The Oscillation of Solutions of Difference Equations

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Abstract—Using the solutions of a suitable inequality, we shall discuss the asymptotic properties of the solutions of a generalized Emden-Fowler difference equation. In the absence of specified constant sign solutions of certain inequalities, oscillatory behavior of all solutions of this equation are also deduced.

Keywords—Emden-Fowler difference equation, Inequalities, Comparison result, Oscillation.

In several recent papers (see e.g., [1-7]), oscillatory behavior of solutions of difference equations and inequalities have been investigated. In this paper, we shall study this property for the solutions of the equation

$$\Delta^2 y_n + \sum_{i=1}^{t} p_{n-s_i} (y_{n-s_i})^{q_i} = 0, \quad n = \tau, \tau + 1, \ldots,$$

where $q : \mathbb{N} \to \mathbb{R}^+$ is a sequence of quotients of odd positive integers, $p_i : \mathbb{N} \to \mathbb{R}^+$, $i = 1, \ldots, t$, $\nu = \max_{1 \leq i \leq t} \{s_i\}$, $s_i$ are fixed nonnegative integers, $t \in \mathbb{N}^\nu$. Some results for suitable difference inequalities will also be presented.

Equation (E) is a discrete analogue of a generalized Emden-Fowler differential equation with retarded arguments. Equations of type (E) with constant power $q_n \equiv q$ (constant), $t = 1, s_1 = -1$ were exhaustively considered in [2]. Some of the results contained therein were generalized in [7]. Sufficient conditions for oscillation of a first order difference equation with delay were studied in [3] (see also [5]). A very good source both for results and literature of the problem is the recent monograph of Agarwal [8].

In this paper, we shall use the following notion: $\mathbb{R}, \mathbb{R}^+, \mathbb{N}, \mathbb{N}_m$ for the sets of reals, nonnegative reals, nonnegative integers, and integers not less than $m$. For any function $x : \mathbb{N} \to \mathbb{R}$, the forward difference operator $\Delta$ is defined by the equality $\Delta x_n = x_{n+1} - x_n, n \in \mathbb{N}$, and for $k \geq 1 : \Delta^k x_n = \Delta(\Delta^{k-1} x_n), n \in \mathbb{N}$. Furthermore, we suppose

$$\sum_{j=k}^{k-m} x_j := 0, \quad \text{for any } k, m \in \mathbb{N}, \ m \geq 1.$$

We call the function $x : \mathbb{N} \to \mathbb{R}$ oscillatory, if there exists a infinite, increasing sequence $\{n_r\}_{r=1}^{\infty}$ of positive integers such that $x_{n_r}, x_{n_r+1} \leq 0$ for all $r \geq 1, r \in \mathbb{N}$. By a solution (ordinary solution), generalized $N_r$ solution, generalized solution of (E), we mean real sequence $y := \{y_n\}_{n=\tau-s}^{\infty}$ satisfying (E): for all $n \in \mathbb{N}_r$, for all $n \in N_r$ for all sufficiently large argument without specification of the initial value for which the equation is satisfied, respectively.
We start our investigations with a useful lemma, given here without proof, which is very simple.

**Lemma 1.** Let \( p^i : \mathbb{R}_+ \to \mathbb{R}_+ \) and for every \( m \in \mathbb{N} \), there is \( \sup_{m \geq m} \sum_{i=1}^{t} p^i_n > 0 \). If \( \{x_n\}_{n \geq \tau - \sigma} \) is a solution of the inequality

\[
\Delta^2 x_n + \sum_{i=1}^{t} p^i_{n-s_i} (x_{n-s_i})^q_n \leq 0, \quad n \in \mathbb{N},
\]

such that

\[
x_n > 0,
\]

for all \( n \geq \nu \) and some \( \nu \in \mathbb{N}_\tau \), then \( \Delta x_n > 0 \) for all \( n \geq \nu + \sigma \).

**Remark 1.** Similar result holds for the reversed inequality

\[
\Delta^2 x_n + \sum_{i=1}^{t} p^i_{n-s_i} (x_{n-s_i})^q_n \geq 0, \quad n \in \mathbb{N},
\]

**Theorem 1.** Let \( p^i : \mathbb{N}_0 \to \mathbb{R}_+ \) be such that \( \sup_{n \geq m} \sum_{i=1}^{t} p^i_n > 0 \) for every \( m \in \mathbb{N}_\tau \). If \( \{x_n\}_{n \geq \tau - \sigma} \) is a solution of the inequality (II) such that (I) holds, then equation (E) has the generalized \( \mathbb{N}_\nu \) solution \( y_n (\mu := \nu + \sigma) \) such that

\[
0 < y_n \leq x_n, \quad n \in \mathbb{N}_\nu,
\]

\[
0 < \Delta y_n \leq \Delta x_n, \quad n \in \mathbb{N}_\mu,
\]

\[
\lim_{n \to \infty} \Delta y_n = 0,
\]

\[
\sum_{j=\mu}^{\infty} p^j_{j-s_i} (y_{j-s_i})^q_j \quad \text{converges for all } i = 1, \ldots, t.
\]

**Proof.** By Lemma 1, we get

\[
\Delta x_n > 0, \quad \text{for } n \geq \mu.
\]

Let us take some \( m, n \in \mathbb{N}_\mu \), \( m \geq n \). Then, from (II), we obtain

\[
\Delta x_{m+1} - \Delta x_n + \sum_{j=n}^{m} \sum_{i=1}^{t} p^i_{j-s_i} (x_{j-s_i})^q_j \leq 0.
\]

Hence, by (3), we obtain

\[
\sum_{j=n}^{m} \sum_{i=1}^{t} p^i_{j-s_i} (x_{j-s_i})^q_j < \Delta x_n,
\]

from this, we get

\[
\sum_{j=n}^{\infty} \sum_{i=1}^{t} p^i_{j-s_i} (x_{j-s_i})^q_j \leq \Delta x_n, \quad \text{for } n \in \mathbb{N}_\mu.
\]

Let us consider the set \( S \) of the sequences \( \{u_n\}_{n \geq \tau - \sigma} \), such that

\[
\begin{cases}
  u_{n} - x_{n}, & \text{for } n = \tau - \sigma, \ldots, \mu \\
  u_{n} \geq 0, & \text{for } n > \mu \\
  \Delta u_{n} \leq \Delta x_{n}, & \text{for } n \geq \mu.
\end{cases}
\]

For any \( u \in S \), we have

\[
\begin{cases}
  u_{n} \leq x_{n}, & \text{for } n \in \mathbb{N}_\tau \sigma.
\end{cases}
\]
By (5) and (6), we get from (4)

$$\sum_{k=\mu}^{n-1} \sum_{j=k}^{\infty} \sum_{i=1}^{t} p_{j-s_i}^i (u_{j-s_i})^{q_j} \leq \sum_{k=\mu}^{n-1} \sum_{j=k}^{\infty} \sum_{i=1}^{t} p_{j-s_i}^i (x_{j-s_i})^{q_j} \leq x_n - x_\mu, \quad \text{for } n \in N_\mu. \quad (7)$$

On the set $S$, we define an operator $T$ by the formula $Tu = z = \{z_n\}_{n \geq -\sigma}$, where

$$\begin{cases} 
  z_n = x_n, & \text{for } n = \tau - \sigma, \ldots, \mu \\
  z_n = x_\mu + \sum_{k=\mu}^{n-1} \sum_{j=k}^{\infty} \sum_{i=1}^{t} p_{j-s_i}^i (u_{j-s_i})^{q_j}, & \text{for } n > \mu.
\end{cases} \quad (8)$$

It can be observed that by (7), the operator $T$ is well defined on $S$. Furthermore,

$$z_n \geq x_\mu > 0$$

and

$$\Delta z_n = \sum_{j=\mu}^{n-1} \sum_{i=1}^{t} p_{j-s_i}^i (u_{j-s_i})^{q_j} \leq \sum_{j=\mu}^{n-1} \sum_{i=1}^{t} p_{j-s_i}^i (x_{j-s_i})^{q_j} = \Delta x_n, \quad n \in N_\mu.$$

So $T : S \to S$. Now let $u = \{u_n\}_{n \geq -\sigma}, v = \{v_n\}_{n \geq -\sigma}$ be any two elements of $S$. We shall say $u < v$ if

$$u_n \leq v_n \quad \text{and} \quad \Delta u_n \leq \Delta v_n, \quad n \in N_{-\sigma}.$$  

One can check that operator $T$ is monotonic on $S$, i.e., if $u, v \in S, u < v$, then $Tu < Tv$. Let us consider the sequence $\{y^r\}_{r \in N}$ of elements of the set $S$ defined as follows:

$$y^0 = \{x_n\}_{n \geq -\sigma} : y^{r+1} = Ty^r, \quad \text{for } r > 0.$$  

We shall prove that $y^1 < y^0$. By (8), we get $y^1_n = y^0_n, n = \tau - \sigma, \ldots, \mu$ and hence, $\Delta y^1_n = \Delta y^0_n, n = \tau - \sigma, \ldots, \mu - 1$. Furthermore, by (4)

$$\Delta y^1_n = \sum_{j=\mu}^{\infty} \sum_{i=1}^{t} p_{j-s_i}^i (y^0_{j-s_i})^{q_j}$$

$$= \sum_{j=\mu}^{\infty} \sum_{i=1}^{t} p_{j-s_i}^i (x_{j-s_i})^{q_j} \leq \Delta x_n = \Delta y^0_n = \Delta y^0_n, \quad \text{for } n \geq \mu;$$

from this, it follows $y^1_n \leq y^0_n$ for $n > \mu$. Hence, we get $y^1 < y^0$. Since $y^1 \in S$ and $T$ is monotonic on $S$, we have $y^2 = Ty^1 < Ty^0 = y^1$. Following this way, we obtain $y^{r+1} < y^r$ for all $t \in N$. For arbitrary $n \in N_{-\sigma}$, the sequence $\{y^r_n\}_{r \in N}$ is nonincreasing and bounded from below by $x_n$ for $n = \tau - \sigma, \ldots, \mu$ and by $x_\mu$ for $n > \mu$. So there exist finite limits

$$y_n = \lim_{r \to \infty} y^r_n, \quad \text{for every } n \in N_{-\sigma}.$$  

Hence, by (8), we have

$$y_n = \lim_{r \to \infty} y^{r+1}_n = x_\mu + \lim_{r \to \infty} \sum_{k=\mu}^{n-1} \sum_{j=k}^{\infty} \sum_{i=1}^{t} p_{j-s_i}^i (y^r_{j-s_i})^{q_j}, \quad n > \mu.$$  

From the above, taking into account that $y_\mu = x_\mu$, we get

$$\Delta y_n = \lim_{r \to \infty} \sum_{j=\mu}^{n-1} \sum_{i=1}^{t} p_{j-s_i}^i (y^r_{j-s_i})^{q_j}, \quad \text{for } n > \mu. \quad (9)$$
and consequently,

$$\Delta^2 y_n = \lim_{r \to \infty} \sum_{j=n+1}^{t} \sum_{i=1}^{t} p_{j-s_i}(y_{j-s_i})^{q_i} - \lim_{r \to \infty} \sum_{j=n}^{t} \sum_{i=1}^{t} p_{j-s_i}(y_{j-s_i})^{q_i}$$

$$= \lim_{r \to \infty} \left[ - \sum_{i=1}^{t} p_{n-s_i}(y_{n-s_i})^{q_i} \right] = - \sum_{i=1}^{t} p_{n-s_i}(y_{n-s_i})^{q_i},$$

for $n \geq \mu$, so the sequence $\{y_n\}_{n \geq \tau}$ is any generalized $N_{\mu}$ solution of (E). Since for every $r \in \mathbb{N}$ and $n \geq \mu$, there is $0 < x_{n} = y_{n} \leq x_{n}$, then $0 < x_{\mu} \leq y_{\mu} \leq x_{\mu}$ for $n \geq \mu$. Furthermore, $0 < x_{n} = y_{n} = y_{n}$ for $n = \nu, \nu + 1, \ldots, \mu$, and hence,

$$0 < y_{n} \leq x_{n}, \quad \text{for all } n \in \mathbb{N}_{\nu},$$

that is, (2i) holds. By (4), (9), and $0 \leq y_{n} \leq x_{n}$, we get

$$0 \leq \Delta y_{n} \leq \sum_{j=\nu}^{\infty} \sum_{i=1}^{t} p_{j-s_i}(x_{j-s_i})^{q_i} \leq \Delta x_{\mu}, \quad \text{for all } n \in \mathbb{N}_{\mu}.$$  \hspace{1cm} (11)

This means that the sequence $\{y_n\}_{n \in \mathbb{N}}$ is nondecreasing at least for $n \geq \mu$. Applying just proved condition (2i), we obtain from (E) $\Delta^2 y_{n} \leq 0$ for $n \geq \mu$, so the sequence $\{\Delta y_{n}\}_{n \geq \mu}$ is nondecreasing and by (11) nonnegative, furthermore, by $\sup_{n>\nu} \sum_{i=1}^{t} p_{n}^{i} > 0$. This sequence contains some strictly decreasing subsequence so $\Delta y_{n} > 0$ for $n \in \mathbb{N}_{\mu}$. Moreover, by (4) the series $\sum_{j=\mu}^{\infty} \sum_{i=1}^{t} p_{j-s_i}(x_{j-s_i})^{q_i}$ is convergent, therefore, by (11), we get (2ii). Also, by (10), we obtain

$$\sum_{j=\mu}^{\infty} \sum_{i=1}^{t} p_{j-s_i}(y_{j-s_i})^{q_i} \leq \sum_{j=\nu}^{\infty} \sum_{i=1}^{t} p_{j-s_i}(x_{j-s_i})^{q_i} \leq \sum_{j=\nu}^{\infty} \sum_{i=1}^{t} p_{j-s_i}(x_{j-s_i})^{q_i} \leq \Delta x_{\mu}.$$  \hspace{1cm} (12)

Hence, the condition (2iv) is valid.

REMARK 2. Similar results for the inequality (12) can be expressed as follows: let $p_{i} : \mathbb{N}_{\tau-s_{i}} + \mathbb{R}_{+}$ be such that $\sup_{n \geq \nu} \sum_{i=1}^{t} p_{n}^{i} > 0$ for every $m \in \mathbb{N}_{\tau}$. If $\{x_{n}\}_{n \geq \tau-s_{-\sigma}}$ is a solution of the inequality (12), such that (1) holds, then equation (E) has the generalized $N_{\mu}$ solution $y_{i} (\mu := \nu + \sigma)$, such that

$$0 > y_{n} \geq x_{n}, \quad \text{for all } n \in \mathbb{N}_{\nu},$$

$$0 > \Delta y_{n} \geq \Delta x_{n}, \quad \text{for all } n \in \mathbb{N}_{\mu},$$

$$\lim_{n \to \infty} \Delta y_{n} = 0,$$  \hspace{1cm} (13)

$$\sum_{j=\mu}^{\infty} \sum_{i=1}^{t} p_{j-s_i}(y_{j-s_i})^{q_i}$$

converges for all $i = 1, \ldots, t$.

REMARK 3. Relation (2iv) describes the fact that the solution of (E) is $q$ summable with weight functions $p_{i}$.

REMARK 4. To get similar results for an ordinary solution, we should have in the assumption (1) $\nu = \tau$. The problem is can we extend a generalized solution up to $n = \tau$, or in other words, does there exist an ordinary solution of (E) which coincides for $n \geq \nu$ with the generalized solution obtained in Theorem 1? Consider the case $\sigma > 0$. Let $I := \{i \in \{1, \ldots, t\} : s_{i} = \sigma\}$. Rewrite equation (E) in the form

$$\sum_{i \in I} p_{n-s_{i}}(y_{n-s_{i}})^{q_i} = -\Delta^2 y_{n} - \sum_{i=1, i \in I} \sum_{i=1}^{t} p_{n-s_{i}}(y_{n-s_{i}})^{q_i}.$$
From this, if \( \sum_{i \in I} p_{n-i}^i \neq 0 \), we obtain

\[
Y_{n-\sigma} = \left\{ \frac{\Delta^2 y_n \left( \sum_{i=1}^{t} p_{n-s_i}^i (y_{n-s_i})^{q_n} \right)}{\sum_{i \in I} p_{n-\sigma}^i} \right\}^{1/q_n}.
\]

(12)

Notice that all arguments of the terms of the sequence \( y \) which are on the right hand side of (12) are greater then \( n - \sigma \). Therefore, if we have any generalized \( N_k \) solution \( v \) that is the sequence \( \{v_n\}_{n \geq k-\sigma} \) which fulfills (E) for \( n \geq k \), and for all \( n = \tau - \sigma, \ldots, k - \sigma - 1 \) there is \( \sum_{i \in I} p_{n}^i \neq 0 \), we can find (at first putting \( n = k-1 \) and \( y_i = y_{k-1} \) for \( i > k-1 \) in formula (12)) the term \( v_{k-\sigma-1} \) and next, following this way, step by step any ordinary solution of (E) which coincides with the given one for \( n \geq k - \sigma \). If \( \sigma = 0 \), then equation (E) can be rewritten in the form

\[
y_n + \gamma q_n \sum_{i=1}^{t} p_{n}^i = -y_{n+2} + 2y_{n+1}.
\]

Now, we can proceed in a similar way provided the equation

\[
z + q^{k-1} \sum_{i=1}^{t} p_{n-k}^i = -v_{k+1} + 2v_k, \quad n < k
\]

has solutions. If this procedure can be repeated up to \( n = \tau - \sigma \), we obtain an ordinary solution of (E).

**THEOREM 2.** Let \( p^i : N_{\tau-s_i} \to \mathbb{R}_+ \) be such that one of the following conditions hold

\[
\limsup_{n \to \infty} \sum_{i=1}^{t} p_{n}^i > 0, \quad (13i)
\]

there exist \( K \subset N \) and \( \alpha > 0 \), such that \( n \sum_{i=1}^{t} p_{nK}^i \geq \alpha \), for all \( n \in N_{\tau}, \) \( (13ii) \)

there exists positive, increasing sequence \( \{\varphi_n\} \), such that \( \lim_{n \to \infty} \varphi_n = \infty, \)

and \( \limsup_{n \to \infty} \left( \frac{1}{\varphi_n} \right) \sum_{j=m}^{n} \varphi_j \sum_{i=1}^{t} p_{i}^j > 0, \) for \( m \in N_{\tau}. \) \( (13iii) \)

Let \( q : N_{\tau} \to \mathbb{R}_+ \) be a bounded from above sequence of quotients of odd positive integers. Then, inequality (11) does not possess any solution which is eventually positive.

**PROOF.** Suppose contrarily that the inequality (11) possesses a solution \( x \) which is eventually positive, say \( x_n > 0 \) for all \( n \geq \nu \) and some \( \nu \in N_{\tau-\sigma} \). Since the assumptions of Theorem 1 are satisfied, equation (E) possesses a positive generalized \( N_{\mu} \) solution \( \{y_n\}_{n \geq \nu} (\mu := \nu + \sigma) \) with the properties

\[
\Delta y_n \to 0, \quad as \ n \to \infty, \quad \Delta y_n > 0, \quad for \ n \geq \mu. \quad (14)
\]

Since the sequence \( y \) is increasing for \( n \geq \mu, y_{n-s_i} \geq y_{n-\sigma} \) for \( n \geq \xi := \mu + \sigma \). Hence, we obtain

\[
\sum_{i=1}^{t} p_{n-s_i}^i (y_{n-s_i})^{q_n} \geq \left( \sum_{i=1}^{t} p_{n-s_i}^i \right) y_{n-\sigma}^{q_n}, \quad for \ n \geq \xi.
\]
From this, by (E) the sequence \( y \) satisfies the inequality
\[
\Delta^2 y_n + \left( \sum_{i=1}^{t} p_{n-i} \right) y_{n-\sigma}^{q_n} \leq 0, \quad n \geq \xi. \tag{15}
\]

Considering the asymptotic behavior of the sequence \( y \), we see that the following three cases are possible

(i) \( y_n \to a \in (0, 1) \),
(ii) \( y_n \to a > 1 \),
(iii) \( y_n \to \infty \),
as \( n \to \infty \).

Suppose at first, that the sequences \( p_i \) satisfy condition (13i). Let us observe that in the case (i), we have
\[
y_{n-\sigma}^{q_n} \geq y_{n-\sigma}^{q_\infty} \geq y_{\mu}^{q_\infty}, \quad \text{for } n \geq \xi,
\]
where \( q_\infty = \sup q_n \).

Therefore, we obtain from (15)
\[
\left( \sum_{i=1}^{t} p_{n-i} \right) y_{\mu}^{q_\infty} \leq \left( \sum_{i=1}^{t} p_{n-i} \right) y_{n-\sigma}^{q_n} \leq -\Delta^2 y_n.
\]

On the ground of (13i), we have
\[
\limsup_{n \to \infty} \left( \sum_{i=1}^{t} p_{n-i} \right) y_{\mu}^{q_\infty} > 0,
\]
while on the other hand \( \Delta^2 y_n \to 0 \). The obtained contradiction completes the proof in the case (i). In the cases (ii), (iii), there exists \( n_1 \geq \mu + \sigma \), such that \( y_{n-\sigma} \geq 1 \) for all \( n \geq n_1 \). Thus, for \( n \geq n_1 \), we obtain from (15)
\[
\left( \sum_{i=1}^{t} p_{n-i} \right) \leq \left( \sum_{i=1}^{t} p_{n-i} \right) y_{n-\sigma}^{q_n} \leq -\Delta^2 y_n
\]
and similar reasoning leads us to the contradiction. The case (13i) is proved. Rewrite inequality (15) in the form
\[
\Delta^2 y_{j+\sigma} \leq -a \left( \sum_{i=1}^{t} p_{j} \right) y_{j+\sigma}^{q_n}.
\]

Similar reasoning as in the previous case allows us to get the above inequality from the following one
\[
\Delta^2 y_{j+\sigma} \leq -a \left( \sum_{i=1}^{t} p_{j} \right), \quad \text{for } j \geq k, jj \geq k, \tag{16}
\]
where
\[
a = y_{\mu}^{q_\infty}, \quad k = \mu \quad \text{in the case (i)}
\]
\[
a = 1, \quad k = n_1 + \sigma \quad \text{in the cases (ii), (iii)}.
\]

Multiplying (16) by \( j \), summing next from \( k \) to \( n \) and applying the method of summation by parts, we obtain
\[
n \Delta y_{n+\sigma+1} - k \Delta y_{\sigma+k} - \sum_{j=k}^{n-1} \Delta y_{j+k+1} \leq -a \sum_{j=k}^{n} j \left( \sum_{i=1}^{t} p_{j} \right),
\]
from this and (13ii), we have
\[ n\Delta y_{n+\sigma+1} - k\Delta y_{\sigma+k} - y_{n+\sigma+1} + y_{\sigma+k+1} \]
\[ \leq -a \sum_{j=E(n/K)}^{E(n/K)+1} jK \left( \sum_{i=1}^{\ell} p_{j,K}^i \right) \leq -aK \left[ E\left( \frac{n}{K} \right) - E\left( \frac{k}{K} \right) \right], \]
where \( E(m) \) denotes the entire part of \( m \). Dividing the last inequality by \( n \), we get
\[ \Delta y_{n+\sigma+1} - \frac{k\Delta y_{\sigma+k}}{n} - \frac{y_{n+\sigma+1}}{n} + \frac{y_{\sigma+k+1}}{n} \leq -\alpha K \frac{E\left( \frac{n}{K} \right) - E\left( \frac{k}{K} \right)}{n}. \] (17)

Notice that
\[ \lim_{n \to \infty} -\alpha K \frac{E\left( \frac{n}{K} \right) - E\left( \frac{k}{K} \right)}{n} = -\alpha \alpha. \]

Considering the cases (i) and (ii), we immediately obtain by (14) that all the terms on the left-hand side of (17) tend to zero. In the case (iii), the same holds with first, second, and fourth terms. For the third term by the Stolz Theorem and again by (14), we obtain
\[ \lim_{n \to \infty} \left( \frac{y_{n+\sigma+1}}{n} \right) = \lim_{n \to \infty} \left( \frac{\Delta y_{n+\sigma+1}}{1} \right) = 0. \]

So in all the cases (i)-(iii), we obtain the contradiction
\[ 0 \leq -\alpha \alpha < 0. \]

The case (13ii) is proven. To see that conditions (13iii) also gives us a contradiction, we proceed in a similar way as in the previous case. Now multiply inequality (16) by \( \varphi_j \), sum from \( k \) to \( n \), and divide by \( \varphi_n \) to obtain
\[ \Delta y_{n+\sigma+1} - \frac{\varphi_k \Delta y_{\sigma+k}}{\varphi_n} - \left( \frac{1}{\varphi_n} \right) \sum_{j=k}^{n-1} (\Delta y_{j+\sigma+1})(\Delta \varphi_j) \leq -a \left( \frac{1}{\varphi_n} \right) \sum_{j=k}^{n} \varphi_j \left( \sum_{i=1}^{\ell} p_{j}^i \right). \] (18)

Of course, \( \lim_{n \to \infty} \Delta y_{n+\sigma+1} = 0 \) and \( \lim_{n \to \infty} (\varphi_k \Delta y_{\sigma+k})/\varphi_n = 0 \). To determine the limit at infinity the third term on the left-hand side, let us observe that \( \Delta y_j > 0, \Delta \varphi_j > 0 \). Therefore, we have two possible cases
\[ \lim_{n \to \infty} \sum_{j=k}^{n-1} (\Delta y_{j+\sigma+1})(\Delta \varphi_j) = b > 0, \quad \text{for any constant } b, \text{ or} \]
\[ \lim_{n \to \infty} \sum_{j=k}^{n-1} (\Delta y_{j+\sigma+1})(\Delta \varphi_j) = \infty. \]

In the first case, the considered term tends to zero. In the second case, again an application of the Stolz Theorem gives, by (14), the same limit. Therefore, in all cases (i)-(iii) the left-hand side of (18) tends to zero. In contrary to this, condition (13iii) follows
\[ \liminf_{n \to \infty} -a \left( \frac{1}{\varphi_n} \right) \sum_{j=k}^{n} \varphi_j \left( \sum_{i=1}^{\ell} p_{j}^i \right) < 0, \]
and we conclude that our assumption on positivity of the solution \( z \) falls down in all cases (13).  

**Remark 5.** As a sequence \( \{\varphi_n\} \) defined in condition (13iii), we can take for example sequences ... \( \{n^3\}, \{n^2\}, \{n\}, \{n^{1/2}\}, \{n^{1/3}\}, \ldots, \{\ln(n)\}, \{\ln(\ln(n))\}, \ldots \), each of which gives in turn
a larger class of sequences $p$ for which conditions (13iii) are satisfied. In fact, if we have two sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \), such that $\alpha_n \leq \beta_n$ and $\Delta(\beta_n/\alpha_n) \geq 0$ for sufficiently large $n$ (say for $n \geq m$), then

\[
0 < \limsup_{n \to \infty} \left( \frac{1}{\beta_n} \right) \sum_{j=m}^{n} \beta_j \left( \sum_{i=1}^{t} p_j^i \right) = \limsup_{n \to \infty} \left( \frac{1}{\beta_n} \right) \sum_{j=m}^{n} \alpha_j \frac{\beta_j}{\alpha_j} \left( \sum_{i=1}^{t} p_j^i \right) \leq \limsup_{n \to \infty} \left( \frac{1}{\alpha_n} \right) \sum_{j=m}^{n} \alpha_j \left( \sum_{i=1}^{t} p_j^i \right).
\]

Hence, if condition (13iii) is satisfied with the bigger sequence \( \beta \), then it also holds with the smaller one \( \alpha \).

**REMARK 6.** Similar result for the inequality (12) can be expressed as follows: let $p^i : \mathbb{N}_r \to \mathbb{R}_+$ be such that one of the following conditions hold

- \( \limsup_{n \to \infty} \sum_{i=1}^{t} p_n^i > 0 \);
- there exist $K \in \mathbb{N}$ and \( a > 0 \), such that $n \sum_{i=1}^{t} p_n^i K \geq \alpha$, for all $n \in \mathbb{N}_r$;
- there exists positive, increasing sequence \( \{\varphi_n\} \), such that $\lim_{n \to \infty} \varphi_n = \infty$,
  
  \[
  \text{and } \limsup_{n \to \infty} (1/\varphi_n) \sum_{j=m}^{n} \varphi_j \sum_{i=1}^{t} p_n^i > 0, \quad \text{for } m \in \mathbb{N}_r.
  \]

Let $q : \mathbb{N}_r \to \mathbb{R}_+$ be a bounded from above sequence of quotients of odd positive integers. Then, inequality (12) does not possess any solution which is eventually negative.

**REMARK 7.** Condition (13i) is insufficient even in the stronger form, i.e.,

\[
\lim_{n \to \infty} \sum_{i=1}^{t} p_n^i = \infty
\]

for the statement of Theorem 2 to be satisfied in the case $q$ is unbounded sequence. The following example confirms assertion of this remark.

**EXAMPLE.** Consider the inequality

\[
\Delta^2 x_n + \left( \frac{2n+2}{(3n)^3} \right) x_{n-1}^{2n} \leq 0, \quad n \in \mathbb{N}_5,
\]

where

\[
q_n = \begin{cases} n, & \text{for } n \text{ odd} \\ n + 1, & \text{for } n \text{ even}. \end{cases}
\]

It is evident that here $\lim_{n \to \infty} \sum_{i=1}^{t} p_n^i = \lim_{n \to \infty} \left( \frac{2n+2}{(3n)^3} \right) = \infty$. On the other hand, this inequality possesses positive solution $x_n = n/(2n + 1)$.

Since every solution of \( E \) is the solution of (11) and (12), then as a consequence of Theorem 2 and Remark 6, we have the following theorem.
THEOREM 3. Let $p_i : \mathbb{N}_+ \rightarrow \mathbb{R}_+$ be such that one of the following conditions hold:

\[
\limsup_{n \to \infty} \sum_{i=1}^{t} p_i n > 0;
\]

there exist $K \in \mathbb{N}$ and $\alpha > 0$, such that $n \sum_{i=1}^{t} p_i n K \geq \alpha$ for all $n \in \mathbb{N}_+$;

there exists positive, increasing sequence $\{\varphi_n\}$, such that $\lim_{n \to \infty} \varphi_n = 0$ and $\limsup_{n \to \infty} (1 / \varphi_n) \sum_{j=m}^{n} \varphi_j \sum_{i=1}^{t} p_i n > 0$ for $m \in \mathbb{N}_+$.

Let $q : \mathbb{N}_+ \rightarrow \mathbb{R}_+$ be a bounded from above sequence of quotients of odd positive integers. Then every solution of (E) is oscillatory.

This work is motivated by the paper of Ladas, Philos and Sficas [2], where these authors considered the equation

\[
\Delta y_n + p_n y_{n-k} = 0,
\]

and where they have given sufficient conditions for all solutions of (E1) to be oscillatory.

REFERENCES


