Alexandria Engineering Journal (2016) 55, 2855–2862

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Alexandria Engineering Journal
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ORIGINAL ARTICLE

Chaikin's perturbation subdivision scheme in non-stationary forms

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Received 31 March 2016; revised 3 July 2016; accepted 11 July 2016
Available online 30 July 2016

KEYWORDS

Binary subdivision schemes;
Non-stationary scheme;
Approximating;
Curvature

Abstract In this paper two non-stationary forms of Chaikin's perturbation subdivision scheme, mentioned in Dyn et al. (2004), have been proposed with tension parameter ω. Comparison among the proposed subdivision schemes and the existing non-stationary subdivision scheme depicts that the trigonometric form is more efficient in the reproduction of circles and ellipses and the hyperbolic form is more suitable for the construction of many analytical curves.

1. Introduction

Subdivision schemes ascend from modeling and interrogation of curves and surfaces, image reconstruction and decomposition, and the construction problems of compact supported wavelet. These schemes are being developed in geometric modeling with great potentiality in computer graphics, CAM/CAD and image processing. Subdivision schemes are widely used in garment CAD, jewelry CAD and computer graphics industry. These schemes are also important in fractal generation by computer particularly [2,3]. Subdivision schemes are used to construct the required curves and surfaces from scattered data directly through stated subdivision rules. Mask of the subdivision schemes is simply averaging rules corresponding to odd and even subsequences of finitely supported sequence of real numbers. In case of level dependent subdivision schemes [4–9] mask varies from one level to another; generally, it allows to generate larger variety of limiting curves having several useful properties *e.g.* reproduction of conics and spirals etc. New methods of convergence of non-stationary schemes have been introduced in [18,19]. In [18] asymptotic similarity has been used instead of asymptotic equivalence. In [19] spectral radius approach has been used along with the asymptotic similarity for convergence. Different properties of the non-stationary subdivision schemes *e.g.* approximation order and reproduction properties have been analyzed in [20–23].

Some numerical schemes have been presented by P. Das and S. Natesan to solve singularly perturbed reaction diffusion differential equations in [15–17].

In this paper, Chaikin's perturbation subdivision scheme [1] has been presented in trigonometric and hyperbolic forms with the abilities to reproduce conic-sections and many analytical curves.

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Peer review under responsibility of Faculty of Engineering, Alexandria University.

http://dx.doi.org/10.1016/j.aej.2016.07.002
1110-0168

The following basic results of the non-stationary subdivision schemes are considered to prove the convergence/ smoothness of the proposed non-stationary subdivision schemes.

Definition 1.1. Given a set of initial control points $P^0 = \{p_i^0 \in R^d\}_{i=-1}^{k+1}$, a binary subdivision scheme generates new set of control points $P^n = \{p_i^n\}_{i=-1}^{2^n k+1}$ at level $n (n \geqslant 0, k \in Z)$ by the subdivision rule

$$p_i^{n+1} = \sum_{j \in z} a_{i-2j}^n p_j^n, \quad i \in Z,$$

where the set of coefficients $a^{(n)} = \{a_i^{(n)}, i \in Z\}$ in above equation is termed as the mask of the subdivision scheme at n^{th} subdivision step. The Laurent polynomial associated with the non-stationary subdivision scheme $\{S_{a^n}\}$ having mask $a^{(n)}$ is

$$a^n(z) = \sum_{i \in Z} a_i^{(n)} z^i, \quad n \geqslant 1.$$

Definition 1.2 [10]. A binary subdivision scheme $\{S_{a^n}\}$ is said to be C^m if for every initial data $p^0 = \{p_i^0 : i \in Z\}$ there exists a limit function $f \in C^m$ such that for any closed interval $K = [a, b] \subset \mathbb{R}$,

$$\lim_{n \to \infty} \max_{i \in \mathbb{Z} \bigcap 2^n K} |p_i^n - f(2^{-n} i)| = 0.$$

Obviously $f = S^\infty p^0 \neq 0$ for some initial data p^0 and also p_i^n are the control points at level n.

Definition 1.3 [10]. Two binary subdivision schemes $\{S_{a^n}\}$ and $\{S_{b^n}\}$ are asymptotically equivalent if

$$\sum_{n=1}^{\infty} \|S_{a^n} - S_{b^n}\| < \infty,$$

where $\|S_{a^n}\|_\infty = \max\{\sum_{i \in Z} |a_{2i}^{(n)}|, \sum_{i \in Z} |a_{2i+1}^{(n)}|\}$.

Theorem 1.1 [11]. *Let $\{S_{a^n}\}$ and $\{S_a\}$ be the two asymptotically equivalent subdivision schemes having finite masks of the same support. Suppose $\{S_{a^n}\}$ is a level dependent subdivision scheme and $\{S_a\}$ is a stationary subdivision scheme. If $\{S_a\}$ is C^m and*

$$\sum_{n=0}^{\infty} 2^{mn} \|S_{a^n} - S_a\| < \infty,$$

then the non-stationary subdivision scheme $\{S_{a^n}\}$ is C^m.

The organization of paper is as follows. In Section 2, Chaikin's perturbation (binary four point approximating) subdivision scheme has been recalled. In Section 3, the trigonometric and hyperbolic forms of Chaikin's perturbation subdivision scheme have been presented. Convergence analysis of the proposed schemes has also been discussed in Section 3. The normalization of the proposed schemes has been given in Section 4 as these schemes do not observe the affine invariance property. In Section 5, some properties of the proposed schemes have been discussed. Graphical behavior of the proposed schemes has been exhibited along with their comparison in Section 6.

2. Chaikin's perturbation subdivision scheme

Given a set of control points $f^0 = \{f_i^0\}_{i \in \mathbb{Z}}$ at level 0, Chaikin's perturbation subdivision scheme [1] generates a new set of control points $\{f_i^n\}_{i \in \mathbb{Z}}$ at the level $n+1$ by applying the following subdivision rules:

$$\begin{cases} f_{2i}^{n+1} = -7\omega f_{i-1}^n + \left(\frac{3}{4} + 9\omega\right) f_i^n + \left(\frac{1}{4} + 3\omega\right) f_{i+1}^n - 5\omega f_{i+2}^n, \\ f_{2i+1}^{n+1} = -5\omega f_{i-1}^n + \left(\frac{1}{4} + 3\omega\right) f_i^n + \left(\frac{3}{4} + 9\omega\right) f_{i+1}^n - 7\omega f_{i+2}^n, \end{cases} \quad (1)$$

with $\omega = 0$ corresponds to the Chaikin's scheme [12]. The scheme gives C^1- continuous limit curves for $\omega = 0$ and C^2- continuous limit curves for $0 < \omega < \frac{\sqrt{6}-1}{80}$.

3. Non-stationary schemes for uniform trigonometric and hyperbolic spline curves

In this section, trigonometric and hyperbolic forms of Chaikin's perturbation subdivision scheme [1] have been presented.

3.1. Trigonometric form

The four point non-stationary subdivision scheme is

$$\begin{cases} f_{2i}^{n+1} = \beta_0^n f_{i-1}^n + \beta_1^n f_i^n + \beta_2^n f_{i+1}^n + \beta_3^n f_{i+2}^n, \\ f_{2i+1}^{n+1} = \beta_3^n f_{i-1}^n + \beta_2^n f_i^n + \beta_1^n f_{i+1}^n + \beta_0^n f_{i+2}^n, \end{cases} \quad (2)$$

where

$$\beta_0^n = -7\omega,$$

$$\beta_1^n = \frac{s\left(\frac{3\alpha}{2^{n+2}}\right)}{s\left(\frac{\alpha}{2^n}\right)} + 9\omega,$$

$$\beta_2^n = \frac{s\left(\frac{\alpha}{2^{n+2}}\right)}{s\left(\frac{\alpha}{2^n}\right)} + 3\omega,$$

$$\beta_3^n = -5\omega,$$

where $s(t) = \sin(t)$ and $c(t) = \cos(t)$.

3.2. Hyperbolic form

The four point non-stationary subdivision scheme is

$$\begin{cases} f_{2i}^{n+1} = \gamma_0^n f_{i-1}^n + \gamma_1^n f_i^n + \gamma_2^n f_{i+1}^n + \gamma_3^n f_{i+2}^n, \\ f_{2i+1}^{n+1} = \gamma_3^n f_{i-1}^n + \gamma_2^n f_i^n + \gamma_1^n f_{i+1}^n + \gamma_0^n f_{i+2}^n, \end{cases} \quad (3)$$

where

$$\gamma_0^n = -7\omega,$$

$$\gamma_1^n = \frac{s'\left(\frac{3\alpha}{2^{n+2}}\right)}{s'\left(\frac{\alpha}{2^n}\right)} + 9\omega,$$

$$\gamma_2^n = \frac{s'\left(\frac{\alpha}{2^{n+2}}\right)}{s'\left(\frac{\alpha}{2^n}\right)} + 3\omega,$$

$$\gamma_3^n = -5\omega,$$

where $s'(t) = \sinh(t)$ and $c'(t) = \cosh(t)$.

3.3. Continuity analysis

Asymptotic equivalence is needed to be established for continuity analysis.

For this, the schemes (2) and (3) will be denoted by $\{S_{d^n}\}$ and $\{S_{d'^n}\}$, where d^n and d'^n are the Laurent polynomials $d^n = \sum_{i\in\mathbb{Z}} d_i^n z^i$ and $d'^n = \sum_{i\in\mathbb{Z}} d_i'^n z^i$ respectively. Firstly, some estimations of β_i^n and $\gamma_i^n, i = 0,\dots,3$ should be given and for this the following six inequalities are needed:

$$\frac{s(x)}{s(y)} \geqslant \frac{x}{y}, \quad 0 < x \leqslant y < \frac{\pi}{2},$$

$$\theta\csc(\theta) < t\csc(t), \quad 0 < \theta < t < \frac{\pi}{2}$$

$$c(x) < \frac{s(x)}{x}, \quad 0 < x < \frac{\pi}{2}.$$

$$\frac{s'(x)}{s'(y)} \leqslant \frac{x}{y}, \quad 0 < x \leqslant y, \quad x, y \in \mathbb{R}^+ \setminus n'\pi,$$

$$\theta\operatorname{csch}(\theta) > t\operatorname{csch}(t), \quad 0 < \theta < t, \quad \theta, t \in \mathbb{R}^+ \setminus n'\pi$$

$$c'(x) > \frac{s'(x)}{x}, \quad x \in \mathbb{R}^+ \setminus n'\pi.$$

Lemma 3.1. *For $n > 0$ and $0 < \alpha < \frac{\pi}{3}$*

i. $\beta_0^n = -7\omega$,
ii. $\frac{3}{4} + 9\omega \leqslant \beta_1^n \leqslant \frac{3}{4}\frac{1}{c\left(\frac{\alpha}{2^n}\right)} + 9\omega$,
iii. $\frac{1}{4} + 3\omega \leqslant \beta_2^n \leqslant \frac{1}{4}\frac{1}{c\left(\frac{\alpha}{2^n}\right)} + 3\omega$,
iv. $\beta_3^n = -5\omega$,
v. $\gamma_0^n = -7\omega$,
vi. $\frac{3}{4} + 9\omega \geqslant \gamma_1^n > \frac{3}{4}\frac{1}{c'\left(\frac{\alpha}{2^n}\right)} + 9\omega$,
vii. $\frac{1}{4} + 3\omega \geqslant \gamma_2^n > \frac{1}{4}\frac{1}{c'\left(\frac{\alpha}{2^n}\right)} + 3\omega$,
viii. $\gamma_3^n = -5\omega$.

Proof.

$$\beta_1^n = \frac{s\left(\frac{3\alpha}{2^{n+2}}\right)}{s\left(\frac{\alpha}{2^n}\right)} + 9\omega \geqslant \frac{\left(\frac{3\alpha}{2^{n+2}}\right)}{\left(\frac{\alpha}{2^n}\right)} + 9\omega,$$
$$\geqslant \frac{3}{4} + 9\omega$$

and

$$\beta_1^n \leqslant \frac{3\alpha}{2^{n+2}}\frac{1}{\frac{\alpha}{2^n}\,c\left(\frac{\alpha}{2^n}\right)} + 9\omega,$$
$$= \frac{3}{4}\frac{1}{c\left(\frac{\alpha}{2^n}\right)} + 9\omega\cdot$$

This proves (ii). The other proofs are similar. □

Lemma 3.2. *For the constants C_i, $i = 1,\dots,4$, independent of n, the following inequalities hold:*

i. $\left|\beta_0^n + 7\omega\right| = 0$,
ii. $\left|\beta_1^n - \frac{3}{4} - 9\omega\right| \leqslant C_1\frac{1}{2^{2n}}$,
iii. $\left|\beta_2^n - \frac{1}{4} - 3\omega\right| \leqslant C_2\frac{1}{2^{2n}}$,
iv. $\left|\beta_3^n + 5\omega\right| = 0$,
v. $\left|\gamma_0^n + 7\omega\right| = 0$,
vi. $\left|\gamma_1^n - \frac{3}{4} - 9\omega\right| < C_3\frac{1}{2^{2n}}$,
vii. $\left|\gamma_2^n - \frac{1}{4} - 3\omega\right| < C_4\frac{1}{2^{2n}}$,
viii. $\left|\gamma_3^n + 5\omega\right| = 0$,

Proof. Using above lemmas, inequality (i) can be proved by taking

$$\beta_1^n \leqslant \frac{3}{4}\frac{1}{c\left(\frac{\alpha}{2^n}\right)} + 9\omega,$$

$$\beta_1^n - \frac{3}{4} - 9\omega \leqslant \frac{3}{4}\left(\frac{1 - c\left(\frac{\alpha}{2^n}\right)}{c\left(\frac{\alpha}{2^n}\right)}\right)$$
$$= \frac{3}{2}\left(\frac{s^2\left(\frac{\alpha}{2^{n+1}}\right)}{c\left(\frac{\alpha}{2^n}\right)}\right)$$
$$\left|\beta_1^n - \frac{3}{4} - 9\omega\right| \leqslant \frac{3}{8}\frac{\alpha^2}{c(\alpha)}\frac{1}{2^{2n}}$$

where $C_1 = \frac{3}{8}\frac{\alpha^2}{c(\alpha)}$, this proves (ii). □

Lemma 3.3. *The symbol $d^n(z)$ corresponding to the n^{th} level of the non-stationary subdivision scheme $\{S_{d^n}\}$ can be written in the form of*

$$d^n(z) = \left(\frac{1+z}{2}\right)e^n(z)$$

where

$$e^n(z) = 2\{\beta_3^n z^{-4} + (\beta_0^n - \beta_3^n)z^{-3} - (\beta_0^n - \beta_2^n - \beta_3^n)z^{-2} + (\beta_0^n - \beta_2^n + \beta_1^n - \beta_3^n)z^{-1} + (-\beta_0^n + \beta_3^n + \beta_2^n) + (\beta_0^n - \beta_3^n)z + \beta_3^n z^2\}$$

Corollary 3.1. *The symbol $d'^n(z)$ corresponding to the n^{th} level of the non-stationary scheme $\{S_{d'^n}\}$ can be written in the form of*

$$d'^n(z) = \left(\frac{1+z}{2}\right)e'^n(z)$$

where

$$e'^n(z) = 2\{\gamma_3^n z^{-4} + (\gamma_0^n - \gamma_3^n)z^{-3} - (\gamma_0^n - \gamma_2^n - \gamma_3^n)z^{-2} + (\gamma_0^n - \gamma_2^n + \gamma_1^n - \gamma_3^n)z^{-1} + (-\gamma_0^n + \gamma_3^n + \gamma_2^n) + (\gamma_0^n - \gamma_3^n)z + \gamma_3^n z^2\}.$$

In view of [13], it is sufficient to show that the schemes $\{S_{d^n}\}$ and $\{S_{d'^n}\}$ associated with $\{d^n(z)\}$ and $\{d'^n(z)\}$ are C^1 in order to prove that the proposed schemes are C^2. The result of Theorem 8(a) of [14] will be used to prove this. In order to use the result of Theorem 8(a) given in [14], the schemes $\{S_{d^n}\}$ and $\{S_{d'^n}\}$ will be compared with $\{S_d\}$ which is defined in the following lemma.

Lemma 3.4. *The stationary subdivision scheme S_d associated with the symbol*

$$d(z) = 2\left\{5\omega z^{-4} + 2\omega z^{-3} - \frac{1}{4}(1 + 20\omega)z^{-2} + \frac{1}{2}(-1 - 8\omega)z^{-1} + \frac{1}{4}(-1 - 20\omega) + 2\omega z + 5\omega z^2\right\}$$

is C^2 for the range $\omega \in \left]0, \frac{\sqrt{6}-1}{80}\right[$.

Theorem 3.1. *The non-stationary subdivision scheme defined in (2) is asymptotically equivalent to the stationary scheme (1). Hence, it generates C^2 limit curves.*

Proof.

$$\sum_{n=0}^{\infty} 2^n \|S_{d^n} - S_d\|_\infty < \infty$$

where

$$\|S_{d^n} - S_d\|_\infty = \max\left\{\sum_{j\in\mathbb{Z}} |d^n_{i+2j} - d_{i+2j}| : i = 0, 1\right\} = \max\left\{\sum_{j\in\mathbb{Z}} |d^n_{2j} - d_{2j}|, \sum_{j\in\mathbb{Z}} |d^n_{1+2j} - d_{1+2j}|\right\},$$

$$\sum_{j\in\mathbb{Z}} |d^n_{1+2j} - d_{1+2j}| = |\beta^n_0 + 7\omega| + \left|\beta^n_1 - \frac{3}{4} - 9\omega\right| + \left|\beta^n_2 - \frac{1}{2} - 3\omega\right| + |\beta^n_3 + 5\omega| = \sum_{j\in\mathbb{Z}} |d^n_{2j} - d_{2j}|,$$

$$\sum_{n=0}^{\infty} 2^n \|S_{d^n} - S_d\|_\infty = \sum_{n=0}^{\infty} 2^n \max\left\{|\beta^n_0 + 7\omega| + \left|\beta^n_1 - \frac{3}{4} - 9\omega\right| + \left|\beta^n_2 - \frac{1}{2} - 3\omega\right| + |\beta^n_3 + 5\omega|\right\}.$$

From the results of Lemma (3.2), it can be noted

$$\sum_{n=0}^{\infty} 2^n \left|\beta^n_1 - \frac{3}{4} - 9\omega\right| < \infty, \quad \sum_{n=0}^{\infty} 2^n \left|\beta^n_2 - \frac{1}{2} - 3\omega\right| < \infty,$$

and thus it can be written as

$$\sum_{n=0}^{\infty} 2^n \|S_{d^n} - S_d\|_\infty < \infty$$

and hence $\{S_{d^n}\}$ is C^1 and the scheme (2) is C^2. □

Corollary 3.2. *The hyperbolic form of the Chaikin's perturbation scheme defined in (3) is C^2.*

In the following section normalized schemes corresponding to proposed schemes (2) and (3) have been presented.

4. Normalization

Since the sums of the masks of the proposed four point schemes are not equal to one *i.e.* the sum of the trigonometric form (2) of four point scheme is

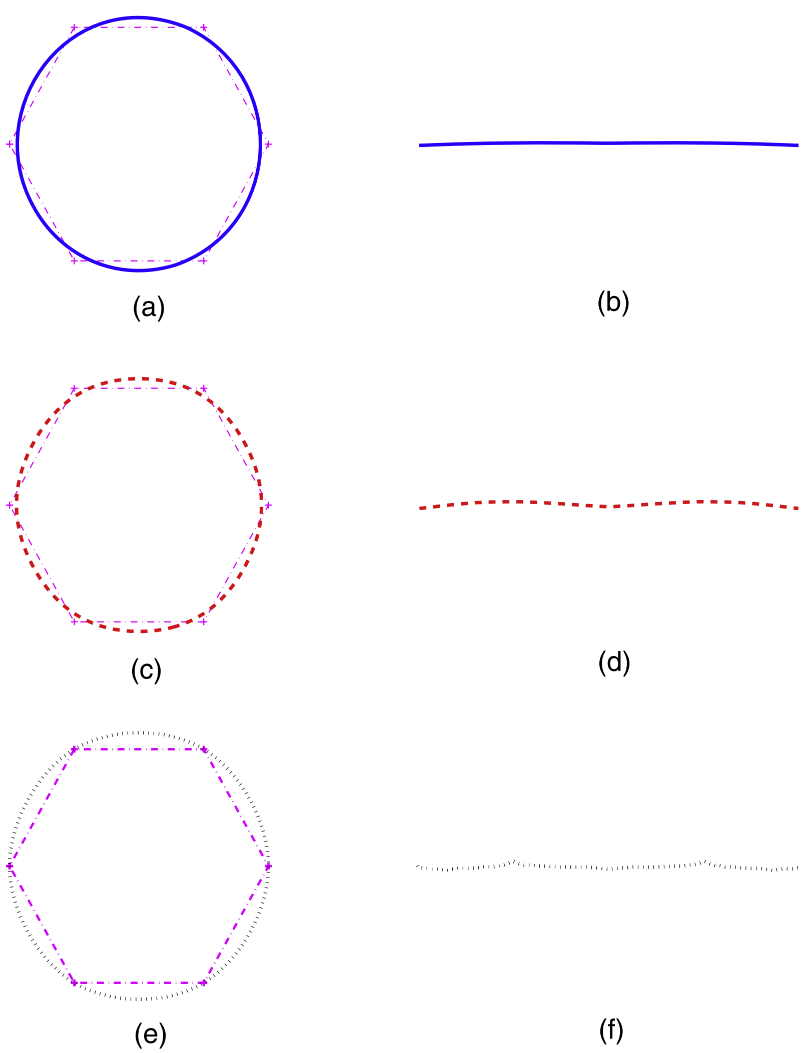

Figure 1 Construction of circles along with the associated curvature plots.

$$\beta^n = \sum_{i=0}^{3} \beta_i^n = \frac{c\left(\frac{\alpha}{2^{n+2}}\right)}{c\left(\frac{\alpha}{2^{n+1}}\right)}.$$

The four point normalized non-stationary subdivision scheme is

$$\begin{cases} f_{2i}^{n+1} = \beta_0'^n f_{i-1}^n + \beta_1'^n f_i^n + \beta_2'^n f_{i+1}^n + \beta_3'^n f_{i+2}^n, \\ f_{2i+1}^{n+1} = \beta_3'^n f_{i-1}^n + \beta_2'^n f_i^n + \beta_1'^n f_{i+1}^n + \beta_0'^n f_{i+2}^n, \end{cases} \tag{4}$$

where

$$\beta_i'^n = \frac{1}{\beta^n} \beta_i^n.$$

The sum of the hyperbolic form (3) of four point scheme is

$$\gamma^n = \sum_{i=0}^{3} \gamma_i^n = \frac{c'\left(\frac{\alpha}{2^{n+2}}\right)}{c'\left(\frac{\alpha}{2^{n+1}}\right)},$$

and the corresponding normalized scheme can be obtained as follows

$$\begin{cases} f_{2i}^{n+1} = \gamma_0'^n f_{i-1}^n + \gamma_1'^n f_i^n + \gamma_2'^n f_{i+1}^n + \gamma_3'^n f_{i+2}^n, \\ f_{2i+1}^{n+1} = \gamma_3'^n f_{i-1}^n + \gamma_2'^n f_i^n + \gamma_1'^n f_{i+1}^n + \gamma_0'^n f_{i+2}^n, \end{cases} \tag{5}$$

with

$$\gamma_j'^n = \frac{1}{\gamma^n} \gamma_j^n, \quad j = 0, \ldots, 3.$$

The convergence analysis of the normalized schemes is similar as in Section 3.

Moreover, the normalized schemes reproduce the function $f(x) = 1$ because $\sum_{i=0}^{3} \beta_i'^n = 1$ and $\sum_{i=0}^{3} \gamma_i'^n = 1$.

5. Properties of the schemes

In this section, the properties of the proposed schemes have been discussed.

The basic limit function denoted by L of the scheme $\{S_{a^n}\}$ is the limit function of the scheme for the data

$$f_i^0 = \begin{cases} 1, & i = 0, \\ 0, & i \neq 0. \end{cases}$$

Let

$$D_k = \left\{ \frac{j}{2^k}, \; j \in \mathbb{Z} \right\}. \tag{6}$$

It is easy to check that restriction of L to D_k satisfies

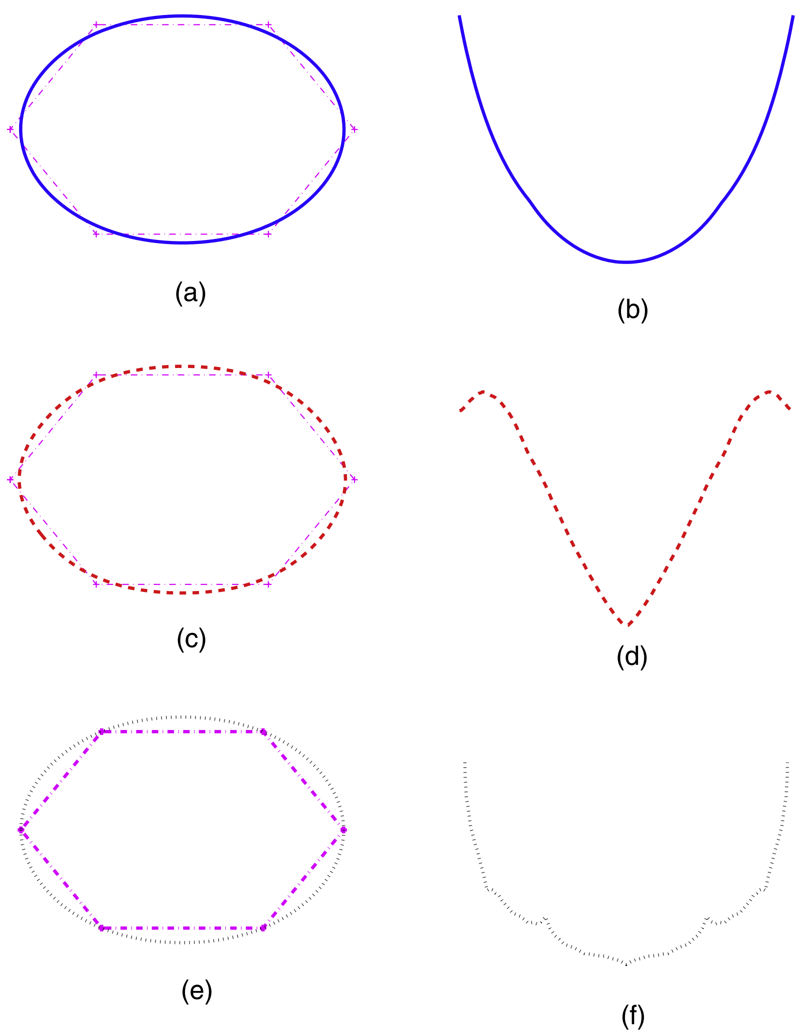

Figure 2 Construction of ellipses along with the associated curvature plots.

$$L\left(\frac{j}{2^k}\right) = f_j^k, \ \forall j. \tag{7}$$

Theorem 5.1. *The basic limit function L of the scheme (2) is symmetric about the Y-axis.*

Proof. Mathematical induction on k is used to prove the theorem.

Since

$$L\left(\frac{j}{2^k}\right) = L\left(-\frac{j}{2^k}\right), \quad \text{for} \quad k = 0, \quad \forall j \in \mathbb{Z}.$$

Therefore it provides basis for induction. Let

$$L\left(\frac{j}{2^n}\right) = L\left(-\frac{j}{2^n}\right), \quad \text{for} \quad n = 1, 2, \ldots, k, \quad \forall j \in \mathbb{Z}.$$

It can be written as

$$L\left(\frac{2j}{2^{k+1}}\right) = L\left(-\frac{2j}{2^{k+1}}\right), \quad \forall j.$$

Moreover,

$$\begin{aligned} L\left(\frac{2j+1}{2^{k+1}}\right) &= \beta_0^k f_{j-1}^k + \beta_1^k f_j^k + \beta_2^k f_{j+1}^k + \beta_3^k f_{j+2}^k, \\ &= \beta_0^k L\left(\frac{j-1}{2^k}\right) + \beta_1^k L\left(\frac{j}{2^k}\right) + \beta_2^k L\left(\frac{j+1}{2^k}\right) \\ &\quad + \beta_3^k L\left(\frac{j+2}{2^k}\right), \\ &= \beta_0^k L\left(\frac{-j+1}{2^k}\right) + \beta_1^k L\left(\frac{-j}{2^k}\right) + \beta_2^k L\left(\frac{-j-1}{2^k}\right) \\ &\quad + \beta_3^k L\left(\frac{-j-2}{2^k}\right), \\ &= L\left(-\frac{2j+1}{2^{k+1}}\right). \end{aligned}$$

Similarly, it can be shown that

$$L\left(\frac{2j}{2^k}\right) = L\left(-\frac{2j}{2^k}\right).$$

Hence,

$$L\left(\frac{j}{2^k}\right) = L\left(-\frac{j}{2^k}\right), \ k \in \mathbb{Z} \text{ and } \forall j.$$

From the continuity of L,

$$L(t) = L(-t), \quad \forall \quad t \in \mathbb{R}.$$

6. Graphical inspection

In this section behavior/comparison among the proposed trigonometric and hyperbolic forms of Chaikin's perturbation subdivision scheme and the binary scheme [11] have been presented. The approach of curvature plots and comparison with the original parametric curves has been used to check out the efficiency of the above listed subdivision schemes.

In Figs. 1 and 2, circles and ellipses have been reconstructed using the proposed non-stationary subdivision schemes (4), (5) and the scheme [11]. Figs. 1(a) and 2(a) are obtained using the proposed trigonometric form (4) along with the associated curvature plots in Figs. 1(b) and 2(b). Figs. 1(c) and 2(c) are obtained from the proposed hyperbolic form (5) and the corresponding curvature plots are given in Figs. 1(d) and 2(d). The Figs. 1(e) and 2(e) are obtained from the subdivision scheme [11] and the corresponding curvature plots are given in Figs. 1 (f) and 2(f). With the help of curvature plots it can be observed that the proposed trigonometric form is more suitable in the

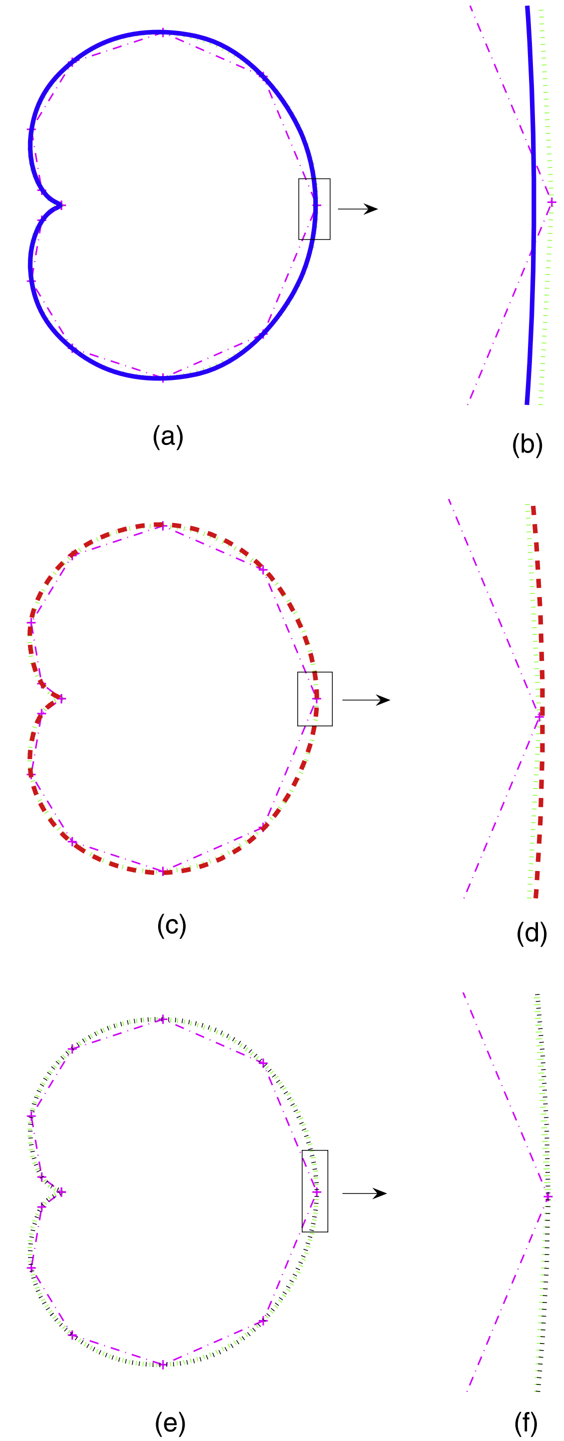

Figure 3 Generation of cardioid along with the corresponding magnified parts.

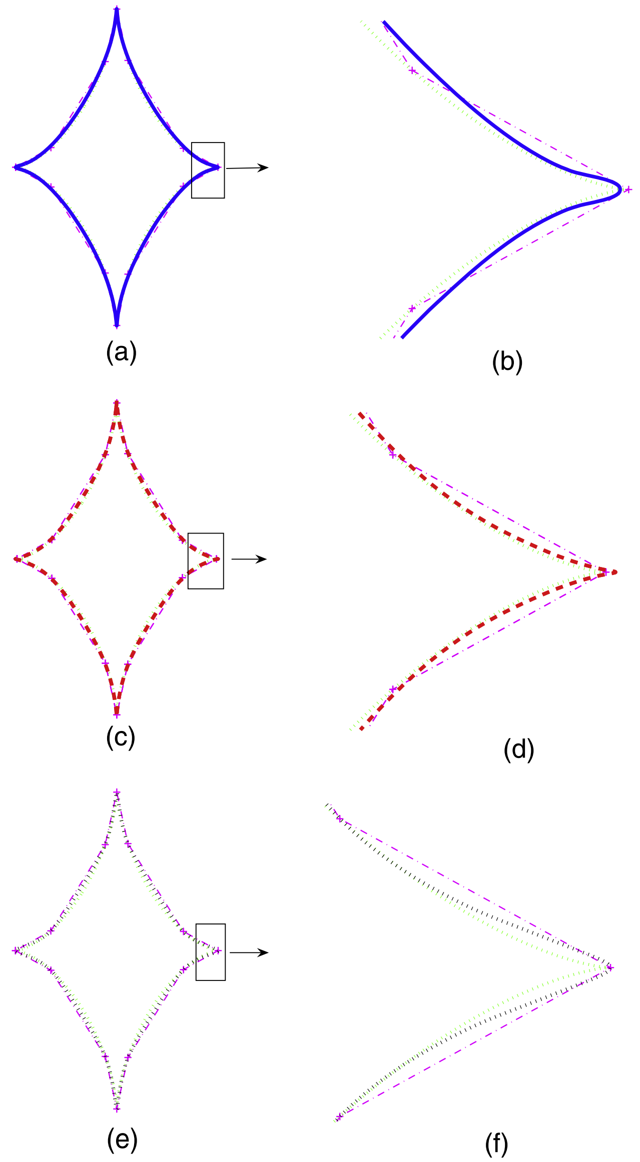

Figure 4 Generation of astroid for $\alpha=\frac{\pi}{5}$, obtained using the proposed trigonometric, hyperbolic form and scheme [11].

construction of circles and ellipses as compared to the proposed hyperbolic form and the scheme [11].

In Figs. 3–5, cardioid, astroid and lemniscate have been generated using the proposed trigonometric form, hyperbolic form and the scheme [11] represented by solid line (in blue), broken line-segments (in red) and dotted line segments (in black).

Comparison among the proposed trigonometric and hyperbolic forms and the scheme [11] with the standard parametric curves of cardioid, astroid and lemniscate is demonstrated in Figs. 3(a,c,e), 4(a,c,e) and 5(a,c,e). Corresponding boxed parts of these Figs. have been magnified to demonstrate better comparison, in Figs. 3(b,d,f), 4(b,d,f) and 5(b,d,f). Parametric equation of cardioid $x=a(1+2\cos t+\cos 2t), y=a(2\sin t+\sin 2t)$ for $a=\frac{1}{2}$, parametric equation of astroid $x=b\cos^3 t$, $y=b\sin^3 t$ for $b=2$, and parametric equation of lemniscate $x=\frac{a\cos(t)}{1+\sin^2(t)}, y=\frac{a\sin(t)\cos(t)}{1+\sin^2(t)}$ for $a=2$, have been used to obtain the standard cardioid, astroid and lemniscate (represented by green dotted line-segments). Through the comparison it can be observed that the cardioid, astroid and lemniscate constructed using the proposed hyperbolic form are more close to the corresponding standard curves. Therefore, it can be said that the proposed hyperbolic form is more suitable in the construction of astroid and lemniscate than the proposed trigonometric form and the scheme [11].

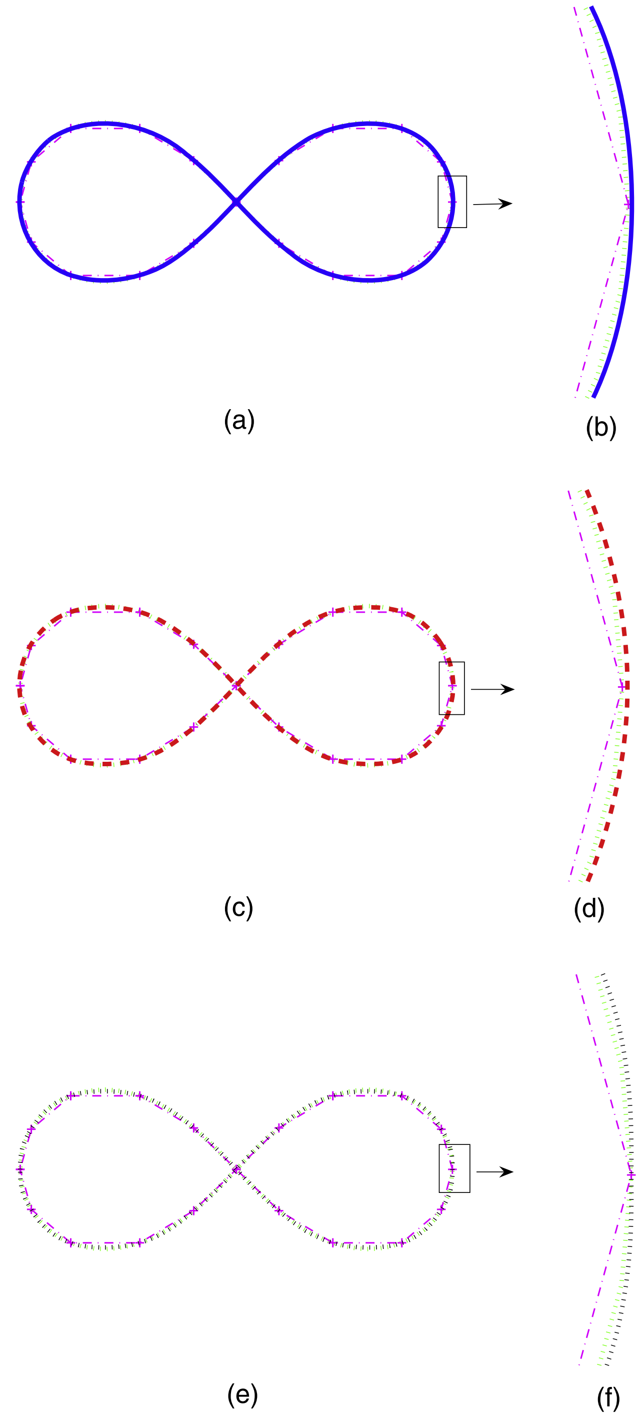

Figure 5 Generation of lemniscate for $\alpha=\frac{3\pi}{20}$.

7. Conclusion

Two non-stationary counterparts of the Chaikin's perturbation subdivision scheme [1] have been developed in trigonometric and hyperbolic forms. The curvature plot approach has been used to check the precision of conics reproduction property (circles and ellipses) of the proposed schemes. Through visual inspection, it can be observed that the proposed trigonometric form gives more precision in reproduction of circles and ellipses as compared to the proposed hyperbolic form and the scheme [11]. On the other hand, hyperbolic form is more precise in the reproduction of cardioid, astroid and lemniscate.

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