = (v_1^\mu, v_I^\mu)$ of two smooth real vector fields defined in some complex neighborhood U of the polycycle D, analytically depending on μ and satisfying

$$d(\log H_\mu)\big(v_1^\mu\big) = 1, \qquad d(\log H_\mu)\big(v_I^\mu\big) = I, \tag{9}$$

where, as before, $H_\mu = \exp(\int \theta_\mu)$. Using these vector fields we can lift smooth curves $\varrho(t) : [0,1] \to \{0 < |h| < h_0\}$ from a small punctured disc $\{0 < |h| < h_0\}$ to U, starting from any point of $H^{-1}(\varrho(0)) \cap U$, provided that the lifted curve does not leave U. We show that for h_0 small enough the lifting does not leave U if the starting point of the lifting lies on the real cycle of integration $\gamma_\mu(h)$, $h = \varrho(0) \in \mathbb{R}_+$. This allows to construct point-wise transport of $\gamma_\mu(h)$ along any such curve $\varrho(t)$ by transporting each point along its own lifting of the curve, and (9) implies that the result of the transport lies on a leaf of the foliation defined by H_μ.

3.0.1. Construction of transport from the vector fields $v^\mu = (v_1^\mu, v_I^\mu)$

Let us recall the construction of the lifting. Choose a point $a \in U$ lying on a leaf $\{H = h \neq 0\}$, and choose a univalued branch of H equal to h at a defined in some small neighborhood W of a. For a vector $\xi \in T_h\mathbb{C} \cong \mathbb{C}$ denote by $\tilde{\xi}_a$ the only real linear combination of $v_1^\mu(a)$ and $v_I^\mu(a)$ such that $dH(\tilde{\xi}_a) = \xi$:

$$\tilde{\xi}_a = \operatorname{Re}\big(h^{-1}\xi\big)v_1^\mu + \operatorname{Im}\big(h^{-1}\xi\big)v_I^\mu. \tag{10}$$

For a germ of a smooth curve $\varrho(t)$, $t \in (-r, r)$, passing through h and for each point $a' \in H^{-1}(\varrho(t)) \cap W$ we can repeat this construction taking vector $\varrho'(t)$ as ξ. This provides a smooth vector field on real three-dimensional surface $H^{-1}(\varrho((-r,r)))$, and the trajectory $\tilde{\varrho}_a(t)$ of this vector field passing through a is the required lifting. Evidently, $H(\tilde{\varrho}_a(t)) = \varrho(t)$. In other words, this construction provides a transport of points from one leaf of the foliation to another along smooth curves in the plane of values $h \in \mathbb{C}$.

It turns out that for h_0 sufficiently small any path on the universal covering of $\{0 < |h| < h_0\}$ can be lifted to U provided that the starting point a of the lifting lies on the real cycle $\gamma_\mu(h)$ and $|\varrho(t)|' > 0$. This allows to transport the real cycle $\gamma_\mu(h)$ to this universal cover: for any path $\varrho(t)$ in the universal cover we define the transport of $\gamma_\mu(h)$ along this path as a union of liftings of $\varrho(t)$ through all points of $\gamma_\mu(h)$. The result is well defined in a suitable sense: the continuation depends on the paths chosen, but continuations along homotopic paths are homotopic (by lifting of homotopy of the paths). This provides an analytic continuation of the pseudo-abelian integral (3) to a universal covering of a punctured disc $\{0 < |h| < h_0\}$.

Remark 2. The constructed vector fields commute everywhere except in small neighborhoods of the singular points of the polycycle. In fact, in a suitable local holomorphic coordinates we have $v_I^\mu = I v_1^\mu$ everywhere, and v^μ defines a holomorphic (in this new complex structure) vector field everywhere in U except these neighborhoods.

The rest of the section will be devoted to construction of v_μ. It will be constructed first in neighborhoods of singular points of the polycycle using the local normal forms for the first integral near the singular points. Then v^μ will be smoothly extended to neighborhoods of the arcs of the polycycle joining them.

We will repeatedly use the following fact, which is an easy consequence of the Cauchy–Riemann equations. Note that multiplication by i on $\mathbb{C}^2$ gives rise to the *real* linear endomorphism J on tangent vectors.

Lemma 1. *Let ξ be a real tangent vector to $\mathbb{C}^2$, H a holomorphic function and $\log H$ its local branch. If $d(\log H)(\xi) \in \mathbb{R}$ then $d(|H|)(\xi) = 0$.*

Also, to simplify notations we will omit the index μ in v^μ.

3.1. Construction of v in neighborhoods of saddles

Let m_k be a saddle of the polycycle D.

Lemma 2. *The foliation defined by H_μ near a saddle point can be analytically linearized, and the linearization depends analytically on parameters. Linearizing coordinates (x, y) can be chosen in such a way that $H = x^{1/\lambda_1} y^{1/\lambda_2}$, where λ_i are analytic functions of μ.*

This is proved in [2,9], and the proof consists of writing the linearizing coordinates explicitly: if the saddle lies on the intersection of $\{P_{1,\mu} = 0\}$ and $\{P_{2,\mu} = 0\}$ then $P_{1,\mu}$ and $P_{2,\mu}$, multiplied by suitable holomorphic factors invertible near the saddle, give the linearizing coordinates.

Example 3. For the form $\theta_{\varepsilon,\alpha}$ of (2) this can be expressed as

$$H_{\varepsilon,\alpha} = x^{a_1} y^{a_2} \quad \text{for } x = P_1\big(P_3^{a_3} \cdots P_k^{a_k} Q^{\alpha-1/\varepsilon}(Q + \varepsilon R)^{1/\varepsilon}\big)^{1/a_1},\ y = P_2.$$

In the linearizing coordinates the construction of v is easy. Choose some $0 < h < 1$.

Lemma 3. *For a family of linear saddles $\dot{x} = \lambda_1 x$, $\dot{y} = -\lambda_2 y$ in a bidisc $\{|x|, |y| \leqslant 1\}$ with the first integral $H = x^{1/\lambda_1} y^{1/\lambda_2}$ one can construct the pair of vector fields $v = (v_1, v_I)$ defined in $U_s = \{|H| < h < 1\} \cap \{|x|, |y| \leqslant 1\}$, satisfying* (9) *and having the following properties*:

(1) *both the negative flow of v_1 and flow of v_I do not increase $|x|$ and $|y|$*;
(2) *both v_1 and v_I are tangent to lines $\{y = \text{const}\}$ near $(0, 1)$ and to the lines $\{x = \text{const}\}$ near $(1, 0)$.*

Proof. The holomorphic vector field $v_x = \lambda_1 x \partial_x$ preserves y, in particular the transversal $\{y = 1\}$, and satisfies

$$d(\log H)(v_x) = 1, \qquad d(\log x)(v_x) = \lambda_1 > 0.$$

Similarly, the vector field $v_y = \lambda_2 y \partial_y$ preserves x and the transversal $\{x = 1\}$, and satisfies

$$d(\log H)(v_y) = 1, \qquad d(\log y)(v_y) = \lambda_2 > 0.$$

Let ϕ be a smooth function defined in U_s, $0 \leqslant \phi \leqslant 1$, equal to 0 in a neighborhood of $\{x = 1\}$ and equal to 1 in a neighborhood of $\{y = 1\}$. We define v as the pair of the real vector fields $(v_1 = \phi v_x + (1 - \phi) v_y,\ v_I = I v_1)$. One can easily see that v satisfies conditions of the lemma. □

Note that v_1 (and therefore also v_I) are not analytic vector fields, as ϕ is not analytic.

Proposition 1. *Transport of a real curve $\gamma \subset \{|x|, |y| \leqslant 1\} \cap \{H = h_0 \in \mathbb{R}, h_0 < h\}$ along any smooth curve $\varrho(t) : [0, 1] \to \{0 < |z| \leqslant |h_0|\}$ remains in U_s if $|\varrho|'(t) < 0$ for all t. Moreover, the transport intersects the transversals $\{x = 1\}$ for all t if γ intersects it* (*and similarly for $\{y = 1\}$*).

This follows from the fact that lifting of $\varrho(t)$ starting from any point $a \in U_s$ will remain in U_s. Indeed, $|\varrho|'(t) < 0$ is equivalent to $\operatorname{Re}(\varrho(t)^{-1} \varrho'(t)) < 0$, so the coefficient of v_1 in (10) is negative. This implies that $|x|$, $|y|$ do not increase along the lifting of $\varrho(t)$, due to the first claim of the previous lemma.

The second claim follows since v_1, v_I are tangent to both transversals.

3.2. Construction of v in neighborhoods of saddle-nodes

Let m_k be a saddle-node of the polycycle D.

Lemma 4. *There exist two real analytic functions $\varepsilon = \varepsilon(\mu)$ and $\alpha = \alpha(\mu)$ vanishing at $\mu = 0$ and real analytic coordinates (x, y) in some neighborhood of m_k such that the vector field*

$$\begin{aligned} \dot{x} &= -x^2 + \varepsilon^2, \\ \dot{y} &= y\big(1 + \alpha(x - \varepsilon)\big) \end{aligned} \tag{11}$$

generates the foliation $\theta_\mu = 0$ in this neighborhood. The function

$$y(x+\varepsilon)^\alpha \left(\frac{x-\varepsilon}{x+\varepsilon}\right)^{1/2\varepsilon} \tag{12}$$

is a first integral of this vector field.

Remark 3. Normalizing coordinates for the family (2) can be given explicitly: let $y = P_1 P_2^{a_2/a_1} \cdots P_k^{a_k/a_1}$. Then

$$H_{\varepsilon,\alpha}^{1/a_1} = y(Q/R)^{\alpha/a_1} \left(\frac{Q/R+\varepsilon}{Q/R}\right)^{1/a_1\varepsilon},$$

which becomes (12) if we take $X = a_1(-Q/R - \varepsilon/2)$ and rescale ε by $a_1/2$ and α by a_1.

Remark 4. It would seem more natural to use as a local model the full versal deformation of the saddle-node, i.e. the family (11) with ε^2 replaced by ε. However, the family of real polycycles we study extends continuously only to the half of the versal deformation where singular points resulting from splitting of the saddle-node remain real. This is the reason for choosing the model (11).

Investigation of another half of the versal deformation is a separate interesting problem.

Proof of Lemma 4. The fact that the first integral is preserved by the vector field is a direct computation. Existence of normalizing coordinates follows from the general theory of bifurcation of saddle-nodes. Indeed, from [6] it follows that (11) is the local *formal* normal form, and it is well known that for closed forms, due to vanishing of the moduli of analytic classification, the formal normal form and the analytic orbital normal form coincide. □

Until the end of this section we will work in the normalizing coordinates and will denote by $H = H_{\varepsilon,\alpha}(x, y)$ the first integral (12) of the model family (11),

$$\frac{dH}{H} = \frac{dy}{y} + \frac{d\widetilde{H}}{\widetilde{H}}, \quad \text{where } \frac{d\widetilde{H}}{\widetilde{H}} = \frac{1+\alpha(x-\varepsilon)}{x^2-\varepsilon^2}\,dx. \tag{13}$$

In other words,

$$\widetilde{H}(x) = (x+\varepsilon)^\alpha \left(\frac{x-\varepsilon}{x+\varepsilon}\right)^{\frac{1}{2\varepsilon}} \tag{14}$$

for $\varepsilon \neq 0$ and $\widetilde{H}(x) = x^\alpha e^{-1/x}$ for $\varepsilon = 0$.

We consider this model in the unitary bidisc $\{|x| \leqslant 1, |y| \leqslant 1\}$.

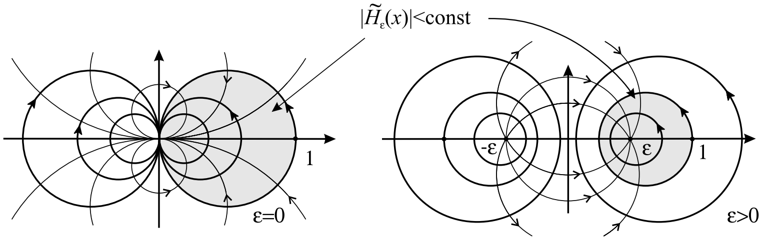

Fig. 2. Flow of the real and imaginary parts of the vector fields v_x.

Lemma 5. *For the model family above there exists a pair* $v = (v_1, v_I)$ *of vector fields* v *defined in* $\{|x|, |y| < 1\}$ (*except in a small neighborhood of* $(1, 1)$) *and satisfying* (9)*. Both the negative flow of* v_1 *and flow of* v_I *do not increase* $|x|$ *and* $|\widetilde{H}|$*. Both* v_1 *and* v_I *are tangent to lines* $\{y = \text{const}\}$ *near* $(0, 1)$ *and to the lines* $\{x = \text{const}\}$ *near* $(1, 0)$.

Proof. We consider only the case $\varepsilon \neq 0$, and the case $\varepsilon = 0$ is obtained by taking the limit.

Let

$$v_x = \frac{x^2 - \varepsilon^2}{1 + \alpha(x - \varepsilon)} \partial_x, \qquad v_y = y \partial_y \tag{15}$$

be two vector fields in the bidisc. We have

$$d(\log H)(v_x) = d(\log H)(v_y) = 1,$$
$$d(\log \widetilde{H})(v_x) = 1, \qquad d(\log \widetilde{H})(v_y) = 0,$$
$$d(\log y)(v_x) = 0, \qquad d(\log y)(v_y) = 1.$$

Let ϕ be a smooth function defined in the bidisc, $0 \leqslant \phi \leqslant 1$, equal to 0 in a neighborhood of $\{x = 1\}$ and equal to 1 in a neighborhood of $\{y = 1\}$. One can easily check that the pair of two real vector fields $(v_1 = \phi v_x + (1 - \phi) v_y, v_I = I v_1)$ satisfies conditions of the lemma. □

The following is a saddle-node analogue of Proposition 1.

Proposition 2. *Let* $\gamma(h_0, \varepsilon, \alpha)$ *be a relative cycle in the unitary bidisc lying on* $\{H_{\varepsilon,\alpha}(x, y) = h_0 \neq 0\}$ *taken modulo the two transversals* $\{y = 1\}$ *and* $\{x = 1\}$*. Assume in addition that the cycle lies entirely in the bidisc*

$$\{|\widetilde{H}(x)| \leqslant |\widetilde{H}(1)|\} \times \{|y| \leqslant 1\}. \tag{16}$$

Then the relative cycle $\gamma(h_0, \varepsilon, \alpha)$ *transports in relative cycles along any curve* $\varrho(t) : [0, 1] \to \{0 < |z| \leqslant |h_0|\}$, $\varrho(0) = h_0$, *remains in* (16) *provided* $|\varrho|'(t) < 0$ *for all* t.

Note that unlike the previous case of saddles, the lifting does not preserve the whole bidisc $\{|x|, |y| \leqslant 1\}$, but only the bidisc (16) (see Fig. 2). However, the parts of the *real* cycles $\gamma_{\varepsilon,\alpha}(h)$ passing near the saddle-node lie in (16), so satisfy the conditions of the lemma.

Proof of Proposition 2. Indeed, since v preserves the transversals $\{y = 1\}$ and $\{x = 1\}$, the endpoints of $\gamma(h, \varepsilon, \alpha)$ still lie on them. Similarly to the proof of Proposition 1, from $\text{Re}(\varrho(t)^{-1}\varrho'(t)) < 0$ we conclude that both $|\widetilde{H}|$ and $|y|$ only decrease along lifting of $\varrho(t)$, so the points of bidisc (16) remain in it when transported along $\varrho(t)$. □

3.3. Gluing a global transport

Here we extend the vector fields constructed above to a vector field defined in a whole neighborhood of the polycycle D.

Proposition 3. *There exist a complex neighborhood U of D_μ and a pair of real vector fields $v = (v_1, v_I)$ in U satisfying* (9)*. Moreover, transport of real cycles $\gamma_\mu(h)$ along any curve $\varrho(t) \subset \{0 < |z| \leqslant |h_0|\}$ remains in U provided $|\varrho|'(t) < 0$ for all t.*

Proof. For each singular points of the polycycle we defined two transversals intersecting the polycycle. They are given by $\{x = 1\}$ and $\{y = 1\}$ in the normalizing chart of the singular point.

For an arc of the polycycle joining two singular points m_1, m_2 consider two transversals Γ_1, Γ_2 to this arc, lying in normalizing charts W_1 and W_2 of m_1 and m_2 correspondingly, and let K be a compact piece of the arc joining Γ_1 and Γ_2. Let U_K be a neighborhood of K in the leaf of the foliation containing K.

To fix notations, assume that in the normalizing coordinates in W_1 the leaf containing K is contained in $\{x = 0\}$. The family $\mathcal{F}_1$ of discs given by $\{y = \text{const}\}$ is transversal to U_K and invariant under the flow of vector fields v_1, v_I constructed before (assuming U_K is sufficiently small). Similar transversal family $\mathcal{F}_2$ of invariant discs exists on the other end of K.

Our immediate goal is to embed these two families of discs into one smooth family $\mathcal{F}$ of smooth real two-dimensional discs transversal to U_K and filling some neighborhood of K in $\mathbb{C}^2$. Let g_1 be a Riemannian metric defined in W_1 which in normalizing coordinates is just a standard Euclidean metric in $\mathbb{R}^4$, so the leaf containing K and the discs of $\mathcal{F}_1$ lie in orthogonal affine planes. Let g_2 be a similar metric in W_2, and continue smoothly these two metrics to a metric g defined in some neighborhood of U_K in $\mathbb{C}^4$. We can assume that g preserves the complex structure of U_K. The exponential mapping $\exp_g$ maps diffeomorphically some neighborhood $\widetilde{W} \subset NU_K$ of U_K in its normal bundle NU_K onto some neighborhood W of U_K in $\mathbb{C}^2$, in such a way that the images of fibers N_xU_K are mapped into the leaves of $\mathcal{F}_i$ for $x \in U_K \cap W_i$, $i = 1, 2$. We define $\mathcal{F}$ as the family $\mathcal{F}(x) = \exp_g(B_x)$, $x \in U_K$, where $B_x = N_xU_K \cap \widetilde{W}$ are small discs (symmetry of g with respect to conjugation assures that for $x \in K$ the leaves $\mathcal{F}(x)$ intersect $\mathbb{R}^2$ by a smooth curve transversal to K).

Shrinking B_x, we can assume that $\mathcal{F}(x)$ is transversal to the leaves $\{H = h\}$ for all sufficiently small h (because $\mathcal{F}(x)$ is transversal to the leaf containing K). We define $v = (v_1, v_I)$ in the neighborhood W_K of U_K in $\mathbb{C}^2$ as the two vector fields tangent to $\mathcal{F}(x)$ and satisfying (9).

Evidently, $\mathcal{F}$ coincide with $\mathcal{F}_i$ in W_i. Since (9) define uniquely the pair of vector fields tangent to a real two-dimensional surface transversal to $\{H = h\}$, we conclude that thus constructed v is a smooth extension of the vector fields constructed before.

Dynamics of v on each leaf $\mathcal{F}(x)$ is conjugated to dynamics on the transversal $\{y = 1\}$ of v constructed in either Lemma 3 (when one of two singular points is a saddle) or Lemma 5 (for a connection between two saddle-nodes), with conjugation map being just the flow from one transversal to another. We use here the fact of smoothness of Q: it implies that the weak manifolds of saddle-nodes of the polycycle D join them to saddles, and two saddle-nodes can be connected by their strong manifolds only.

Let a be a real point lying on $\mathcal{F}(x) \cap \gamma_\mu(h)$ for h sufficiently small. Then the lifting of any curve $\varrho(t) \subset \{0 < |z| \leqslant |h|\}$, $\varrho(0) = h$, starting from a remains in $\{|H| \leqslant h\} \cap \mathcal{F}(x)$, which is contained in W_K provided that h is sufficiently small.

Repeating the construction for all arcs of the polycycle, we get the pair $v = (v_1, v_I)$ defined in the neighborhood U of the polycycle, where U is the union of normalizing charts W_i and the neighborhoods W_K of all singular points and arcs of the polycycle. □

4. Pushing cycles away from the weak manifold

Recall that $L_E^{\mathbb{R}}$ is the union of edges D contained in the zero level curve $Q = 0$. Any arc of D lying on $\{Q = 0\}$ joins two saddle-nodes, and is the strong manifold of both.

The aim of this section is to prove the following proposition:

Proposition 4. *There exist a neighborhood $U(L_E^{\mathbb{R}})$ of $L_E^{\mathbb{R}}$ in $\mathbb{C}^2$ and neighborhoods $U_k^{\mathbb{C}}$ of the central varieties of saddle-nodes $m_l \in D$ such that in the open set $E_L = U(L_E^{\mathbb{R}}) \setminus \bigcup \overline{U_l^{\mathbb{C}}}$*

(1) *the family θ_μ defines a holomorphic foliation without singularities analytically depending on μ, and*
(2) *the family of cycles*

$$Var_{a_1,\dots,a_k}\,\gamma_\mu(h) \tag{17}$$

is homotopic along the fibers to the family of cycles lying in E_L, where $\gamma_\mu(h)$ are real cycles as in (3).

Shrinking E_L if necessary, we can assume that connected components of E_L are in one-to-one correspondence of the arcs of D lying on $\{Q = 0\}$ and have homotopy type of the figure eight.

We first show that a cycle lying near L_E and in the saddle regions of the saddle-nodes can be pushed away from the central variety while remaining in a neighborhood of L_E. This will be needed to prove that the integral of a meromorphic form $M^{-1}\eta_\mu$ over such cycle depends holomorphically on μ and the transversal coordinate. The transversal coordinate is exactly $\frac{1}{\omega(h,\varepsilon(\mu),\alpha(\mu))}$ for suitable functions $\varepsilon(\mu), \alpha(\mu)$.

Lemma 6. *Using assumptions and notations of Proposition* 2 *let $\gamma = \gamma(h, \varepsilon, \alpha) \subset \{H_{\varepsilon,\alpha} = h\}$ be a relative cycle lying in the bidisc* (16) *and whose boundary is in $\{H_{\varepsilon,\alpha} = h\} \cap \{y = 1\}$. Then γ is homotopy equivalent in $\{H_{\varepsilon,\alpha} = h\}$ to a relative cycle $\tilde{\gamma}$ with the same property and, in addition, not intersecting a neighborhood $\{|y| < \delta\}$ of the x-axis, for a sufficiently small $\delta > 0$ independent of the cycle.*

Proof. Choose a non-negative bump function $\psi(y)$ equal identically to 1 on $\{|y| < \delta\}$, and vanishing outside $\{|y| < 2\delta < 1\}$. Define the vector field $V = \psi(v_y - v_x)$, where v_x and v_y were defined in (15). Evidently, $dH(V) = 0$, so the flow of V preserves the foliation. We can assume that $c = \operatorname{dist}(\gamma, \{y = 0\}) < \delta$. Consider the image $\tau_M\gamma$ of the cycle γ by the M-time flow, where $M = \log\frac{\delta}{c} > 0$. Since $L_V y = y$ in $\{|y| \leqslant \delta\}$, the image $\tau_M\gamma$ lies outside $\{|y| \leqslant ce^M = \delta\}$. Since $L_V(\log\widetilde{H}) = -\psi < 0$, the $|\widetilde{H}|$ is decreased by this flow, so the condition (16) is still satisfied. □

4.1. Flow-box triviality

Consider a neighborhood $U(L_E^{\mathbb{R}})$ of $L_E^{\mathbb{R}}$ in $\mathbb{C}^2$ which is a union of the normalizing charts of the saddle-nodes and of the open set U_K constructed in the proof of Proposition 3 for $L_E^{\mathbb{R}}$. Let $U_k^{\mathbb{C}}$ be neighborhoods of the central variety of each saddle-node m_k as in Lemma 6.

Lemma 7. *The foliation defined by H_μ in the open set E_L is analytic without singularities and depends analytically on sufficiently small parameter μ.*

Proof. Indeed, by construction E_L is covered by several charts, namely neighborhoods of bifurcating saddle-nodes and neighborhoods of compact subsets of separatrices on some positive distance from the saddle-nodes. In each of these sets the foliation defined by H_ε can be brought analytically to a suitable normal form, either to normal form of Lemma 4 or just to the standard flow box. Evidently, E_L lies on a finite distance from singularities. □

Proof of Proposition 4. The cycle γ can be continuously moved to close leafs of the foliation by Proposition 3. It was proved in [2] that the pieces of γ lying near saddles or near separatrices lying on $P_i = 0$ are annihilated by the operator $Var_{a_1,\dots,a_k}$. Therefore the cycle $Var_{a_1,\dots,a_n}\gamma$ is supported in $U(L_E^{\mathbb{R}})$. Moreover, it still lies in (16) in normal coordinates, so by Lemma 6 it is homotopically equivalent along the fibers to a cycle $\gamma'(h)$ lying in E_L. □

5. Proof of Theorem 3

Let z be a holomorphic coordinate on a transversal Γ to $\{Q=0\}$.

Lemma 8. *For the family θ_μ the coordinate z is a holomorphic function of $-\frac{1}{\omega(H_\Gamma,\varepsilon(\mu),\alpha(\mu))}$, where $\varepsilon(\mu)$, $\alpha(\mu)$ are some analytic functions of μ which are the same for any two transversals to the same arc of D.*

Remark 5. Functions $\varepsilon(\mu),\alpha(\mu)$ from Lemmas 4 and 8 coincide.

Proof of Lemma 8. Every transversal can be holomorphically mapped to a transversal lying in a normalizing chart of some saddle-node of the polycycle D, just by the flow of the vector field tangent to the foliation. Therefore, the claim follows from Lemma 4: when restricted to $\{y=1\}$, the first integral (12) becomes (6), up to a linear change of ε,α. □

Remark 6. The parameters $\varepsilon(\mu)$, $\alpha(\mu)$ are intrinsically defined: $1/\varepsilon(\mu)$ is the residue, and $\alpha(\mu)$ is the sum of residues of the restriction to Γ of the form θ_μ. For the family (2) the smooth irreducible double divisor $\{Q=0\}$ is split into two close irreducible smooth curves $\{Q=0\}$ and $\{Q+\varepsilon R=0\}$, with residues $1/\varepsilon$ and $\alpha-1/\varepsilon$ being the same for all transversals. In general, the residues are locally constant along $\{Q=0\}$ (e.g. by closedness of θ_μ), but can be different for different connected components.

Lemma 9. *For sufficiently small ε the mapping $h\mapsto -\frac{1}{\omega(h,\varepsilon,\alpha)}$ is one-to-one on the interval* $[0,1]$.

Let B_μ be some small polydisc, and consider a foliation $\mathcal{F}$ of $E_L\times B_\mu$ by one-dimensional leaves $\{H_\mu=h,\ \mu=\text{const}\}$. According to Lemma 7 this is an analytic foliation without singularities.

Lemma 10. *Let γ be a closed connected curve on a leaf of $\mathcal{F}$ and assume that it can be continuously transported to nearby leafs. Denote the resulting family by $\gamma_\mu(z)$, where z is the coordinate of a point of the intersection of the cycle and some fixed transversal to $\{Q=0\}$. Let η_μ be a meromorphic one-form in $E_L\times B_\mu$ such that $Q_{1,\mu}Q_{2,\mu}\eta_\mu$ is holomorphic. Then there exist two analytic functions $\varepsilon(\mu),\alpha(\mu)$ such that the integral $I_\mu(z)=\int_{\gamma_\mu(z)}\eta_\mu$ is a meromorphic function of z and depends analytically on μ.*

Proof. A connected component of the open set $E_L\times B_{\varepsilon,\alpha}$ containing $\gamma_\mu(z)$ is covered by two normalizing charts of neighborhoods of saddle-nodes (with a neighborhood of weak manifold removed) and a neighborhood of the connection between saddle-nodes. In each of the above charts leafs of our foliation are graphs of (multivalued) functions $x(y,h)$ of the coordinate y along the leaf $\{Q=0\}$. Therefore in each chart the curve γ can be written as a curve $(x(t),y(t),\mu)$, and we can define its projection curve $(0,y(t),0)$ lying on $\{Q=\mu=0\}$. It is important here that by Proposition 4 we can keep the cycle away from the weak manifold where the projection is not regular.

We can join γ and its projection by a continuous family of closed curves lying on leaves of foliation using the explicit normalizing charts. We can do it in each normalizing chart, and the condition of trivial holonomy of γ guarantees that these pieces will glue together. This implies that the holonomy of the projection curve is trivial, so γ can be continued from L to all nearby leaves. Therefore $I(z)$ is univalued in a neighborhood of $z=\mu=0$. Since the length of the continuation is bounded, the growth of $I(h)$ is at most polynomial. □

Lemma 11. *Define the functions $g_\beta(z,\varepsilon,\alpha)$ by*

$$g_\beta\left(-\frac{1}{\omega(he^{i\beta},\varepsilon,\alpha)},\varepsilon,\alpha\right)=-\frac{1}{\omega(h,\varepsilon,\alpha)}.$$

Then for any $\beta_0 > 0$ and any neighborhood $W \subset \mathbb{C}$ of the origin there exists a small tridisc $W' \subset \mathbb{C}^3_{z,\varepsilon,\alpha}$ near the origin such that the function $g_{\beta'}(z,\varepsilon,\alpha)$ maps $W' \times \{|\beta| < \beta_0\}$ holomorphically into W.

Proof. The function $g_\beta(z,\varepsilon,\alpha)$ is the $i\beta$-time flow of the vector field $\tilde{v}' = \frac{z^2+\varepsilon z}{1+\alpha z}\partial_z$, which is just the vector field v_x of (15) up to an affine change of variables. Therefore the claim follows from the fact that $x = 0$ is a fixed point of $\tilde{v}'$ for $\varepsilon = \alpha = 0$ and analytic dependence of the solution of ODE on the initial conditions and parameters. □

Proof of Theorem 3. By Proposition 4 and the definition of the operator Var_{a_i} the cycle $\gamma' = Var_{a_1,\dots,a_n}\gamma(h,\mu)$ is a union of several disjoint cycles γ_i' lying in E_L on leaves $\{H = he^{i\beta_i}\}$, for finitely many $\beta_i \in \mathbb{R}$. Since γ' can be continued by h, the cycles γ_i' also can be continued by h. Therefore by Lemmas 10 and 8 the function $Var_{a_1,\dots,a_n} I_\mu$ is a finite sum of $f_i(-\frac{1}{\omega(he^{i\beta_i},\varepsilon_i(\mu),\alpha_i(\mu))},\mu)$, where each f_i is holomorphic in some bidisc at the origin. Then Lemma 11 implies it is an analytic function of $-\frac{1}{\omega(h,\varepsilon,\alpha)}$. □

6. Proof of Theorem 2

Proposition 5. *Application of the operator Var_a decreases the number of zeros of $I_\mu(h)$ by at most some finite number uniformly bounded from above and depending on the family Θ only.*

Proof. To prove the proposition, consider the sector $\{r < |h| < 1,\ |\arg h| \leqslant \alpha\pi\}$. Proposition 6 guarantees that the zeros of $I(h)$ do not accumulate to 0, so for r small enough this sector includes all zeros of $I(h)$ on $(0,1)$. To count the number of zeros of $I(h)$ in this sector apply the argument principle. As in [2,9], the increment of argument of $I(h)$ on the counterclockwise arc $\{|h| = 1,\ |\arg h| \leqslant \alpha\pi\}$ passed counterclockwise is uniformly bounded from above by Gabrielov's theorem [4]. Here we need the analytic dependence of the compensator function $\omega(u,\varepsilon,\alpha)$ on the parameters ε, α, when $|u| = \text{const}$. This is proved in Proposition 7.

Proposition 6 below implies that the increment of argument along the small arc $\{|h| = r,\ |\arg h| \leqslant \alpha\pi\}$ passed clockwise is uniformly bounded from above as well. The classical Petrov's argument now shows that the increment of argument of $I(h)$ along the segments $\{r < |h| < 1,\ |\arg h| = \pm\alpha\pi\}$ is bounded from above by the number of zeros of $Var_\alpha I(h)$, which proves the proposition. □

End of the proof of Theorem 2. Theorem 2 follows from Proposition 5, Theorem 3 and the fact that the number of zeros of

$$f = \sum_i f_i\left(-\frac{1}{\omega(h,\varepsilon_i(\mu),\alpha_i(\mu))},\varepsilon_i(\mu),\alpha_i(\mu),\mu\right),$$

i.e. of the right-hand side of (8), on some interval $(0,r)$ is uniformly bounded for all sufficiently small μ. The latter claim is a direct application of fewnomials theory of Khovanskii [8]: since all $-\frac{1}{\omega(h,\varepsilon,\alpha)}$ are Pfaffian functions, see (7), the upper bound for this number of zeros can be given, using Rolle–Khovanskii arguments of [7], in terms of the number of zeros of some polynomials in F_i and their derivatives. The latter are uniformly bounded by Gabrielov's theorem [4]. □

The aim of the following Proposition 6 is to describe the asymptotics of the pseudo-abelian integral $I(h)$ and its variation at $h = 0$. This justifies the application of Petrov's argument in the proof of Theorem 1.

The regular form of the singularity together with a priori bound for the growth of the integral $I(h)$ gives us an estimate for the increment of the argument along arcs of small circles around $h = 0$. Note that the singularity at $\varepsilon \neq 0$ case was already investigated [2,9]. Thus, it remains to investigate the non-trivial exponential case $\varepsilon = 0$.

Proposition 6. *Let $I(h)$ be a non-zero multivalued holomorphic function on a neighborhood of $h = 0$ verifying the iterated variation relation (8) for some k and satisfying the a priori bound*

$$|I| \leqslant C|h|^{-N} \tag{18}$$

in sectors $\{|\arg h| \leqslant A\}$.

Then $I(h)$ has a leading term of the form $h^\alpha(\log h)^k$ or of the form $(\log h)^{-k}(\log(\log h))^l$, with $k, l > 0$. Moreover, for any $N' > N$ the increment of argument of $I(h)$ along the arc $C_0 = \{re^{i\phi} \mid \phi \in [-A, A]\}$ traveled clockwise can be estimated from above

$$\Delta \operatorname{Arg}_{C_0} I \leqslant 2N'A, \tag{19}$$

for all sufficiently small $r > 0$.

7. Generalized Roussarie–Ecalle compensator

In this section we prove the existence of the generalized Roussarie–Ecalle compensator (6). We start with the following, general statement.

Lemma 12. *Let $r(x)$ be a rational function. There exists a holomorphic, multivalued, endlessly continuable function $\omega(z)$ which satisfies the following equation*

$$\omega'(z) = r\big(\omega(z)\big). \tag{20}$$

The ramification set of the function $\omega(z)$ is discrete along any path.

Proof. Consider the Riemann sphere $\overline{\mathbb{C}}$ with small disjoint, open discs $D_1, \ldots, D_k$ centered at zeroes and poles of $r(x)$. Let the initial condition $x_0 \in \overline{\mathbb{C}}$ be chosen away from these discs. Let $z = l(s)$, $s \in [0, 1]$, $l(0) = 0$, be a path in $\mathbb{C}$ starting at $z = 0$. Since the domain $\overline{\mathbb{C}} \setminus (\bigsqcup_j D_j)$ is compact, the solution of the equation $\omega' = r(\omega)$ is well defined along l at least until it enters to some disc D_j, i.e. for $s \in [0, s_j]$. The solution can be extended to a holomorphic function in a neighborhood of this segment of l.

In a disc D_j there exists a holomorphic coordinate ξ such that the equation takes the following (normal) form

$$\xi' = \begin{cases} a\xi^n, & a \in \mathbb{C}^* \text{ for } n \leqslant -1, \\ a\xi, & a \in \mathbb{C}^*, \\ (r\xi^{-1} + a\xi^{-n})^{-1}, & a \in \mathbb{C}^*,\ r \in \mathbb{C} \text{ for } n \geqslant 2. \end{cases}$$

The solution reads respectively

$$t - t_0 = \begin{cases} a^{-1}(1-n)^{-1}\xi^{1-n}, \\ a^{-1}\log\xi, \\ r\log\xi + \frac{a}{1-n}\xi^{1-n}. \end{cases}$$

Now, if $n \geqslant 1$, the solution ω cannot reach the singular point $\xi = 0$, so it either leaves the disc D_j or stays inside (and is well defined) for $s \in [s_j, 1]$. If $n \leqslant -1$, then the singular point $\xi = 0$ corresponds to the ramification of the solution ω. $\square$

Now we return to the particular problem related to the existence of the compensator. One checks that the compensator function ω in the logarithmic coordinate $u = \log h$ must satisfy the following differential equation

$$\omega'(u) = \frac{\omega(\omega - \varepsilon)}{1 + \alpha(\omega - \varepsilon)}. \tag{21}$$

Thus, by Lemma 12, *for fixed* ε the solution is a well defined multivalued holomorphic function. The dependence on ε is not automatically analytic since in Eq. (21) the collision of two zeroes (at $\omega = 0$) and the collision of zero and pole (at $\omega = \infty$) occur for $\varepsilon = 0$. We overcome these difficulties by taking respective blow-ups. More precisely, the following proposition holds. Recall that the logarithmic chart $u = \log h$ assumed.

Proposition 7. *There exist a positive constant l_0 and three functions $F_S(\varepsilon, s)$, $F_E(\varepsilon, u)$, $F_N(\varepsilon, w)$ analytic in ε, analytic multivalued in s, u, w respectively such that in a neighborhood of any u_0 the compensator $\omega(\varepsilon; u)$ has one of the following forms (depending on the value $\omega(\varepsilon; u_0)$)*

$$\omega\big(e^u, \varepsilon, \alpha\big) = \begin{cases} \varepsilon F_S(\varepsilon, \varepsilon(u - u_0)), \\ F_E(\varepsilon, u - u_0), \\ \alpha^{-1}\, 1/F_N(\varepsilon, \alpha^{-1}(u - u_0)). \end{cases} \tag{22}$$

Moreover, these expressions are valid for all paths starting at u_0, of length bounded by l_0.

Remark 7. The indices S, E, N of functions come from the south pole, equator and north pole on the Riemann sphere.

Proof of Proposition 7. In the whole proof the logarithmic chart is assumed $u = \log h$. We will use the notation $\omega(u, \varepsilon)$. One easily observes that Eq. (21) has the following singularities: zeros of order 1 at $\omega = 0$, $\omega = \varepsilon$ and $\omega = \infty$ and pole of order 1 at $\omega = -\alpha^{-1} + \varepsilon$. For $\varepsilon = 0$ they degenerate to a single pole of order 2 at $\omega = 0$. Let two discs centered at 0 and ∞ respectively, both of radius $r_0/2$ contain all these singularities for $|\varepsilon| < \varepsilon_0$. Thus, on the ring $R = R(r_0, r_0^{-1})$ the rational function $\frac{\omega(\omega-\varepsilon)}{1+\alpha(\omega-\varepsilon)}$ is bounded by a constant M. Let $\omega(u_0, \varepsilon) = \omega_0 \in R$ and $\operatorname{dist}(\omega_0, \partial R) = \delta$. Analytic continuation of ω along any path l starting at t_1, of length $\leqslant \delta/M$ is so contained in R and satisfies estimate $|\omega - \omega_0| \leqslant M\,|l|$. Moreover, this solution depends analytically on ε. Defining the "base" solution F_E on the ring R by the initial condition $F_E(\varepsilon, 0) = 1$ we get

$$\omega(u, \varepsilon) = F_E(\varepsilon, u - u_0),$$

where $u_0 = u_1 - \int_1^{\omega_0} \frac{1+\alpha(\omega-\varepsilon)}{\omega(\omega-\varepsilon)}\, d\omega$.

Now, we consider the lower semi-sphere $|\omega| < 1$ in the Riemann sphere $\overline{\mathbb{C}}$. We make the following blow-up transformation

$$\omega = \varepsilon y, \qquad s = \varepsilon u.$$

Eq. (21) takes the form

$$y' = \frac{y(y-1)}{1 + \varepsilon\alpha(y-1)}.$$

The solution $y = F_S(\varepsilon, s)$, fixed by the initial condition $F_S(\varepsilon, 0) = \frac{1}{2}\varepsilon^{-1}$, is ε-analytic as far as it remains in a safe distance from "upper" singularities, e.g. if $|y| < 2/\varepsilon$. Thus, the compensator reads

$\omega(u,\varepsilon)=\varepsilon F_S(\varepsilon,\varepsilon(u-u_0))$ and this formula is valid along any path of length bounded by $1/M$, provided $|\omega_0|<1$.

Finally, on the upper semi-sphere $|\omega|>1$, the blow-up map $x=\alpha^{-1}/z$, $s=\alpha^{-1}u$ transforms Eq. (21) to

$$z'=-\frac{z(1-\alpha\varepsilon z)}{1+z-\alpha\varepsilon z}.$$

We fix the solution $F_N(\varepsilon,s)$ which is ε-analytic in the region $|\omega_0|>1/2$. Thus, the following formula for compensator remains valid along any path of length bounded by $1/M$, provided $|\omega_0|>1$. □

8. Proof of Proposition 6

Note that it is enough to prove the statement point-wise with respect to all parameters, in particular ε. As the case $\varepsilon\neq 0$ was already investigated [2], it remains to prove the claim in the non-trivial exponential case $\varepsilon=0$.

The general strategy of the proof is the following. We construct explicitly a particular solution of the variation equation (8). Since solutions of the corresponding homogeneous variation equation (i.e. $Var_{a_1,\dots,a_k} I\equiv 0$) were already considered in [2], this gives us a description of the general solution. To construct a particular solution of (8) we first solve it explicitly up to a sufficiently small remainder on the right-hand side (Lemma 13). Next the solution to the new equation is found in terms of convergent series (Lemma 14).

Remark 8. This strategy is in the spirit of the two steps construction of a solution of the homological equation associated to the normal form problem for diffeomorphisms and vector fields [6,12].

In this section we will work in the logarithmic chart $u=\log h$. In this coordinate the variation operator Var_a (5) becomes a difference operator

$$\Delta_a f=f(u+ia\pi)-f(u-ia\pi). \tag{23}$$

We introduce also the notation for the iterated differences

$$\Delta_{a_1,\dots,a_k}:=\Delta_{a_1}\cdots\Delta_{a_k}.$$

The multivalued functions defined in a punctured neighborhood of $h=0$ become functions holomorphic in the half-planes $\mathcal{H}_{L-}=\{\operatorname{Re}u<-L\ll 0\}$. All functions below are assumed to be of this type.

Let $\mathcal{P}(u)$ be the space $\mathbb{C}[u,\frac{1}{u},\log u]$ of polynomials in $\log u$ and Laurent polynomials in u.

Lemma 13. *Assume that $f(\frac{1}{u},\frac{\log u}{u})$ is a holomorphic function of the second variable $\frac{\log u}{u}$ and meromorphic function of $\frac{1}{u}$.*

(1) *For any real $A\in\mathbb{R}$ there exists a polynomial $p\in\mathcal{P}(u)$ such that*

$$\bigl|(f-p)\bigr|\leqslant M|u|^{-A} \tag{24}$$

for some constant M.

(2) *The space $\mathcal{P}$ is closed under the integration operation, i.e. for any $p\in\mathcal{P}$ there exists $P\in\mathcal{P}(u)$ such that $P'=p$.*

(3) *For any real $A \in \mathbb{R}$ there exists a function $P_f \in \mathcal{P}$ such that*

$$\left|(f - \Delta_{a_1,\dots,a_k} P_f)\right| \leqslant M|u|^{-A} \tag{25}$$

for some constant M.

Proof. (1) The function f has the following power series expansion

$$f = \sum_{m \geqslant 0,\, l \geqslant -l_0} a_{ml} \frac{\log^m u}{u^{m+l}}.$$

We define p to be the sum of all terms with $m + l \leqslant A + 1$; this sum is finite, so $p \in \mathcal{P}(u)$.

(2) We use the induction by $(\log u)$-degree of p. If p is a Laurent polynomial in u, the integral $\int p$ is a sum of a Laurent polynomial in u and a term $a \log u$, $a \in \mathbb{C}$. Consider relations

$$\left(u^l \log^m u\right)' = l u^{l-1} \log^m u + m u^{l-1} \log^{m-1} u, \qquad \left(\log^m u\right)' = m u^{-1} \log^{m-1}. \tag{26}$$

Let $p \in \mathcal{P}(u)$ be an element of $(\log u)$-degree $\leqslant m$. The integral $\int p$ is a sum of terms of $(\log u)$-degree $\leqslant m$ and $a \log^{m+1} u$, $a \in \mathbb{C}$.

(3) Points (1), (2) and simple induction reduce problem to the following observation. For any $p \in \mathcal{P}(u)$ the leading term of the solution to the difference equation $\Delta_a F = p$ is given by the integral $P = \int p$, i.e.

$$|p| \leqslant M|u|^{-A} \quad \Rightarrow \quad \left|p - \Delta_a \frac{1}{2\pi i a} P\right| \leqslant \widetilde{M}|u|^{-(A+1)}.$$

We estimate

$$\left|p - \Delta_a \frac{1}{2\pi i a} P\right| = \left|p(u) - \frac{1}{2\pi i a} \int_{u-\pi i a}^{u+\pi i a} p(s)\right| = \left|\frac{1}{2\pi i a} \int_{u-\pi i a}^{u+\pi i a} \big(p(s) - p(u)\big)\right|$$

$$= \left|\frac{1}{2\pi i a} \int_{u-\pi i a}^{u+\pi i a} p'(u + \xi_s)\right| \leqslant M\left|u^{-(A+1)}\right|.$$

The last inequality follows from the estimate $|p'| \leqslant M'|u|^{-(A+1)}$ valid for arbitrary $p \in \mathcal{P}(u)$ satisfying $|p| \leqslant M|u|^{-A}$. □

Let Q_+ (respectively Q_-) be an upper-left (respectively lower-left) quarter-plane defined as follows $Q_+ = \{u \in \mathbb{C}\colon\ \operatorname{Re} u < -L,\ \operatorname{Im} u > -K\}$ and $Q_- = \{u \in \mathbb{C}\colon\ \operatorname{Re} u < -L,\ \operatorname{Im} u < K\}$ for some positive constants K, L. We construct here a solution of the variation equation in Q_+. This is sufficient for our purposes, since for application of the Petrov's argument we need only estimates in a half-strip $\{\operatorname{Re} u < -L,\ |\operatorname{Im} u| < K\}$ with some finite $L, K > 0$.

Lemma 14. *Let f be a holomorphic function on $Q_\pm$ which satisfies the estimate $|f(u)| \leqslant M|u|^{-A}$ on $Q_\pm$ for some constant M. Assume that $A > n$. Then the following series*

$$F_\pm = (\mp 1)^k \sum_{m_1,\dots,m_k > 0} f\big(u \pm 2\pi i(a_1 m_1 + \dots + a_k m_k) \mp \pi i(a_1 + \dots + a_k)\big) \tag{27}$$

converges and solves the difference equation on $Q_\pm$

$$\Delta_{a_1,\dots,a_k} F_\pm = f, \quad a_j > 0. \tag{28}$$

Moreover, the solution $F_\pm$ *is of order* $A - n - \varepsilon$*, i.e. for all* $B < A - n$ *the solution* $F_\pm$ *satisfies the estimate*

$$|F_\pm| \leqslant M_{A-n-B} |u|^{-B}. \tag{29}$$

Proof. By induction, it is enough to prove the following statement. Let $|f| \leqslant M|u|^{-A}$, $A > 1$ on $Q_\pm$. Then the formula

$$F_\pm = \mp \sum_{m=1}^{\infty} f\big(u \pm (2\pi iam - \pi ia)\big) \tag{30}$$

solves the difference equation $\Delta_a F_\pm = f$ and $F_\pm$ satisfies the estimate

$$|F_\pm| \leqslant M_{A-1-B} |u|^{-B} \tag{31}$$

for $B < A - 1$.

The series (30) is convergent, so the function $F_\pm$ is well defined. A direct computation shows that it satisfies the difference equation. We estimate

$$\begin{aligned} |F_\pm| &\leqslant \sum_m \big| f\big(u \pm (2\pi iam - \pi ia)\big)\big| \\ &\leqslant M|u|^{-B} \sum_m \left| \frac{u}{u \pm (2\pi iam - \pi ia)} \right|^B \big|u \pm (2\pi iam - \pi ia)\big|^{B-A}. \end{aligned}$$

The function $|\frac{u}{u \pm (2\pi iam - \pi ia)}| \leqslant M_\pm$ is bounded on $Q_\pm$ (not true on the whole half-plane $\mathcal{H}_-$!) and the series $\sum_m |u \pm (2\pi iam - \pi ia)|^{B-A}$ converges since $B - A < -1$. This shows the estimate (31). □

Remark 9. Note that formula (30) for $F_\pm$ defines a holomorphic function on the whole half-plane $\mathcal{H}_-$. The difference

$$F_- - F_+ = \sum_{m \in \mathbb{Z}} f(u + \pi ia + 2\pi iam)$$

defines a $2\pi ia$-periodic function on $\mathcal{H}_-$. However, the estimate (31) does not extend to $\mathcal{H}_-$. Passing to the variable $\tilde{h} = e^u/a$ the difference $(F_- - F_+)(\tilde{h})$ defines a germ of a meromorphic function at the origin. This situation is in the spirit of the functional cochain [5].

Corollary 1. *Using Lemmas* 13 *and* 14 *we can solve explicitly the difference equation* $\Delta_a F = f$*, where* $f(\frac{1}{u}, \frac{\log u}{u})$ *is as in Lemma* 13*. Indeed, the general solution consists of* 3 *terms: principal part, given by* $P \in \mathcal{P}(u)$*, remainder given by series* (30) *and an arbitrary solution to the homogeneous equation* $\Delta_a F_H \equiv 0$*. The latter one is given by a series* $\sum_l a_l e^{lu/a}$*.*

In the next lemma we investigate the analytic properties of the generalized compensator $\omega(h, \varepsilon, \alpha)$ (see (6)) for $\varepsilon = 0$. Recall that $\omega(h, \varepsilon, \alpha = 0)$ is the Roussarie compensator. Below we study the case with $\varepsilon = 0$ and arbitrary α in the logarithmic coordinate $u = \log h$. We denote

$$w = -\frac{1}{\omega(e^u, 0, \alpha)}, \tag{32}$$

so $w^\alpha e^{-1/w} = e^u$.

Lemma 15. *For* $\operatorname{Re} u \ll 0$ *we have*

$$w = \frac{1}{u}\left(a + g_\alpha\left(\frac{1}{u}, \frac{\log u}{u}\right)\right), \tag{33}$$

where $\mathbb{C} \ni a \neq 0$, $g_\alpha(\cdot,\cdot)$ *is an analytic function and* $g_\alpha(0,0) = 0$.

Proof. Indeed, writing $w = -\frac{w_1}{u}$, we get

$$z_1\alpha \log w_1 - \alpha z_2 + \frac{1}{w_1} = 1, \quad z_1 = \frac{1}{u},\ z_2 = \frac{\log(-u)}{u}.$$

The left-hand side of this equation is an analytic function $F = F(w_1, z_1, z_2)$ in a neighborhood of $(1,0,0)$, and $F(1,0,0) = 1$. Since $\frac{\partial F}{\partial w_1}|_{(1,0,0)} = 1$, by implicit function theorem we get

$$w = -\frac{1}{u}\left(1 + g\left(\frac{1}{u}, \frac{\log(-u)}{u}\right)\right). \qquad \square$$

Proof of Proposition 6. Note that the main difficulty in the proof is to control the form of the singularity of the function I at $h = 0$. Indeed, consider, as a toy example, the special case when I is a meromorphic function of h. Then, the moderate growth bound (18) restricts the order of pole at $h = 0$ to N and so the increment of argument satisfies (19). To prove a proposition in the general case it is enough to show that the form of singularity which is allowed by the variation relation (8) together with the moderate growth estimation forces an explicit bound for the increment of argument in terms of N only. Due to this idea, it is enough to work point-wise with respect to all parameters (i.e. $\varepsilon, \alpha, \ldots$). The case $\varepsilon \neq 0$ was already investigated in [2]. The conclusion was that the leading term of the integral $I(h)$ at $h = 0$ is a monomial $h^A \log^k h$, with positive integer k. Thus, the same estimate as in the meromorphic case holds.

First we give a proof in a special case $\alpha = 0$ (compare (2)). It contains the essence of the general case with much less technical details.

The $\boldsymbol{\alpha = 0}$ case. The function w given by formula (32) reads $w = -\frac{1}{\log h}$. We use the logarithmic chart $u = \log h$. By Lemma 13, there exists a polynomial $P \in \mathbb{C}[\log u, u, \frac{1}{u}]$ (leading term) such that

$$|F - \Delta_{a_1,\ldots,a_k} P| \leqslant M|u|^{-A},$$

for some $A > n$ and a positive constant M. Thus, the iterated variation (difference) of $I - P$ is of sufficiently high order and a solution F_+ defined in Q_+ is given by the iterated sum formula (27). Moreover, it is of lower order than P.

Now, the iterated difference vanishes identically

$$\Delta_{a_1,\ldots,a_k}(I - P - F_+) \equiv 0.$$

Thus, by Lemma 4.8 from [2], the principal term of $I - P - F_+$ has the form $h^\alpha \log^m h$. Finally, the principal term of I is either a monomial $h^\alpha \log^m h$, $\alpha \geqslant -N$, $m \in \mathbb{Z}$, $m \geqslant 0$, or $\log^l h \log^m(\log h)$, $m, l \in \mathbb{Z}$, $m \geqslant 0$. In both cases the upper bound (19) holds.

The general case ($\boldsymbol{\alpha \neq 0}$). By Lemma 15 we know that the function w has the following form

$$w = \frac{1}{u}\left(a + g\left(\frac{1}{u}, \frac{\log u}{u}\right)\right), \quad a \neq 0,$$

and g is a holomorphic function, $g(0,0)=0$. For arbitrary meromorphic function $F(\cdot)$, the composition $F(w)$ has the following expansion

$$F(w)=\sum_{k\geqslant -k_0}\left(\frac{1}{u}\right)^k q_k(\log u),$$

where q_k is a polynomial. Now we can repeat the argument used in the special case $\alpha=0$. We take the principal part P_F of $F(w)$ up to order $A>n$. It is a polynomial in $\log u$ and Laurent polynomial in u. We can solve the iterated difference equation explicitly, up to terms of higher order (Lemma 13). Then, by Lemma 14, a solution F_+ to the iterated difference equation for $(I-P_F)$ is given by the iterated sum formula (27). Finally, we obtain that the leading term of I is a monomial $h^\alpha\log^m h$, $\alpha\geqslant -N$, $m\in\mathbb{Z}$, $m\geqslant 0$, or $\log^l h\log^m(\log h)$, $m,l\in\mathbb{Z}$, $m\geqslant 0$. In both cases the upper bound (19) holds. □

Remark 10. In the above proof one can replace the iterated sum solution F_+ by F_-, which is well defined over Q_-. The remaining part of the proof works as well with F_-.

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