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Axisymmetric flow due to a stretching sheet with partial slip

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Abstract

The steady, laminar, axisymmetric flow of a Newtonian fluid due to a stretching sheet when there is a partial slip of the fluid past the sheet has been investigated. The flow is governed by a third-order non-linear boundary value problem whose exact numerical solution has been obtained non-iteratively in terms of the non-dimensional slip parameter λ. A perturbation solution valid for small λ and an asymptotic solution valid for large λ have been derived. Finally a solution based upon He's homotopy perturbation method has been developed. The latter, being analytical, is elegant but is sufficiently accurate for all values of λ.

Keywords: Axisymmetric flow; Partial slip; Homotopy perturbation method; Ackroyd's method; Samuel and Hall method; Perturbation solution; Asymptotic solution

1. Introduction

It is a well known fact that a viscous fluid normally sticks to a boundary, i.e., there is no slip of the fluid relative to the boundary. However, there are numerous situations where there may be a partial slip between the fluid and the boundary, e.g., the fluid may be particulate or that it could be a rarefied gas with a suitable value of the Knudsen number. For such fluids the motion is still governed by the Navier–Stokes equations, but the usual no-slip condition at the boundary is replaced by the slip condition (Beavers and Joseph [1])

$$u_T = k\tau_T, \tag{1}$$

where u_T and τ_T are the tangential components of the velocity and the stress at the boundary.

Sparrow and his co-workers [2–4] analyzed a number of flow problems taking the velocity slip into account in 1960's. Wang has revived an interest in problems involving partial slip by considering the stagnation point flows [5], flow due to a rotating disk [6] and the flow past a stretching sheet [7]. For the stagnation point flow Wang derived the solutions for both the cases: (i) plane two-dimensional flow and (ii) axisymmetric flow, but for the flow past a stretching sheet he only obtained the solution for the plane two-dimensional case. Moreover he did not give an exact analytical solution of the problem, which, in general, characterizes most of the problems associated with the two-dimensional flow past a stretching sheet (Gupta and Gupta [8], Anderssen [9], Troy et al. [10], Ariel [11] etc.), and which was derived by Anderssen [12] independently. The axisymmetric flow due a stretching sheet, when there is a partial slip at the boundary, is an equally important and interesting problem that remains unexplored so far.

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The aim of the present paper is to investigate the steady, laminar flow of an incompressible, viscous, Newtonian fluid due to the radial stretching of a sheet, when there is a partial slip at the boundary. Due to the boundary condition (1), a non-dimensional parameter λ representing the amount of slip past the sheet, characterizes the problem. The governing equations of motion are reduced to a third-order non-linear boundary value problem (BVP) in non-dimensional stream function ϕ. Unlike the case of the two-dimensional flow, the present problem does not admit an exact solution. This affords a number of interesting techniques and algorithms for obtaining the solution.

There is no doubt that the present problem can be solved by a shooting method used by Wang [7] to obtain the solution of the corresponding two-dimensional problem. However, there are two aspects of the shooting method which need to be addressed. Firstly, the BVP for the present problem in non-linear so its solution by the shooting method has to be necessarily iterative. Secondly, the domain of integration is semi-infinite, which means that a finite number, usually referred to as numerical infinity, has to be used to truncate the domain. This number usually depends upon the physical parameters of the problem and its value needs to be adjusted as the values of the parameters change. It would be nice and aesthetically pleasing if an algorithm could be developed which would obviate both of the above-mentioned features.

For the moving boundaries Ackroyd [13,14] has given an algorithm, first mooted by Benton [15], for the flow of a fluid past a rotating disk, that eliminates the need of introducing the numerical infinity. The idea is to use a series comprising of the exponentially decaying functions for various physical quantities. This ensures that the conditions at infinity are satisfied automatically. The drawback of the method is that the conditions at the physical boundary need to be satisfied, which would require a system of non-linear algebraic equations be solved iteratively. Ackroyd pointed out that the need for iterations can be eliminated if the problem is reformulated in accordance with the scheme suggested by Samuel and Hall [16]. Eventually one has to solve an initial value problem (IVP) consisting of a system of equations in a finite domain to obtain the solution. We found that a similar approach can be used for the present problem—the difference in our approach being that rather than solving the IVP by an integration routine, the solution is obtained by using a power series consisting of a small number of terms.

During the last century, particularly in the first half, in the absence of fast computing devices, much attention of the researchers was focused in obtaining some form of an analytical solution of the non-linear problems. A hallmark of a good solution has been its simplicity and accuracy. The accuracy in a strict sense could only be judged by comparing the analytical solution with the numerical solution, though, in most cases the numerical solution was too labor intensive to derive. In such cases a sequence of analytical solutions was developed and the accuracy was judged by the "difference" of the solutions between two successive members of the sequence.

With the advent of the digital computers the emphasis was shifted to the numerical solutions, which, because of the digital nature of the computer, could only be listed at a discrete set of values in the domain of interest. If the solution was sought at a point not belonging to the discrete set, some kind of interpolation was necessary. An analytical solution has the advantage of generating the solution at an arbitrarily chosen point in the domain. Further, for the non-linear problems a numerical solution is iterative in general. Therefore in order to start the iterations some starting value is required. The closer the starting value to the exact solution the lesser the number of iterations is required, and more the efficient the numerical scheme is. Because of this and other reasons, analytical solutions still continue to demand the attention of researchers.

There are two techniques of recent vintage which have become quite popular in the literature. Both the techniques have the attractive feature of finding a totally analytical solution of a non-linear BVP. Liao [17–21] devised and elaborated the homotopy analysis method (HAM) which is based upon the familiar ideas of homotopy. He introduced an auxiliary parameter and a function, the proper choices of which can go a long way to accelerate the convergence of the sequence of the functions generated by the HAM. Liao demonstrated the usefulness of the method by applying it to a number of problems in fluid dynamics and other areas of engineering. He [22–28], on the other hand, formulated the homotopy perturbation method (HPM), the idea of which was to use p, the parameter characterizing the homotopy, also as the perturbation parameter. However, the originality of the method was in the introduction of another parameter which was to be chosen in a manner to enforce some highly desirable features of the solution, such as, the absence of the secular terms etc. A succinct account of the method has been given by He [28] recently. Ariel et al. [29] applied HPM successfully to the problem of axisymmetric flow past a stretching sheet with or without suction and magnetic field, taken separately or jointly. They demonstrated that whereas the standard numerical techniques such as shooting methods, homotopy method due to Watson [30] can compute the solution efficiently only up to some moderate values of the physical parameters representing suction or magnetic field, the HPM can effortlessly handle these situations—

in fact in most cases the performance improves vastly as the larger values of the parameters are considered. In the present paper we show that HPM is equally adapt at handling all values of the slip parameters.

If a physical parameter characterizes the problem it might be possible to obtain analytical solutions for small and large values of the physical parameters. For the present problem we give both of these solutions. It may be mentioned here that Wang [7] also gave a perturbation solution valid for small λ. However, his solution contains the secular terms. On the other hand, by stretching the independent variable, we are able to give a solution that is free of the secular terms. For large values of λ Wang [7] alluded to an asymptotic behavior, without, however, deriving an analytical form of the solution. In the present work we give an *analytical* solution for large values of λ.

Finally we compare the solutions obtained by various techniques and draw the appropriate conclusions.

2. Equations of motion

Consider an axially stretching boundary for which the lateral surface velocity U is proportional to the distance r from the axis, i.e.,

$$U = Ar, \tag{2}$$

A being the constant of proportionality.

The fluid occupies the upper half of the space $z \geq 0$. The entire motion is caused by the stretching of the sheet. Let (u, w) be the velocity components in the cylindrical coordinate system in the r- and z-directions. On account of the velocity slip, the boundary condition on the sheet takes the form

$$u(r, 0) - U = k\nu \frac{\partial u}{\partial r}(r, 0). \tag{3}$$

It has been shown in the literature (Ariel [31], Ariel et al. [29]) that the choice of the variables

$$u = Ar\phi'(\zeta), \quad w = -2\sqrt{A\nu}\phi(\zeta), \tag{4}$$

where

$$\zeta = \sqrt{\frac{A}{\nu}} z, \tag{5}$$

(a prime denotes the derivative with respect to ζ) transform the Navier–Stokes equations into the ordinary differential equation

$$\phi''' + 2\phi\phi'' - \phi'^2 = 0. \tag{6}$$

The boundary conditions become

$$\phi(0) = 0, \quad \phi'(0) - 1 = \lambda\phi''(0), \quad \phi'(\infty) = 1, \tag{7a, b, c}$$

where

$$\lambda = k\sqrt{A\nu}, \tag{8}$$

is the dimensionless parameter representing the velocity partial slip. When $\lambda = 0$, the fluid sticks to the boundary, and when $\lambda \to \infty$, the fluid slips past the boundary without any resistance. Note that the BVP (6), (7), unlike the two-dimensional case, does not admit a closed form analytical solution. We are thus compelled to seek alternate solutions. In the following sections we present different solutions based upon varying methodologies.

3. A series solution involving exponentials

For moving boundaries Ackroyd [13] has demonstrated that a series solution of the form

$$\phi'(\zeta) = \sum_{n=1}^{\infty} a_n e^{-nc\zeta}, \tag{9}$$

epitomizes Eq. (6).

Integrating Eq. (9) we obtain

$$\phi(\zeta) = \phi_\infty - \frac{1}{c}\sum_{n=1}^{\infty}\frac{a_n}{n}e^{-nc\zeta}, \tag{10}$$

ϕ_∞ being the normal velocity at infinity.

Substituting for ϕ and its derivatives into Eq. (6), the following recurrence relation is readily obtained for a_ns:

$$a_n = -\frac{1}{n(n-1)c^2}\sum_{m=1}^{n-1}\left(\frac{2n}{m}-3\right)a_m a_{n-m}, \quad n \geq 2. \tag{11}$$

Also equating the coefficient of $e^{-c\zeta}$ leads to

$$c = 2\phi_\infty, \tag{12}$$

and boundary conditions (7) result into

$$\sum_{n=1}^{\infty}(1+\lambda nc)a_n = 1, \tag{13}$$

and

$$\sum_{n=1}^{\infty}\frac{a_n}{n} = c\phi_\infty,$$

which in view of Eq. (12) takes the form

$$\sum_{n=1}^{\infty}\frac{a_n}{n} = \frac{1}{2}c^2. \tag{14}$$

Now if a_1 and c are known then all a_ns are known, and the solution for the flow described by Eqs. (9) and (10) can be considered complete. The values of a_1 and c are determined from Eqs. (13) and (14). Thus we have at hand a zero-finding problem for two parameters. For this any suitable algorithm can be chosen. In particular the two-dimensional version of the secant method is ideally suited and has been employed successfully by the author for several problems. Note that Ackroyd's method neatly eliminates the problem of approximating infinity by a finite number, but in the bargain *two* values of the parameters need to be determined as opposed to *one* when the shooting method is used to find the missing initial condition $\phi''(0)$. In either case the solution can only be obtained iteratively. We now present a non-iterative solution of the problem using the technique of Samuel and Hall [16].

4. A non-iterative solution

Following Ackroyd [13] we introduce the new set of coefficients A_n defined by

$$a_n = c^2\left(\frac{a_1}{c^2}\right)^n A_n, \tag{15}$$

implying

$$A_1 = 1, \tag{16}$$

and, from Eq. (11)

$$A_n = -\frac{1}{n(n-1)}\sum_{m=1}^{n-1}\left(\frac{2n}{m}-3\right)A_m A_{n-m}, \quad n \geq 2. \tag{17}$$

An immediate consequence of Eqs. (16) and (17) is that A_ns are computed once and for all—they need not be computed afresh for each iteration.

Next we introduce the variable

$$Z = Z_0 e^{-c\zeta}, \tag{18}$$

where

$$Z_0 = \frac{a_1}{c^2}. \tag{19}$$

Boundary conditions (13) and (14) then transform to

$$\sum_{n=1}^{\infty} (1 + \lambda nc) c^2 Z_0^n A_n = 1, \tag{20}$$

and

$$\sum_{n=1}^{\infty} \frac{A_n}{n} Z_0^n = \frac{1}{2}. \tag{21}$$

Since A_ns are known, Z_0 can be readily obtained from Eq. (21) and then c can be determined from Eq. (20). The solution can again be considered to be complete—the only iterations involved here are in the determination of the value of Z_0. The major bottleneck in the computation, namely the calculation of a_ns at each iteration thus has been disposed of. A further refinement due to Samuel and Hall [16] finally gets rid of the last need of iterations. We introduce the variables

$$\phi = c\left(\frac{1}{2} - f(Z)\right), \quad \phi' = c^2 g(Z), \quad \phi'' = -c^3 h(Z), \tag{22}$$

which when substituted in Eq. (6) lead to the following IVP

$$\frac{\mathrm{d}f}{\mathrm{d}Z} = \frac{g}{Z}, \quad f(0) = 0, \tag{23}$$

$$\frac{\mathrm{d}g}{\mathrm{d}Z} = \frac{h}{Z}, \quad g(0) = 0, \tag{24}$$

$$\frac{\mathrm{d}h}{\mathrm{d}Z} = \frac{(1-2f)h + g^2}{Z}, \quad h(0) = 0. \tag{25}$$

Because of the singularity at $Z = 0$, we need to take the limits in Eqs. (23)–(25) in order to start the integration. We obtain

$$\frac{\mathrm{d}f(0)}{\mathrm{d}Z} = 1, \quad \frac{\mathrm{d}g(0)}{\mathrm{d}Z} = 1, \quad \frac{\mathrm{d}h(0)}{\mathrm{d}Z} = 1. \tag{26}$$

The integration of the system (23)–(25) now proceeds smoothly till f becomes equal to $\frac{1}{2}$, at which point ϕ becomes zero and we reach the sheet, which means that here Z equals Z_0. The boundary condition (7b), in conjunction with (22), gives

$$\lambda h(Z_0) c^3 + g(Z_0) c^2 - 1 = 0. \tag{27}$$

Since the values of h and g would be known from integration at $Z = Z_0$, c can be computed from Eq. (27) for any given value of λ. The most significant advantage of the procedure is that the flow is similar for all values of λ. It is only the value of c (dependent upon λ) that distinguishes one member of the family from another. Thus one can come up with the universal curves for f, g and h, and from them the flow can be computed for any given value of λ by computing c from Eq. (27).

Note that in the integration scheme presented above, one has to monitor the value of f. A final artifice, suggested by Ackroyd, even eliminates the need of that. We use f as the independent variable rather than Z. This transforms the system (23)–(26) into

$$\frac{\mathrm{d}Z}{\mathrm{d}f} = \frac{Z}{g}, \quad Z(0) = 0, \tag{28}$$

$$\frac{\mathrm{d}g}{\mathrm{d}f} = \frac{h}{g}, \quad g(0) = 0, \tag{29}$$

$$\frac{\mathrm{d}h}{\mathrm{d}f} = \frac{(1-2f)h + g^2}{g}, \quad h(0) = 0, \tag{30}$$

$$\frac{\mathrm{d}Z(0)}{\mathrm{d}f} = 1, \quad \frac{\mathrm{d}g(0)}{\mathrm{d}f} = 1, \quad \frac{\mathrm{d}h(0)}{\mathrm{d}f} = 1. \tag{31}$$

Ackroyd [13] recommends integrating Eqs. (28)–(30) by using some integration routine. We instead suggest using the power series expansions for Z, g and h in terms of f as under

$$Z = f + Z_2 f^2 + Z_3 f^3 + \cdots, \tag{32}$$

$$g = f + g_2 f^2 + g_3 f^3 + \cdots, \tag{33}$$

$$h = f + h_2 f^2 + h_3 f^3 + \cdots. \tag{34}$$

Substitution of Z, g and h from Eqs. (32) to (34) into Eqs. (28)–(30) leads to the following recurrence relations ($n > 2$):

$$(n-1)Z_n = -\sum_{m=2}^{n-1} m Z_m g_{n+1-m}, \tag{35}$$

$$(n+1)g_n - h_n = -\sum_{m=2}^{n-1} m g_m g_{n+1-m}, \tag{36}$$

$$g_n + (n-1)h_n = -\sum_{m=2}^{n-1} m h_m g_{n+1-m} + \sum_{m=2}^{n-1} g_m g_{n-m} + g_{n-1} - 2h_{n-1}. \tag{37}$$

For $n = 2$, we obtain

$$Z_2 = \frac{1}{4}, \quad g_2 = -\frac{1}{4}, \quad h_2 = -\frac{3}{4}. \tag{38}$$

Eqs. (36) and (37) can be readily solved for g_n and h_n, after which Z_n can be obtained from Eq. (35). The values of the coefficients are listed in Table 1. Note the rapid decaying of the coefficients.

The functions Z, g and h can now be computed for $0 \le f \le \frac{1}{2}$ from Eqs. (32) to (34). The graphs of f, g, and h are depicted in Fig. 1. As remarked earlier, these are universal functions, and use can be made of these for computing the flow for any value of λ. The values of these functions are also listed in Table 2.

The value of c is given by Eq. (27), which in view of the values of g and h at $f = \frac{1}{2}$, can be rewritten as

$$0.3456949134\lambda c^3 + 0.4426754872c^2 - 1 = 0.$$

The above equation can be explicitly solved for c giving

$$c = \begin{cases} 1.3016310342 \ \sec\left[\dfrac{1}{3}\cos^{-1}(3.0494159309\lambda)\right], & \lambda \le 0.3279316507 \\ 1.3016310342 \ \mathrm{sech}\left[\dfrac{1}{3}\cosh^{-1}(3.0494159309\lambda)\right], & \lambda > 0.3279316507. \end{cases} \tag{39}$$

A similar expression was obtained by Anderssen [12] for the two-dimensional flow.

Table 1
The values of Z_n, g_n and h_n

n	Z_n	g_n	h_n
1	1.0000000000	1.0000000000	1.0000000000
2	0.2500000000	−0.2500000000	−0.7500000000
3	0.0416666667	0.0416666667	0.2916666667
4	0.0034722222	0.0000000000	−0.0520833333
5	−0.0001736111	−0.0010416667	−0.0010416667
6	−0.0000462963	−0.0000434028	0.0015190972
7	0.0000119481	0.0000469459	0.0001151502
8	0.0000033144	0.0000057022	−0.0000705849
9	−0.0000004536	−0.0000022483	−0.0000117518
10	−0.0000002384	−0.0000005471	0.0000032749
11	0.0000000074	0.0000000912	0.0000010363
12	0.0000000162	0.0000000447	−0.0000001149
13	0.0000000011	−0.0000000016	−0.0000000807
14	−0.0000000010	−0.0000000032	−0.0000000004
15	−0.0000000002	−0.0000000002	0.0000000056
16	0.0000000001	0.0000000002	0.0000000006
17	0.0000000000	0.0000000000	−0.0000000003
18	0.0000000000	0.0000000000	−0.0000000001

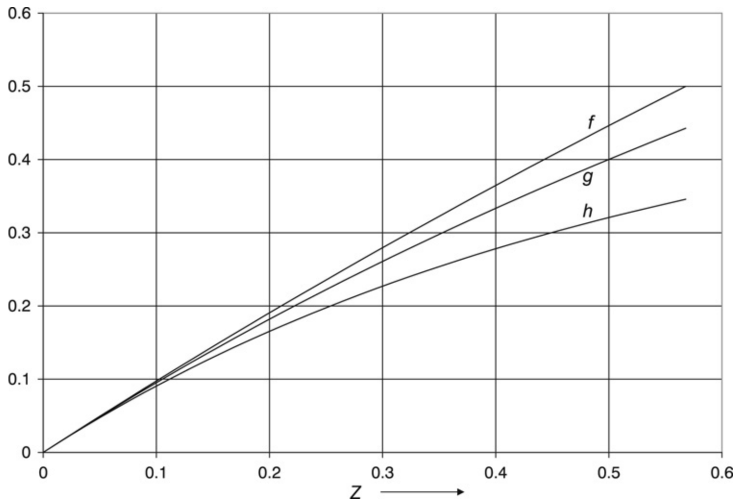

Fig. 1. Illustrating the variation of the functions f, g and h occurring in Section 4 with Z, the variable defined by Eq. (18).

A quantity of immense physical interest is the skin-friction at the wall, and is represented by $-\phi''(0)$. It is given by c^3h when $f = \frac{1}{2}$. Thus we have

$$-\phi''(0) = \begin{cases} 0.7623539828\ \sec^3\left[\dfrac{1}{3}\cos^{-1}(3.0494159309\lambda)\right], & \lambda \leq 0.3279316507 \\ 0.7623539828\ \mathrm{sech}^3\left[\dfrac{1}{3}\cosh^{-1}(3.0494159309\lambda)\right], & \lambda > 0.3279316507. \end{cases} \tag{40}$$

This is a very important by-product of the non-iterative method. It permits the skin-friction to be expressed analytically as a function of λ, the slip parameter. In contrast, for the MHD flow in the absence of velocity slip, no such analytical expression could be developed for the skin-friction in terms of the Hartmann number (Ariel [32]). In Fig. 2, $-\phi''(0)$ and c have been plotted against λ on a log–log graph. The asymptotic nature of $-\phi''(0)$ and c for large values of λ is rather obvious from the figure. We shall return back to this theme in a later section when we consider the asymptotic solution for large λ.

Table 2
Illustrating the variation of Z, g and h with f

f	Z	g	h
0	0	0	0
0.025	0.0251569024	0.0248444010	0.0245357869
0.050	0.0506302300	0.0493802080	0.0481611325
0.075	0.0764239376	0.0736113256	0.0709026467
0.100	0.1025420121	0.0975416562	0.0927864494
0.125	0.1289884724	0.1211750983	0.1138381699
0.150	0.1557673691	0.1445155455	0.1340829462
0.175	0.1828827841	0.1675668853	0.1535454257
0.200	0.2103388305	0.1903329978	0.1722497652
0.225	0.2381396526	0.2128177545	0.1902196310
0.250	0.2662894250	0.2350250168	0.2074782007
0.275	0.2947923530	0.2569586351	0.2240481633
0.300	0.3236526722	0.2786224477	0.2399517215
0.325	0.3528746478	0.3000202790	0.2552105928
0.350	0.3824625748	0.3211559388	0.2698460117
0.375	0.4124207776	0.3420332210	0.2838787318
0.400	0.4427536098	0.3626559022	0.2973290286
0.425	0.4734654534	0.3830277409	0.3102167020
0.450	0.5045607196	0.4031524761	0.3225610793
0.475	0.5360438473	0.4230338259	0.3343810189
0.500	0.5679193037	0.4426754872	0.3456949134

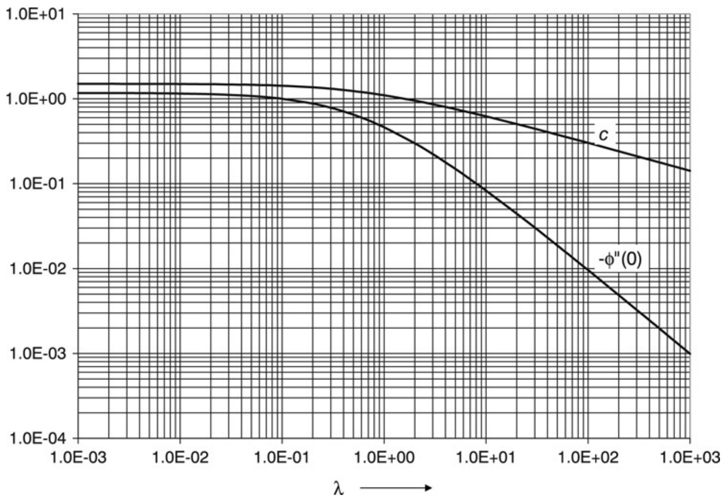

Fig. 2. Illustrating the variation of c, the scaling parameter, and $-\varphi''(0)$, a dimensionless measure of the skin-friction at the sheet, with λ, a measure of the partial slip.

Once c is found, and Z, g and h computed for various values of f, the velocity profiles can be immediately obtained from Eq. (22). The asymptotic value of the velocity component normal to the sheet is given from Eq. (12). In Figs. 3 and 4, the velocity components normal and along the sheet have been plotted against ζ for various values of λ. Consistent with the two-dimensional case, both the velocity components decrease as the value of λ is increased.

5. Homotopy perturbation method

For the flow without the partial slip at the boundary, HPM has been shown to be an excellent technique for obtaining an analytical solution that is elegant and also sufficiently accurate (Ariel et al. [29]). Now we apply the method to

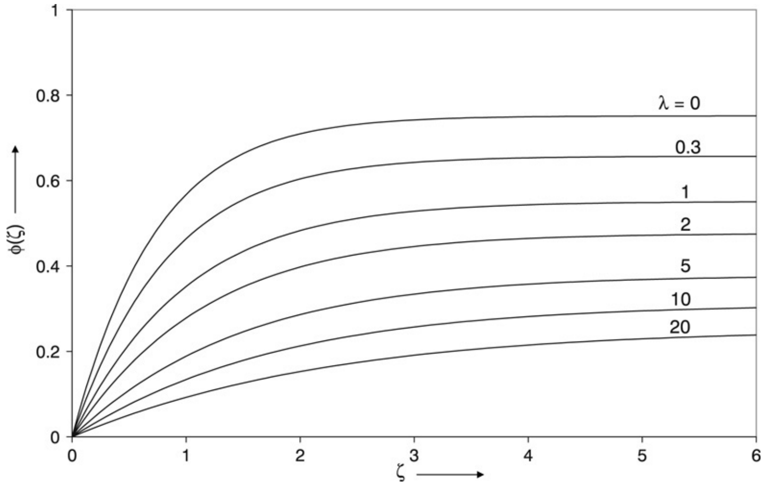

Fig. 3. Illustrating the variation of $\phi(\zeta)$, the velocity normal to the sheet, with ζ, the dimensionless distance normal to the sheet, for various values of λ, a measure of the partial slip.

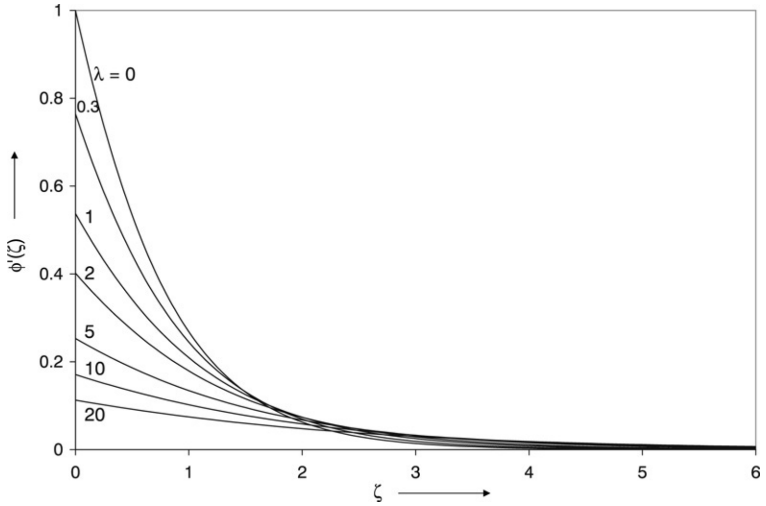

Fig. 4. Illustrating the variation of $\phi(\zeta)$, the mainstream velocity, with ζ, the dimensionless distance normal to the sheet, for various values of λ, a measure of the partial slip.

derive an analytical solution when there is a partial slip at the boundary. We reformulate differential equation (6) as

$$\phi''' - b^2\phi' + p(2\phi\phi'' - \phi'^2 + b^2\phi') = 0, \tag{41}$$

where p is the perturbation parameter and b is an additional adjustable parameter, which needs to be chosen to satisfy some suitable requirement of the solution. When $p = 1$, the original differential equation is recovered.

We seek a perturbation solution for ϕ as under

$$\phi = \phi_0 + p\phi_1 + p^2\phi_2 + \cdots. \tag{42}$$

Substituting for ϕ from Eq. (42) into (41) and equating like powers of p, we obtain the following systems of BVPs

Zeroth order:

$$\phi_0''' - b^2\phi_0' = 0, \tag{43}$$

$$\phi_0(0) = 0, \quad \phi_0'(0) - 1 = \lambda\phi_0'', \quad \phi_0'(\infty) = 0. \tag{44}$$

Higher order:

$$\phi_n''' - b^2\phi_n' + \sum_{m=0}^{n-1}(2\phi_m\phi_{n-m-1}'' - \phi_m'\phi_{n-m-1}') + b^2\phi_{n-1}' = 0, \quad n \geq 1, \tag{45}$$

$$\phi_n(0) = 0, \quad \phi_n'(0) = \lambda\phi_n''(0), \quad \phi_n'(\infty) = 0. \tag{46}$$

The solution of the zeroth-order system is

$$\phi_0 = \frac{1}{b(1+\lambda b)}(1 - \mathrm{e}^{-b\zeta}). \tag{47}$$

When the value of ϕ_0 is substituted in the first-order system, the following differential equation for ϕ_1 results:

$$\phi_1''' - b^2\phi_1' + \left[-\frac{2}{(1+\lambda b)^2} + \frac{b^2}{1+\lambda b}\right]\mathrm{e}^{-b\zeta} + \frac{1}{(1+\lambda b)^2}\mathrm{e}^{-2b\zeta} = 0. \tag{48}$$

Now He recommends that the adjustable parameter must be so chosen that the solution may be free of the secular term. This can be ensured if we choose

$$b^2(1+\lambda b) - 2 = 0. \tag{49}$$

Eq. (49) can be explicitly solved for b to yield

$$b = \begin{cases} \sqrt{\frac{3}{2}}\ \sec\left(\frac{1}{3}\cos^{-1}\frac{3\sqrt{3}\lambda}{\sqrt{2}}\right), & \lambda \leq \frac{1}{3}\sqrt{\frac{2}{3}} \\ \sqrt{\frac{3}{2}}\ \mathrm{sech}\left(\frac{1}{3}\cosh^{-1}\frac{3\sqrt{3}\lambda}{\sqrt{2}}\right), & \lambda > \frac{1}{3}\sqrt{\frac{2}{3}}. \end{cases} \tag{50}$$

With this choice of b, Eq. (48) reduces to

$$\phi_1''' - b^2\phi_1' + \frac{1}{(1+\lambda b)^2}\mathrm{e}^{-2b\zeta} = 0, \tag{51}$$

the solution of which, subject to the boundary conditions (46) for $n = 1$, is

$$\phi_1 = \frac{1 - \mathrm{e}^{-b\zeta}}{6b^3(1+\lambda b)^3}\left[(1 + 3\lambda b) - (1+\lambda b)\mathrm{e}^{-b\zeta}\right]. \tag{52}$$

As pointed out in [29], the second-order solution cannot be derived without getting the secular terms. However, it has been demonstrated repeatedly that even the first-order perturbation in HPM gives very good results. Therefore, we write the final solution as

$$\phi = \phi_0 + \phi_1, \tag{53}$$

which is obtained after truncating the solution after the first degree terms in p in Eq. (42) and setting $p = 1$. Thus we have

$$\phi(\zeta) = \frac{1 - \mathrm{e}^{-b\zeta}}{b(1+\lambda b)}\left[1 + \frac{(1+3\lambda b) - (1+\lambda b)\mathrm{e}^{-b\zeta}}{6b^2(1+\lambda b)}\right], \tag{54}$$

where b is given by Eq. (50).

From Eq. (54) we derive

$$-\phi''(0) = \frac{b}{1+\lambda b}\left[1 - \frac{1}{3b^2(1+\lambda b)^2}\right]. \tag{55}$$

To see how accurate our solution using the HPM is, we compare the value of $-\phi''(0)$ derived above with the exact solution (40). This is done in Section 8, where a comparison has been made of the results obtained by other techniques also. At this point, we may point out that when $\lambda = 0$, i.e., when there is no partial slip Ariel et al. [29] have already shown that the value of $-\phi''(0)$ obtained by the HPM method is within 0.4% of the true value.

6. A perturbation solution valid for small λ

When $\lambda = 0$, i.e., in the absence of a partial slip, a series solution for $\phi(\zeta)$ in exponentials $e^{-nc\zeta}$ has been derived by Ariel [32,33]. When $\lambda \neq 0$, i.e., when there is a partial slip at the boundary, it has been shown in Section 3 that a similar series solution still characterizes the flow. For small values of λ it seems natural to assume that c and a_ns would be only marginally affected by the presence of λ. In view of the nature of the boundary condition (7b) we seek a solution of the form

$$c = c_0 + \lambda c_1 + \lambda^2 c_2 + \cdots, \tag{56}$$

$$a_n = a_{n0} + \lambda a_{n1} + \lambda^2 a_{n2} + \cdots. \tag{57}$$

It has been demonstrated in [33] that a solution of the above form, which essentially strains the coordinate, is superior to the usual technique of writing the system of BVPs by expanding ϕ in the power series of λ, as the perturbed solution often includes the secular terms (Wang [34]).

Substituting for c and a_n from (56) and (57) in Eqs. (11), (13) and (14), we obtain the following system of algebraic equations:

Zeroth order:

$$a_{n0} = -\frac{1}{n(n-1)c_0^2} \sum_{m=1}^{n-1} \left(\frac{2n}{m} - 3\right) a_{m0} a_{n-m,0}, \tag{58}$$

$$\sum_{n=1}^{\infty} a_{n0} = 1, \quad \sum_{n=1}^{\infty} \frac{a_{n0}}{n} = \frac{1}{2} c_0^2. \tag{59}$$

First order:

$$a_{n1} = -\frac{1}{n(n-1)c_0^2} \sum_{m=1}^{n-1} \left(\frac{2n}{m} - 3\right) (a_{m1} a_{n-m,0} + a_{m0} a_{n-m,1}) - 2\frac{c_1}{c_0} a_{n0}, \tag{60}$$

$$\sum_{n=1}^{\infty} (a_{n1} + n c_0 a_{n0}) = 0, \quad \sum_{n=1}^{\infty} \frac{a_{n1}}{n} = c_0 c_1. \tag{61}$$

Second order:

$$\begin{aligned} a_{n2} = {} & -\frac{1}{n(n-1)c_0^2} \sum_{m=1}^{n-1} \left(\frac{2n}{m} - 3\right) (a_{m2} a_{n-m,0} + a_{m1} a_{n-m,1} \\ & + a_{m0} a_{n-m,2}) - \left(2\frac{c_2}{c_0} + \frac{c_1^2}{c_0^2}\right) a_{n0} - 2\frac{c_1}{c_0} a_{n1}, \end{aligned} \tag{62}$$

$$\sum_{n=1}^{\infty} (a_{n2} + n c_0 a_{n1} + n c_1 a_{n0}) = 0, \quad \sum_{n=1}^{\infty} \frac{a_{n2}}{n} = c_0 c_2 + \frac{1}{2} c_1^2. \tag{63}$$

Only the zeroth-order system is non-linear. It has already been reported in the literature (Ariel [32,33]). All the higher order systems are linear and can be solved easily (Ariel [33]). In Table 3, the values of a_{ni} $(i = 0, 1, 2)$ are presented. Only seventeen terms are required for a ten-digit accuracy.

The values of c and $-\phi''(0)$ are listed below up to the second degree terms in λ.

$$c = 1.5029940560 - 0.8820476471\lambda + 1.2940970203\lambda^2 + O(\lambda^3), \tag{64}$$

$$-\phi''(0) = 1.1737207392 - 2.0664305601\lambda + 4.2444717075\lambda^2 + O(\lambda^3). \tag{65}$$

Table 3
The values of a_{n0}, a_{n1} and a_{n2} for the perturbation solution

n	a_{n0}	a_{n1}	a_{n2}
1	1.2829246706	−1.5057952948	2.6510747411
2	−0.3642988428	0.4275851077	−0.7527982605
3	0.1034461726	−0.1214169183	0.2137643319
4	−0.0277426203	0.0325620889	−0.0573281983
5	0.0070668364	−0.0082944924	0.0146031266
6	−0.0017290486	0.0020294201	−0.0035729588
7	0.0004102413	−0.0004815088	0.0008477352
8	−0.0000950739	0.0001115902	−0.0001964636
9	0.0000216362	−0.0000253947	0.0000447096
10	−0.0000048538	0.0000056971	−0.0000100301
11	0.0000010765	−0.0000012635	0.0000022246
12	−0.0000002365	0.0000002777	−0.0000004888
13	0.0000000516	−0.0000000605	0.0000001066
14	−0.0000000112	0.0000000131	−0.0000000231
15	0.0000000024	−0.0000000028	0.0000000050
16	−0.0000000005	0.0000000006	−0.0000000011
17	0.0000000001	−0.0000000001	0.0000000003
18	0.0000000000	0.0000000001	0.0000000000

It may be remarked here that the above expansions could also have been deduced directly from Eqs. (39) and (40) by seeking a Taylor series expansion around $\lambda = 0$. However, we have preferred the present approach as it is direct and also gives the coefficients occurring in the series expansion of ϕ, which would allow the velocity field to be computed conveniently. It is worth remarking that the coefficients a_{ni} $(i = 0, 1, 2)$ in Eq. (57), as well as those in Eqs. (64) and (65) are increasing as we move to the higher order system. This does not auger well for the perturbation solution as we shall see presently.

7. An asymptotic solution valid for large λ

It is instructive to note from Fig. 2, that for large values of λ, both c and $-\phi''(0)$ exhibit a power relationship with λ. In particular, we note that $c = O(\lambda^{-1/3})$—a fact which can also be deduced from Eq. (39). From Eq. (14), then, we can conclude that $a_n = O(\lambda^{-2/3})$. Thus we can formally write

$$c = \lambda^{-1/3}\left(c_0 + \frac{c_1}{\lambda^q} + \frac{c_2}{\lambda^{2q}} + \cdots\right),$$
$$a_n = \lambda^{-2/3}\left(a_{n0} + \frac{a_{n1}}{\lambda^q} + \frac{a_{n2}}{\lambda^{2q}} + \cdots\right),$$

where q is a constant (the constants c_is and a_{ni} $(i = 0, 1, 2 \cdots)$ are different here than those in the preceding section).

If we substitute for c and a_n from above into Eq. (13) and compare the powers of λ, we find $q = \frac{2}{3}$. Therefore, we must have

$$c = \lambda^{-1/3}\left(c_0 + \frac{c_1}{\lambda^{2/3}} + \frac{c_2}{\lambda^{4/3}} + \cdots\right), \tag{66}$$

$$a_n = \lambda^{-2/3}\left(a_{n0} + \frac{a_{n1}}{\lambda^{2/3}} + \frac{a_{n2}}{\lambda^{4/3}} + \cdots\right). \tag{67}$$

Substituting for c and a_n in Eqs. (11), (13) and (14), we obtain the following system of equations:

Zeroth order:

$$a_{n0} = -\frac{1}{n(n-1)c_0^2}\sum_{m=1}^{n-1}\left(\frac{2n}{m} - 3\right)a_{m0}a_{n-m,0}, \tag{68}$$

$$c_0\sum_{n=1}^{\infty} n a_{n0} = 1, \qquad \sum_{n=1}^{\infty}\frac{a_{n0}}{n} = \frac{1}{2}c_0^2. \tag{69}$$

Table 4
The values of a_{n0}, a_{n1} and a_{n2} for the asymptotic solution

n	a_{n0}	a_{n1}	a_{n2}
1	1.1529876031	−0.6908072078	0.3104204659
2	−0.3274019583	0.1961613742	−0.0881468874
3	0.0929689461	−0.0557019155	0.0250301595
4	−0.0249327946	0.0149383689	−0.0067126912
5	0.0063510936	−0.0038052285	0.0017099138
6	−0.0015539272	0.0009310283	−0.0004183660
7	0.0003686913	−0.0002208997	0.0000992633
8	−0.0000854446	0.0000511938	−0.0000230044
9	0.0000194448	−0.0000116503	0.0000052351
10	−0.0000043622	0.0000026136	−0.0000011745
11	0.0000009675	−0.0000005797	0.0000002604
12	−0.0000002126	0.0000001274	−0.0000000572
13	0.0000000464	−0.0000000277	0.0000000125
14	−0.0000000100	0.0000000060	−0.0000000027
15	0.0000000022	−0.0000000013	0.0000000006
16	−0.0000000005	0.0000000003	−0.0000000001
17	0.0000000001	−0.0000000001	0.0000000000
18	0.0000000000	0.0000000000	0.0000000000

First order:

$$a_{n1} = -\frac{1}{n(n-1)c_0^2}\sum_{m=1}^{n-1}\left(\frac{2n}{m}-3\right)(a_{m1}a_{n-m,0}+a_{m0}a_{n-m,1}) - 2\frac{c_1}{c_0}a_{n0}, \tag{70}$$

$$c_0\sum_{n=1}^{\infty} na_{n1} + \sum_{n=1}^{\infty}(1+nc_1)a_{n0} = 0, \qquad \sum_{n=1}^{\infty}\frac{a_{n1}}{n} = c_0c_1. \tag{71}$$

Second order:

$$\begin{aligned} a_{n2} = &-\frac{1}{n(n-1)c_0^2}\sum_{m=1}^{n-1}\left(\frac{2n}{m}-3\right)(a_{m2}a_{n-m,0}+a_{m1}a_{n-m,1} \\ &+a_{m0}a_{n-m,2}) - \left(2\frac{c_2}{c_0}+\frac{c_1^2}{c_0^2}\right)a_{n0} - 2\frac{c_1}{c_0}a_{n1}, \end{aligned} \tag{72}$$

$$c_0\sum_{n=1}^{\infty} na_{n2} + \sum_{n=1}^{\infty}[(1+nc_1)a_{n1}+nc_2a_{n0}] = 0, \qquad \sum_{n=1}^{\infty}\frac{a_{n2}}{n} = c_0c_2+\frac{1}{2}c_1^2. \tag{73}$$

The set of Eqs. (68)–(73) is almost similar to that for the perturbation solution. Only one of the conditions – corresponding to Eq. (13) – is different. We can, therefore, use the same technique for solving the above systems. The values of a_{ni} $(i = 0, 1$ and $2)$ are listed in Table 4.

The asymptotic expansions for c and $-\phi''(0)$ for large λ are

$$c = \frac{1}{\lambda^{1/3}}\left(1.4248495303 - \frac{0.4268460142}{\lambda^{2/3}} + \frac{0.1278714110}{\lambda^{4/3}}\right) + O\left(\frac{1}{\lambda^{7/3}}\right), \tag{74}$$

$$-\phi''(0) = \frac{1}{\lambda}\left(1 - \frac{0.8987180851}{\lambda^{2/3}} + \frac{0.4728808468}{\lambda^{4/3}}\right) + O\left(\frac{1}{\lambda^{3}}\right). \tag{75}$$

In contrast with the perturbation solution for small λ, here one can see that a_{ni} $(i = 0, 1, 2)$, and also the coefficients in Eqs. (74) and (75) are nicely decreasing as we move on to the higher order systems. Therefore, we expect the asymptotic solution to give good approximations.

Table 5
Illustrating the variation of $-\phi''(0)$ with λ using (i) the exact solution, (ii) the HPM solution (iii) the perturbation solution, and (iv) the asymptotic solution

	$-\phi''(0)$			
λ	Exact	HPM	Perturbation	Asymptotic
0	1.173721	1.178511	1.173721	
0.01	1.153472	1.157311	1.153481	
0.02	1.134017	1.136998	1.134090	
0.05	1.079949	1.080820	1.081010	
0.1	1.001834	1.000308	1.009522	
0.2	0.878425	0.874453	0.930213	
0.5	0.650528	0.645304	1.201623	1.529918
1	0.462510	0.458333		0.574163
2	0.299050	0.296534		0.310753
5	0.149393	0.148454		0.149590
10	0.082912	0.082532		0.082833
20	0.044368	0.044228		0.044337
50	0.018732	0.018698		0.018727
100	0.009594	0.009583		0.009593

8. Comparison of the results

We choose $-\phi''(0)$ as a representative number of the flow to compare the results obtained by various techniques enunciated in the preceding sections. In Table 5, the values of $-\phi''(0)$ are listed using (i) the exact solution given in Section 4, (ii) the HPM solution given in Section 5, (iii) the perturbation solution given in Section 6, and (iv) the asymptotic solution given in Section 7.

As might be expected, the perturbation solution and the asymptotic solution produce the best results for the limiting values of λ, which are either too small or too large. If we somewhat arbitrarily restrict to a maximum error of 1%, we note that the perturbation solution gives acceptable results only up to approximately $\lambda = 0.1$, while the asymptotic solution produces acceptable results for values of λ approximately greater than 3. The performance of the solution produced by the HPM is remarkable in that it consistently produces results which are well within the maximum allowed error. In their respective domains the perturbation method ($0 \leq \lambda \leq 0.05$), and the asymptotic method ($\lambda > 4$) outperform the HPM, however for moderate values of λ ($0.05 < \lambda < 4$), the HPM produces the best results out of the three methods, and even outside this domain, it gives results which are fully consistent with the exact solution. Moreover, the HPM generates an exquisite analytical solution which is simple enough and can be equally used to compute the velocity at any point in the domain as well as the stress at the sheet without the necessity of an interpolation.

9. Conclusion

In the present work we have computed the steady, laminar, axisymmetric flow of an incompressible, viscous, Newtonian fluid past a stretching sheet when there is a partial slip at the boundary. The flow is governed by a non-linear third-order differential equation in a semi-infinite domain. The BVP has been solved to obtain an exact solution using a non-iterative technique. Also the approximate solutions have been obtained by using (i) the HPM proposed by He, (ii) a perturbation solution valid for small λ, the slip parameter, and (iii) an asymptotic solution valid for large λ. A comparison of the results obtained by approximate solutions with those obtained by the exact solution shows that overall the HPM gives the best solution in that the solution is fully analytical and error is consistently within 1% for all values of λ. The HPM, therefore provides an attractive tool for computing the flow due to moving boundaries.

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