Journal of Algebra 349 (2012) 38–49

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Generators of supersymmetric polynomials in positive characteristic

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ARTICLE INFO

Article history:
Received 19 May 2010
Available online 3 November 2011
Communicated by Efim Zelmanov

Keywords:
Invariants
Supersymmetric polynomials
Pseudosymmetric polynomials
General linear supergroup
Schur superalgebra

ABSTRACT

In Kantor and Trishin (1997) [3], Kantor and Trishin described the algebra of polynomial invariants of the adjoint representation of the Lie superalgebra $gl(m|n)$ and a related algebra A_s of what they called pseudosymmetric polynomials over an algebraically closed field K of characteristic zero. The algebra A_s was investigated earlier by Stembridge (1985) who in [9] called the elements of A_s supersymmetric polynomials and determined generators of A_s.
The case of positive characteristic p of the ground field K has been recently investigated by La Scala and Zubkov (in press) in [6]. We extend their work and give a complete description of generators of polynomial invariants of the adjoint action of the general linear supergroup $GL(m|n)$ and generators of A_s.

Introduction and notation

We will start by recalling a classical problem of finding invariants of conjugacy classes of matrices, a solution of which is known for more than a century. Let K be an infinite field of arbitrary characteristic, $GL(n)$ be the general linear group and $\mathfrak{g}$ be its Lie algebra. A function $f \in K[\mathfrak{g}]$ is called an invariant if it has the same value on each conjugacy class of matrices. For an $n \times n$ matrix M, denote by $\sigma_i(M)$ the i-th coefficient of the characteristic polynomial of M; in particular $\sigma_1(M)$ is the trace of M and $\sigma_n(M)$ is the determinant of M. Chevalley restriction theorem (see Theorem 1.5.7 of [8] or [4]) gives an isomorphism of the ring of invariants $K[\mathfrak{g}]^{GL(n)}$ and the ring of symmetric functions in n variables, say $x_1, \dots, x_n$. This isomorphism is given by restriction on a subset consisting of all diagonal matrices with pairwise different eigenvalues. Generators of $K[\mathfrak{g}]^{GL(n)}$ corresponding

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.10.022

to coefficients of the characteristic polynomial of such matrices are given by elementary symmetric polynomials.

This classical result was extended to the case of the general linear supergroup $GL(m|n)$ in characteristic zero by Kantor and Trishin [3]. Before formulating their results, we will introduce the general linear supergroup $G = GL(m|n)$. Let K be an algebraically closed field of characteristic zero or positive characteristic $p > 2$. Let $K[c_{ij}]$ be a commutative superalgebra freely generated by elements c_{ij} for $1 \leqslant i, j \leqslant m+n$, where c_{ij} is even if either $1 \leqslant i, j \leqslant m$ or $m+1 \leqslant i, j \leqslant m+n$, and c_{ij} is odd otherwise. Denote by C the generic matrix $(c_{ij})_{1\leqslant i,j\leqslant m+n}$ and write it as a block matrix

$$\begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix},$$

where entries of C_{00} and C_{11} are even and entries of C_{01} and C_{10} are odd. The localization of $K[c_{ij}]$ by elements $det(C_{00})$ and $det(C_{11})$ is the coordinate superalgebra $K[G]$ of the general linear supergroup $G = GL(m|n)$. The general linear supergroup $G = GL(m|n)$ is a group functor from the category $SAlg_K$ of commutative superalgebras over K to the category of groups, represented by its coordinate ring $K[G]$, that is $G(A) = Hom_{SAlg_K}(K[G], A)$ for $A \in SAlg_K$. Here, for $g \in G(A)$ and $f \in K[G]$ we define $f(g) = g(f)$. Denote by $Ber(C) = det(C_{00} - C_{01}C_{11}^{-1}C_{10})det(C_{11})^{-1}$ the Berezinian element. The Berezinian plays a role analogous to that of the ordinary determinant in the classical case $GL(n)$.

The algebra R of invariants with respect to the adjoint action of G is a set of functions $f \in K[G]$ satisfying $f(g_1^{-1}g_2g_1) = f(g_2)$ for any $g_1, g_2 \in G(A)$ and any commutative superalgebra A over K. The algebra R_{pol} of polynomial invariants is a subalgebra of R consisting of polynomial functions.

In the case when the characteristic of the ground field K is zero, Kantor and Trishin [3] described generators of R_{pol} using supertraces. To explain their result we will need the following definition.

If V is a G-supermodule with a homogeneous basis $\{v_1, \dots, v_a, v_{a+1}, \dots, v_{a+b}\}$ such that v_i is even for $1 \leqslant i \leqslant a$ and v_i is odd for $a+1 \leqslant i \leqslant a+b$, and the image $\rho_V(v_i)$ of a basis element v_i under a comultiplication ρ_V is given as $\rho_V(v_i) = \sum_{1\leqslant j\leqslant a+b} v_j \otimes f_{ji}$, then the supertrace $Tr(V)$ is defined as $\sum_{1\leqslant i\leqslant a} f_{ii} - \sum_{a+1\leqslant i\leqslant a+b} f_{ii}$.

Let E be the natural G-supermodule given by basis elements $e_1, \dots, e_m$ that are even, and $e_{m+1}, \dots, e_{m+n}$ that are odd, and by comultiplication $\rho_E(e_i) = \sum_{1\leqslant j\leqslant m+n} e_j \otimes c_{ji}$. Denote by $\Lambda^r(E)$ the r-th superexterior power of E and by C_r the supertrace of $\Lambda^r(E)$.

If V is a (polynomial) G-supermodule, then $Tr(V)$ is a (polynomial) invariant of G (see Lemma 5.2 of [6]). Therefore elements $C_r \in R_{pol}$. It was proved in [3] that R_{pol} is generated by C_r and that the algebra R_{pol} is isomorphic to the algebra of pseudosymmetric polynomials $\Omega(m,n)$, which is a subalgebra of the polynomial ring over K in commuting variables $x_1, \dots, x_m, y_1, \dots, y_n$, generated by polynomials $I_k = \sum_{i=1}^m x_i^k - \sum_{j=1}^n y_i^k$ for $k = 0, 1, 2, \dots$. Moreover, it was observed there that this algebra is not finitely generated. The same algebra was investigated earlier by Stembridge in [9], who called it an algebra of supersymmetric polynomials.

The main objective of this paper is to describe generators of invariants of G when the characteristic $p > 2$. As in the case of characteristic zero, all elements C_r are polynomial invariants. However, in our case there are additional polynomial invariants $\sigma_i(C_{00})^p$, $\sigma_j(C_{11})^p$ and $\sigma_n(C_{11})^p Ber(C)^k \in R_{pol}$ for $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ and $0 < k < p$ which cannot be expressed solely in terms of the C_r's.

To show that the elements $\sigma_i(C_{00})^p$ and $\sigma_j(C_{11})^p$ are polynomial invariants of $GL(m|n)$, consider the Frobenius map $F : K[GL(m) \times GL(n)] \to K[GL(m|n)]$ given by $f \mapsto f^p$. Clearly, if f_0 is even and f_1 is odd, then $F(f = f_0 + f_1) = f_0^p$. By computing images of generators c_{ij}, where $1 \leqslant i, j \leqslant m$ and $m+1 \leqslant i, j \leqslant m+n$, it can be verified that the map F is a morphism of Hopf superrings. Since coadjoint actions are defined over the ring of integers, the Frobenius map F sends coadjoint $GL(m) \times GL(n)$-invariants to $GL(m|n)$-invariants. Therefore $F(\sigma_i(C_{00})) = \sigma_i(C_{00})^p \in R_{pol}$ for $1 \leqslant i \leqslant m$ and $F(\sigma_j(C_{11})) = \sigma_j(C_{11})^p \in R_{pol}$ for $1 \leqslant j \leqslant n$.

The element $\sigma_n(C_{11})^p$ is group-like by Lemma 3.3.1a of [7] and $Ber(C)$ is also group-like by [1]. Therefore an element $\sigma_n(C_{11})^p Ber(C)^k$, where $0 < k < p$, generates a one-dimensional simple G-supermodule and it belongs to R. Since the (highest) weight of $\sigma_n(C_{11})^p Ber(C)^k$ is $(k, \dots, k|p-k, \dots, p-k)$, by Theorem 6.5 of [2] it is polynomial. For example, if $m = n = 1$, then $\sigma_n(C_{11})^p Ber(C)^k = c_{11}^k c_{22}^{p-k} - kc_{12}c_{22}^{p-k-1}c_{21}c_{11}^{k-1}$ is polynomial for $1 \leqslant k \leqslant p-1$.

Actually, in the case $m = n = 1$, it is simple to determine the linear basis of R_{pol}: if p divides r, then it is given by elements

$$c_{11}^i c_{22}^{r-i} + (r-i)c_{11}^{i-1}c_{12}c_{21}c_{22}^{r-i-1}$$

for $0 \leqslant i \leqslant r$, and if p does not divide r, then it consists of elements

$$c_{11}^i c_{22}^{r-i} + (r-i)c_{11}^{i-1}c_{12}c_{21}c_{22}^{r-i-1} - c_{11}^{i-1}c_{22}^{r-i+1} + (i-1)c_{11}^{i-2}c_{12}c_{21}c_{22}^{r-i}$$

for $1 \leqslant i \leqslant r$.

Our first result states that the above invariants are generators of algebra R_{pol}.

Theorem 1. *The algebra R_{pol} is generated by elements*

$$C_r, \quad \sigma_i(C_{00})^p, \quad \sigma_j(C_{11})^p, \quad \sigma_n(C_{11})^p Ber(C)^k,$$

where $0 \leqslant r$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ and $0 < k < p$.

A description of the algebra R follows easily from this theorem.

Corollary 1. *The algebra R equals $R_{pol}[\sigma_m(C_{00})^{-p}, \sigma_n(C_{11})^{-p}]$.*

Proof. If $f \in R$, then its multiple by a sufficiently large power of $\sigma_m(C_{00})^p \sigma_n(C_{11})^p$ is a polynomial invariant. □

The main tool used in the proof of the above theorem is (again) the Chevalley map $\phi : K[G] \to A = K[x_1^{\pm 1}, \dots, x_m^{\pm 1}, y_1^{\pm 1}, \dots, y_n^{\pm 1}]$ defined on entries of a generic matrix C by $\phi(c_{ij}) = \delta_{ij}x_i$ for $1 \leqslant i \leqslant m$ and $\phi(c_{ij}) = \delta_{ij}y_{i-m}$ for $m+1 \leqslant i \leqslant m+n$. According to [6] and [3], the restriction of ϕ to R is an injective map and its image is contained in the algebra A_s of supersymmetric polynomials which by definition consists of polynomials $f(x|y) = f(x_1, \dots, x_m, y_1, \dots, y_n)$ that are symmetric in variables $x_1, \dots, x_m$ and $y_1, \dots, y_n$ separately, such that $\frac{d}{dT} f(x|y)|_{x_1=y_1=T}$ vanishes.

We will show that the image $\phi(R_{pol})$ equals A_s, hence $R_{pol} \cong A_s$.

To find images under ϕ of the previously defined elements from R_{pol}, consider the standard maximal torus T in G and a set of characters $X(T)$. Let V be a G-supermodule with weight decomposition $V = \bigoplus_{\lambda \in X(T)} V_\lambda$, where $\lambda = (\lambda_1, \dots, \lambda_{m+n})$, and each V_λ splits into a sum of its even subspace $(V_\lambda)_0$ and odd subspace $(V_\lambda)_1$. The (formal) supercharacter $\chi_{sup}(V)$ of V is defined as

$$\chi_{sup}(V) = \sum_{\lambda \in X(T)} \big(dim(V_\lambda)_0 - dim(V_\lambda)_1\big) x_1^{\lambda_1} \dots x_m^{\lambda_m} y_1^{\lambda_{m+1}} \dots y_n^{\lambda_{m+n}}.$$

Then for any G-supermodule V we have $\phi(Tr(V)) = \chi_{sup}(V)$. In particular, for $0 \leqslant r$ we have

$$\phi(C_r) = c_r = \sum_{0 \leqslant i \leqslant min(r,m)} (-1)^{r-i} \sigma_i(x_1, \dots, x_m) p_{r-i}(y_1, \dots, y_n),$$

where σ_i is the i-th elementary symmetric function and p_j is the j-th complete symmetric function.

The images of the remaining generators of R_{pol} under ϕ,

$$\phi\big(\sigma_i(C_{00})^p\big) = \sigma_i(x_1, \ldots, x_m)^p$$

for $1 \leqslant i \leqslant m$,

$$\phi\big(\sigma_j(C_{11})^p\big) = \sigma_j(y_1, \ldots, y_n)^p$$

for $1 \leqslant j \leqslant n$, and

$$\phi\big(\sigma_n(C_{11})^p Ber(C)^k\big) = u_k(x|y) = \sigma_m(x_1, \ldots, x_m)^k \sigma_n(y_1, \ldots, y_n)^{p-k}$$

for $0 < k < p$ are elements from A_s.

Theorem 1 will follow from the following description of generators of the algebra A_s.

Theorem 2. *The algebra A_s is generated by elements c_r for $r \geqslant 0$, $\sigma_i(x_1, \ldots, x_m)^p$ for $1 \leqslant i \leqslant m$, $\sigma_j(y_1, \ldots, y_n)^p$ for $1 \leqslant j \leqslant n$ and $u_k(x|y)$ for $0 < k < p$.*

We conclude the introduction with the following remarkable observation. It was showed in [3] that $A_s \simeq R_{pol}$ is not finitely generated if the characteristic of the field K equals zero. We will show that if the characteristic of K is positive, then the algebra $A_s \simeq R_{pol}$ is finitely generated.

1. Proof of Theorem 1

In this section we will compare algebras corresponding to different values of m, n and apply the Schur functor. Therefore we adjust the notation slightly to reflect the dependence on m, n. For example, we will write $R(m|n)$ instead of R and $A_s(m|n)$ instead of A_s.

Denote by $R'_{pol}(m|n)$ a subalgebra of $R_{pol}(m|n)$ generated by elements C_r, $\sigma_i(C_{00})^p$, $\sigma_j(C_{11})^p$ and $\sigma_n(C_{11})^p Ber(C)^k$, where $0 \leqslant r$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ and $0 < k < p$.

Further, denote by $A_{ns}(m|n)$ a subalgebra of $A_s(m|n)$ generated by polynomials $c_r(m|n)$, $\sigma_i(x, m)^p = \sigma_i(x_1, \ldots, x_m)^p$, $\sigma_j(y, n)^p = \sigma_j(y_1, \ldots, y_n)^p$ and $u_k(m|n) = \sigma_m(x, m)^k \sigma_n(y, n)^{p-k}$ for $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ and $0 < k < p$.

There is the following commutative diagram

$$\begin{array}{ccc} R_{pol}(m|n) & \xrightarrow{\ \phi\ } & A_s(m|n) \\ \uparrow & & \uparrow \\ R'_{pol}(m|n) & \xrightarrow{\ \phi\ } & A_{ns}(m|n) \end{array}$$

where the vertical maps are inclusions and horizontal maps are given by restrictions of the Chevalley morphism ϕ.

Both horizontal maps in the above diagram are monomorphisms. The bottom map is an epimorphism by definition of $R'_{pol}(m|n)$ and $A_{ns}(m|n)$, hence an isomorphism. We will produce three proofs of Theorem 1. For the first proof, we will show in Proposition 1.3 that $\phi(R_{pol}(m|n)) = A_{ns}(m|n)$ and it implies $R_{pol}(m|n) = R'_{pol}(m|n)$. The second proof uses the equality $A_{ns}(m|n) = A_s(m|n)$ from Theorem 2. From the above diagram it follows that $R_{pol}(m|n) = R'_{pol}(m|n)$. An elementary proof of Theorem 2 will provide the third proof of Theorem 1.

Denote by $A_{ns}(m|n, t)$ the homogeneous component of $A_{ns}(m|n)$ of degree t. For any integers $M \geqslant m$, $N \geqslant n$ there is a graded superalgebra morphism $p_e : K[x_1, \ldots, x_M, y_1, \ldots, y_N] \rightarrow$

$K[x_1,\dots,x_m,y_1,\dots,y_n]$ that maps $x_i \mapsto x_i$ for $i \leqslant m$, $y_j \mapsto y_j$ for $j \leqslant n$ and the remaining generators x_i, y_j to zero. Clearly the image of $A_s(M|N)$ under p_e is a subset of $A_s(m|n)$.

Lemma 1.1. *The morphism p_e maps $A_{ns}(M|N)$ to $A_{ns}(m|n)$.*

Proof. Verify that

$$p_e\big(\sigma_i(x,M)\big)=\sigma_i(x,m) \quad \text{if } i \leqslant m \quad \text{and} \quad p_e\big(\sigma_i(x,M)\big)=0 \quad \text{if } i>m,$$
$$p_e\big(\sigma_j(y,N)\big)=\sigma_j(y,n) \quad \text{if } j \leqslant n \quad \text{and} \quad p_e\big(\sigma_j(y,N)\big)=0 \quad \text{if } j>n,$$
$$p_e\big(c_r(M,N)\big)=c_r(m|n) \quad \text{if } r \leqslant m,n \quad \text{and} \quad p_e\big(c_r(M,N)\big)=0 \quad \text{if } r>m \text{ or } r>n,$$

and

$$p_e\big(u_k(M|N)\big)=0.$$

The claim follows. □

For the integers $M \geqslant m$, $N \geqslant n$ consider the Schur superalgebra $S(M|N,r)$ and its idempotent $e=\sum_\mu \xi_\mu$, where the sum is over all weights μ for which $\mu_i=0$ whenever $m<i\leqslant M$ or $M+n<i\leqslant M+N$. Then $S(m|n,r)\simeq eS(M|N,r)e$ and there is a natural Schur functor $S(M|N,r)-mod \to S(m|n,r)-mod$ given by $V \mapsto eV$. If V is an $S(M|N,r)$-supermodule, then eV is a supersubspace of V and therefore, eV has a canonical $S(m|n,r)$-supermodule structure.

Let $\mathbf{V}=\{V\}$ be a collection of polynomial G-supermodules. Such a collection is called *good* if for any simple polynomial G-supermodule L there is $V\in\mathbf{V}$ such that L is a composition factor of V and the highest weights of all remaining composition factors of V are strictly smaller than the highest weight of L. Clearly, the collection of all simple polynomial G-supermodules is good. The collection of all costandard supermodules is also good. We will use repeatedly Theorem 5.3 from [6] which states that if $\{V\}$ is a good collection of polynomial G-supermodules, then R_{pol} is spanned by $Tr(V)$.

Lemma 1.2. *The map p_e induces an epimorphism of graded algebras $\phi(R_{pol}(M|N)) \to \phi(R_{pol}(m|n))$.*

Proof. Applying the Chevalley map ϕ to the collection of all simple polynomial G-supermodules L and using Theorem 5.3 of [6] we obtain that the algebra $\phi(R_{pol})$ is spanned by the supercharacters $\chi_{sup}(L)$. If λ is the highest weight of L, then $\chi_{sup}(L)$ is a homogeneous polynomial of degree $r=|\lambda|=\sum_{1\leqslant i\leqslant m+n}\lambda_i$. By a standard property of a Schur functor, there is a simple $S(M|N,r)$-supermodule L' such that $eL'\simeq L$. Since $p_e(\chi_{sup}(L'))=\chi_{sup}(L)$, the claim follows. □

Proposition 1.3. *The image $\phi(R_{pol}(m|n))$ equals $A_{ns}(m|n)$.*

Proof. Fix a homogeneous element $f\in\phi(R_{pol}(m|n))$ of degree r and choose $M\geqslant m$ strictly greater than r. By Lemma 1.2, there is a homogeneous polynomial $f'\in\phi(R_{pol}(M|n))$ of degree r such that $p_e(f')=f$. Using the Chevalley map, and applying Theorem 5.3 of [6] to the collection of all costandard polynomial modules $\nabla(\mu)$, we obtain that f' is a linear combination of supercharacters $\chi_{sup}(\nabla(\mu))$, or alternatively of supercharacters $\chi_{sup}(L(\mu))$, where μ runs over polynomial dominant weights with $|\mu|=r$. We can write $\chi_{sup}(\nabla(\mu))=\sum_{\pi\leqslant\mu}c_{\mu,\pi}\chi_{sup}(L(\pi))$ and $\chi_{sup}(L(\mu))=\sum_{\pi\leqslant\mu}d_{\mu,\pi}\chi_{sup}(\nabla(\pi))$, where coefficients $c_{\mu,\pi}$ and $d_{\mu,\pi}$ are non-negative integers, and $c_{\mu,\mu}=d_{\mu,\mu}=1$.

Denote by Γ_r a finite set of all polynomial dominant weights μ such that $|\mu|\leqslant r$. Define a partial order on Γ_r by $\lambda\prec\mu$ if and only if $|\lambda|<|\mu|$ or $\lambda\leqslant\mu$ (recall that $\lambda\leqslant\mu$ implies $|\lambda|=|\mu|$). Then $\chi_{sup}(\nabla(\pi))\in A_{ns}(M|n)$ for all $\pi\prec\mu$ is equivalent to $\chi_{sup}(L(\pi))\in A_{ns}(M|n)$ for all $\pi\prec\mu$.

Consider $\mu \in \Gamma_r$ and assume that $\chi_{sup}(\nabla(\pi)) \in A_{ns}(M|n)$ for any $\pi \prec \mu$, $\pi \neq \mu$. The assumption $M > r$, Theorem 5.4 and Proposition 5.6 of [5] imply that for the highest weight $\mu = (\mu_+|\mu_-)$ we have $\mu_- = p\overline{\mu}$ for some weight $\overline{\mu}$, and $\nabla(\mu) \simeq \nabla(\mu_+|0) \otimes F(\overline{\nabla}(\overline{\mu}))$, where F is the Frobenius map and $\overline{\nabla}(\overline{\mu})$ is the costandard $GL(n)$-module with the highest weight $\overline{\mu}$. Therefore

$$\chi_{sup}\big(\nabla(\mu)\big) = \chi_{sup}\big(\nabla(\mu_+|0)\big)\chi\big(\overline{\nabla}(\overline{\mu})\big)^p$$

and $\chi(\overline{\nabla}(\overline{\mu}))^p$ is a polynomial in $\sigma_j(y,n)^p$. If $\mu_- \neq 0$ then, by the inductive hypothesis, $\chi_{sup}(\nabla(\mu_+|0)) \in A_{ns}(M|n)$. Otherwise, $\mu = (\mu_+|0)$.

An exterior power $\Lambda^t(E(M|n))$ for $t \leqslant M$ has a unique maximal weight $(1^t|0)$. Consequently, an $S(M|n,r)$-supermodule

$$V = \Lambda^M\big(E(M|n)\big)^{\otimes \mu_M} \otimes \Lambda^{M-1}\big(E(M|n)\big)^{\otimes(\mu_{M-1}-\mu_M)} \otimes \cdots \otimes \Lambda^1\big(E(M|n)\big)^{\otimes(\mu_1-\mu_2)}$$

has a unique maximal weight μ and the supercharacter

$$\chi_{sup}(V) = c_1^{\mu_1-\mu_2}\ldots c_{M-1}^{\mu_{M-1}-\mu_M} c_M^{\mu_M}.$$

The module V has a composition series with a unique section that is isomorphic to $L(\mu)$ and the remaining sections isomorphic to $L(\kappa)$, where $\kappa < \mu$. By the inductive hypothesis, all $\chi_{sup}(L(\kappa)) \in A_{ns}(M|n)$ and therefore, $\chi_{sup}(L(\mu)) \in A_{ns}(M|n)$. □

Corollary 1.4. *The morphism p_e maps $A_{ns}(M|N,t)$ onto $A_{ns}(m|n,t)$.*

Proof of Theorem 1. Recall that the restriction of ϕ on R is a monomorphism. Since $\phi(C_r) = c_r$, $\phi(\sigma_i(C_{00})^p) = \sigma_i(x_1,\ldots,x_m)^p$, $\phi(\sigma_j(C_{11})^p) = \sigma_j(y_1,\ldots,y_n)^p$ and $\phi(\sigma_n(C_{11})^p Ber(C)^k) = u_k(x|y)$, the statement follows from Proposition 1.3. □

2. Proof of Theorem 2

We will need the following crucial observation.

Lemma 2.1. *If $f \in A_s(m|n)$ is divisible by x_m, then f is divisible by a nonconstant element of $A_{ns}(m|n)$.*

Proof. We can assume $f \neq 0$ and use the symmetricity of f in variables $x_1,\ldots,x_m$ and $y_1,\ldots,y_n$ to write $f = x_1^a\ldots x_m^a y_1^b \ldots y_n^b g$, where exponents $a > 0$, $b \geqslant 0$, and polynomial g, such that $g|_{x_m=y_n=0} \neq 0$, are unique. Then

$$\begin{aligned} f|_{x_m=y_n=T} &= T^{a+b}x_1^a\ldots x_{m-1}^a y_1^b \ldots y_{n-1}^b g|_{x_m=y_n=T} \\ &= T^{a+b}x_1^a\ldots x_{m-1}^a y_1^b \ldots y_{n-1}^b g_0 + T^{a+b+1}x_1^a\ldots x_{m-1}^a y_1^b \ldots y_{n-1}^b g_1, \end{aligned}$$

where we write $g|_{x_m=y_n=T} = g_0 + Tg_1$. The requirement $g|_{x_m=y_n=0} \neq 0$ implies $g_0 \neq 0$. Since $\frac{d}{dT} f|_{x_m=y_n=T} = 0$, this is only possible if $a+b \equiv 0 \pmod p$. Since $a > 0$, the polynomial $x_1^a \ldots x_m^a y_1^b \ldots y_n^b$ is not constant, and is a product of $\sigma_m(x,m)^p$, $\sigma_n(y,n)^p$ and $u_k(m|n)$, all of which belong to $A_{ns}(m|n)$. In fact, since $a > 0$, we have that f is divisible either by $\sigma_m(x,m)^p$ or by some $u_k(m|n)$. □

Proof of Theorem 2. The statement of the theorem is equivalent to the equality $A_s(m|n) = A_{ns}(m|n)$.

Fix n and assume that m is minimal, such that there exists a polynomial $f \in A_s(m|n) \setminus A_{ns}(m|n)$, and choose f that is homogeneous and of the minimal degree. Then its reduction $f|_{x_m=0} \in A_{ns}(m-1|n)$ is a nonzero polynomial $h(c_t(m-1|n), \sigma_i(x, m-1)^p, \sigma_j(y,n)^p, u_k(m-1|n))$ in elements $c_t(m-1|n)$, $\sigma_i(x,m-1)^p$, $\sigma_j(y,n)^p$ and $u_k(m-1,n)$ where $t \geqslant 0$, $1 \leqslant i \leqslant m-1$, $1 \leqslant j \leqslant n$ and $0 < k < p$. By Corollary 1.4 there are elements $v_k \in A_{ns}(m|n)$ of degree $mk + (p-k)n$ such that $v_k|_{x_m=0} = u_k(m-1|n)$. Since $c_t(m|n)|_{x_m=0} = c_t(m-1|n)$, $\sigma_i(x,m)^p|_{x_m=0} = \sigma_i(x,m-1)^p$ and $\sigma_j(y,n)^p|_{x_m=0} = \sigma_j(y,n)^p$, the polynomial $l = f - h(c_t(m|n), \sigma_i(x,m)^p, \sigma_j(y,n)^p, v_k(m|n))$ satisfies $l|_{x_m=0} = 0$. Since the degree of l does not exceed the degree of f, $l \in A_s(m|n)$ and x_m divides l, Lemma 2.1 implies that $l = l_0 l_1$, where $l_0 \in A_{ns}(m|n)$ and the degree of l_1 is strictly less than the degree of f. But $l_1 \in A_s(m|n) \setminus A_{ns}(m|n)$ which is a contradiction with our choice of f. □

3. Elementary proof of Theorem 2

A closer look at the proof of Theorem 2 reveals that Corollary 1.4 is the only result from Section 1 that was used in the proof of Theorem 2. Actually, only the following weaker statement was required in the proof of Theorem 2.

Proposition 3.1. *For each $0 < k < p$ there is a polynomial $v_k \in A_{ns}(m|n)$ of degree $(m-1)k + (p-k)n$ such that $v_k|_{x_m=0} = u_k(m-1|n)$.*

In this section we give a constructive elementary proof of Proposition 3.1 that bypasses the use of the Schur functor and the results about costandard modules derived in [5].

Fix $0 < k < p$ and denote $s = \lceil \frac{k}{p-k} \rceil$. Then for $i = 0, \ldots, s-1$ define $k_i = (i+1)k - ip > 0$ and $k_p = sp - (s+1)k \geqslant 0$. The relations

$$k_i + (p-k) = k_{i-1}, \qquad k_p + k = s(p-k), \qquad k_i + k_p = (s-i)(p-k)$$

will be used without explicit reference.

A symbol $\mathcal{I}$ will denote a nondecreasing sequence $(i_1 \leqslant \cdots \leqslant i_t)$ of natural numbers, where $0 \leqslant t < s$. We denote $\|\mathcal{I}\| = t$ and $|\mathcal{I}| = \sum_{j=1}^t i_j$. In particular, we allow $\mathcal{I} = \emptyset$ with $\|\emptyset\| = |\emptyset| = 0$. Additionally, denote by $Supp(\mathcal{I})$ the set of all elements (without repetitions) appearing in $\mathcal{I}$. If $i \in Supp(\mathcal{I})$, then by slightly abusing notation we define $\mathcal{I} \setminus i$ to be a sequence obtained from $\mathcal{I}$ by deleting one arbitrary element equal to i and define $\mathcal{I} \cup i$ to be a sequence obtained from $\mathcal{I}$ by adding an extra element equal to i.

Fix an arbitrary sequence $(a_1, \ldots, a_j)$ of length $j \leqslant M$. Denote by Σ_j the symmetric group acting on j symbols, and by Y its Young subgroup which preserves the fibers of the map $j \mapsto a_j$. Then there is a unique symmetric polynomial in $x_1, \ldots, x_M$ that has integral coefficients, with the coefficient of the monomial $x_1^{a_1} \ldots x_j^{a_j}$ equal to 1. This polynomial is denoted $Sym_{x,M}(a_1, \ldots, a_j)$ and is defined as

$$Sym_{x,M}(a_1, \ldots, a_j) = \sum_{\{k_1,\ldots,k_j\} \subset \{1,\ldots,M\}} \sum_{\sigma \in Y \setminus \Sigma_j} x_{k_{\sigma(1)}}^{a_1} \ldots x_{k_{\sigma(j)}}^{a_j},$$

where the first sum is over all subsets $\{k_1, \ldots, k_j\}$ of $\{1, \ldots, M\}$ of cardinality j, and the second sum is over representatives of the left cosets of Σ_j over its Young subgroup Y.

The symmetric polynomial $Sym_{y,N}(b_1, \ldots, b_j)$ in variables $y_1, \ldots, y_N$, that has integral coefficients, with the coefficient of the monomial $y_1^{b_1} \ldots y_j^{b_j}$ equal to 1, is defined analogously.

For simplicity we will use a multiplicative notation, and instead of $Sym_{x,M}(\underbrace{a, \ldots, a}_{m_a}, \ldots, \underbrace{z, \ldots, z}_{m_z})$, we will write $Sym_{x,M}(a^{m_a} \ldots z^{m_z})$. We will use analogous multiplicative notation for $Sym_{y,N}$.

Further, denote

$$A(\mathcal{I},j)_{M,N}=Sym_{x,M}\big(k^{M-t}k_{i_1}\dots k_{i_t}\big)\,Sym_{y,N}\big((p-k)^{N-j-1}k_p\big)$$

for $0\leqslant\|\mathcal{I}\|=t\leqslant M$ and $0\leqslant j<N$, and $A(\mathcal{I},l)_{M,N}=0$ if $t>M$ or $j\geqslant N$;

$$B(\mathcal{I},j)_{M,N}=Sym_{x,M}\big(k^{M-t}k_{i_1}\dots k_{i_t}\big)\,Sym_{y,N}\big((p-k)^{N-j}\big)$$

for $0\leqslant\|\mathcal{I}\|=t\leqslant M$ and $0\leqslant j\leqslant N$, and $B(\mathcal{I},j)_{M,N}=0$ if $t>M$ or $j>N$;

$$C(\mathcal{I},l,j)_{M,N}=Sym_{x,M}\big(k^{M-t-1}(lp-lk)k_{i_1}\dots k_{i_t}\big)\,Sym_{y,N}\big((p-k)^{N-j}\big)$$

for $0\leqslant\|\mathcal{I}\|=t<M$ and $0\leqslant j\leqslant N$ and any l, and $C(\mathcal{I},l,j)_{M,N}=0$ if $t\geqslant M$ or $j>N$.

For simplicity write $A(\mathcal{I},j)$, $B(\mathcal{I},j)$ and $C(\mathcal{I},l,j)$ short for $A(\mathcal{I},j)_{m-1,n-1}$, $B(\mathcal{I},j)_{m-1,n-1}$ and $C(\mathcal{I},l,j)_{m-1,n-1}$.

For $f\in K[x_1,\dots,x_m,y_1,\dots,y_n]$ define $\psi(f)=f|_{x_m=y_n=T}$ and for $g,h\in K[x_1,\dots,x_{m-1},y_1,\dots,y_{n-1},T]$ write $g\equiv h$ if and only if $\frac{d}{dT}(g-h)=0$.

Lemma 3.2. *The following relations are valid*:

$$\psi\big(C(\mathcal{I},l,j)_{m,n}\big)\equiv T^kC(\mathcal{I},l,j-1)+T^{(l+1)(p-k)}B(\mathcal{I},j)+T^{l(p-k)}B(\mathcal{I},j-1)\\+\sum_{i\in Supp(\mathcal{I})}\big(T^{k_i}C(\mathcal{I}\setminus i,l,j-1)+T^{k_{i-1}}C(\mathcal{I}\setminus i,l,j)\big)$$

and

$$\psi\big(A(\mathcal{I},j)_{m,n}\big)\equiv T^kA(\mathcal{I},j-1)+T^{s(p-k)}B(\mathcal{I},j)+\sum_{i\in Supp(\mathcal{I})}\big(T^{k_i}A(\mathcal{I}\setminus i,j-1)+T^{k_{i-1}}A(\mathcal{I}\setminus i,j)\\+T^{(s-i)(p-k)}B(\mathcal{I}\setminus i,j)\big).$$

Proof. It is easy to see that for $j=0,\dots,n$ we have

$$\psi\big(S_{y,n}\big((p-k)^{n-j}\big)\big)=(1-\delta_{j,n})T^{p-k}S_{y,n-1}\big((p-k)^{n-1-j}\big)+(1-\delta_{j,0})S_{y,n-1}\big((p-k)^{n-j}\big)$$

and for $j=0,\dots,n-1$ we have

$$\psi\big(S_{y,n}\big((p-k)^{n-1-j}k_p\big)\big)=T^{k_p}S_{y,n-1}\big((p-k)^{n-1-j}\big)+(1-\delta_{j,n-1})T^{p-k}S_{y,n-1}\big((p-k)^{n-2-j}k_p\big)\\+(1-\delta_{j,0})S_{y,n-1}\big((p-k)^{n-1-j}k_p\big).$$

Assume that $lp-lk$ is different from k and all numbers k_i. Then we can verify that for $t=0,\dots,m$ we have

$$\psi\big(S_{x,m}\big(k^{m-t}k_{i_1}\dots k_{i_t}\big)\big)=(1-\delta_{t,m})T^kS_{x,m-1}\big(k^{m-1-t}k_{i_1}\dots k_{i_t}\big)\\+\sum_{u=1}^{t}T^{k_{i_u}}S_{x,m-1}\big(k^{m-t}k_{i_1}\dots\widehat{k_{i_u}}\dots k_{i_t}\big)$$

and for $t = 0, \dots, m-1$ we have

$$\begin{aligned}
&\psi\big(S_{x,m}\big(k^{m-1-t}(lp-lk)k_{i_1}\dots k_{i_t}\big)\big) \\
&\quad = T^{lp-lk}S_{x,m-1}\big(k^{m-1-t}k_{i_1}\dots k_{i_t}\big) + (1-\delta_{m-1,t})T^k S_{x.m-1}\big(k^{m-2-t}(lp-lk)k_{i_1}\dots k_{i_t}\big) \\
&\qquad + \sum_{u=1}^{t} T^{k_{i_u}} S_{x,m-1}\big(k^{m-1-t}(lp-lk)k_{i_1}\dots \widehat{k_{i_u}}\dots k_{i_t}\big).
\end{aligned}$$

Even when $lp-lk$ coincides with k or one of k_i, the above formulae remain valid.

Using these formulae we obtain readily

$$\begin{aligned}
&\psi\big(C(\mathcal{I},l,j)_{m,n}\big) \\
&\quad = (1-\delta_{j,n})T^{(l+1)(p-k)}S_{x,m-1}\big(k^{m-1-t}k_{i_1}\dots k_{i_t}\big)S_{y,n-1}\big((p-k)^{n-1-j}\big) \\
&\qquad + (1-\delta_{j,n})(1-\delta_{m-1,t})T^p S_{x,m-1}\big(k^{m-2-t}(lp-lk)k_{i_1}\dots k_{i_t}\big)S_{y,m-1}\big((p-k)^{n-j}\big) \\
&\qquad + (1-\delta_{j,n})\sum_{u=1}^{t} T^{p-k+k_{i_u}} S_{x,m-1}\big(k^{m-1-t}(lp-lk)k_{i_1}\dots \widehat{k_{i_u}}\dots k_{i_t}\big)S_{y,n-1}\big((p-k)^{n-1-j}\big) \\
&\qquad + (1-\delta_{j,0})T^{lp-lk}S_{x,m-1}\big(k^{m-1-t}k_{i_1}\dots k_{i_t}\big)S_{y,n-1}\big((p-k)^{n-j}\big) \\
&\qquad + (1-\delta_{j,0})(1-\delta_{m-1,t})T^k S_{x,m-1}\big(k^{m-2-t}(lp-lk)k_{i_1}\dots k_{i_t}\big)S_{y,n-1}\big((p-k)^{n-j}\big) \\
&\qquad + \sum_{u=1}^{t} T^{k_{i_u}} S_{x,m-1}\big(k^{m-1-t}(lp-lk)k_{i_1}\dots \widehat{k_{i_u}}\dots k_{i_t}\big)S_{y,n-1}\big((p-k)^{n-j}\big)
\end{aligned}$$

hence

$$\begin{aligned}
\psi\big(C(\mathcal{I},l,j)_{m,n}\big) \equiv{}& T^{(l+1)(p-k)}B(\mathcal{I},j) + \sum_{u=1}^{t} T^{p-k+k_{i_u}} C(\mathcal{I}\setminus i_u,l,j) + T^{l(p-k)}B(\mathcal{I},j-1) \\
&+ T^k C(\mathcal{I},j,j-1) + \sum_{u=1}^{t} T^{k_{i_u}} C(\mathcal{I}\setminus i_u,l,j-1)
\end{aligned}$$

and the formula for $\psi(C(\mathcal{I},l,j)_{m,n})$ follows.

Additionally, we obtain

$$\begin{aligned}
\psi\big(A(\mathcal{I},j)_{m,n}\big) ={}& (1-\delta_{t,m})T^{k+k_p}S_{x,m-1}\big(k^{m-1-t}k_{i_1}\dots k_{i_t}\big)S_{y,n-1}\big((p-k)^{n-1-j}\big) \\
&+ (1-\delta_{t,m})(1-\delta_{n-1,j})T^p S_{x,m-1}\big(k^{m-1-t}k_{i_1}\dots k_{i_t}\big)S_{y,n-1}\big((p-k)^{n-2-j}k_p\big) \\
&+ (1-\delta_{t,m})(1-\delta_{0,j})T^k S_{x,m-1}\big(k^{m-1-t}k_{i_1}\dots k_{i_t}\big)S_{y,n-1}\big((p-k)^{n-1-j}k_p\big) \\
&+ \sum_{u=1}^{t} T^{k_{i_u}+k_p} S_{x,m-1}\big(k^{m-t}k_{i_1}\dots \widehat{k_{i_u}}\dots k_{i_t}\big)S_{y,n-1}\big((p-k)^{n-1-j}\big) \\
&+ (1-\delta_{n-1,j})\sum_{u=1}^{t} T^{k_{i_u}+p-k} S_{x,m-1}\big(k^{m-t}k_{i_1}\dots \widehat{k_{i_u}}\dots k_{i_t}\big)S_{y,n-1}\big((p-k)^{n-2-j}k_p\big)
\end{aligned}$$

$$+ (1-\delta_{0,j}) \sum_{u=1}^{t} T^{k_u} S_{x,m-1}\big(k^{m-t} k_{i_1} \dots \widehat{k_{i_u}} \dots k_{i_t}\big) S_{y,n-1}\big((p-k)^{n-1-j} k_p\big)$$

hence

$$\psi\big(A(\mathcal{I}, j)_{m,n}\big) \equiv T^{s(p-k)} B(\mathcal{I}, j) + T^k A(\mathcal{I}, j-1) + \sum_{u=1}^{t} T^{k_{i_u}+k_p} B(\mathcal{I} \setminus i_u, j)$$
$$+ \sum_{u=1}^{t} T^{k_{i_u}+(p-k)} A(\mathcal{I} \setminus i_u, j) + \sum_{u=1}^{t} T^{k_{i_u}} A(\mathcal{I} \setminus i_u, j-1)$$

and the formula for $\psi(A(\mathcal{I}, j)_{m,n})$ follows. □

Let us define

$$w = \sum_{l=1}^{s-1} \sum_{0 \leqslant |\mathcal{I}| \leqslant l} (-1)^{|\mathcal{I}|+s+l}(s-l) C\big(\mathcal{I}, l, l-|\mathcal{I}|\big)_{m,n} + \sum_{0 \leqslant |\mathcal{I}| < s} (-1)^{|\mathcal{I}|} A\big(\mathcal{I}, s-1-|\mathcal{I}|\big)_{m,n}.$$

Then

$$\psi(w) = \sum_{l=1}^{s-1} \sum_{0 \leqslant |\mathcal{I}| \leqslant l} (-1)^{|\mathcal{I}|+s+l}(s-l) \psi\big(C\big(\mathcal{I}, l, l-|\mathcal{I}|\big)_{m,n}\big) + \sum_{0 \leqslant |\mathcal{I}| < s} (-1)^{|\mathcal{I}|} \psi\big(A\big(\mathcal{I}, s-1-|\mathcal{I}|\big)_{m,n}\big)$$

and by Lemma 3.2

$$\psi(w) \equiv \sum_{l=1}^{s-1} \sum_{0 \leqslant |\mathcal{I}| \leqslant l} (-1)^{s+l+|\mathcal{I}|}(s-l) \Big(T^k C\big(\mathcal{I}, l, l-|\mathcal{I}|-1\big) + T^{(l+1)(p-k)} B\big(\mathcal{I}, l-|\mathcal{I}|\big)$$
$$+ T^{l(p-k)} B\big(\mathcal{I}, l-|\mathcal{I}|-1\big) + \sum_{i \in Supp(\mathcal{I})} \big(T^{k_i} C\big(\mathcal{I} \setminus i, l, l-|\mathcal{I}|-1\big) + T^{k_{i-1}} C\big(\mathcal{I} \setminus i, l, l-|\mathcal{I}|\big)\big)\Big)$$
$$+ \sum_{0 \leqslant |\mathcal{I}| < s} (-1)^{|\mathcal{I}|} \Big(T^k A\big(\mathcal{I}, s-|\mathcal{I}|-2\big) + T^{s(p-k)} B\big(\mathcal{I}, s-1-|\mathcal{I}|\big)$$
$$+ \sum_{i \in Supp(\mathcal{I})} \big(T^{k_i} A\big(\mathcal{I} \setminus i, s-|\mathcal{I}|-2\big) + T^{k_{i-1}} A\big(\mathcal{I} \setminus i, s-|\mathcal{I}|-1\big)$$
$$+ T^{(s-i)(p-k)} B\big(\mathcal{I} \setminus i, s-|\mathcal{I}|-1\big)\big)\Big).$$

Lemma 3.3. *The element $\psi(w)$ is described by*

$$\psi(w) \equiv (-1)^{s+1} s T^{p-k} B(\emptyset, 0).$$

Proof. If $s = 1$, then $\psi(w) \equiv T^{p-k} B(\emptyset, 0)$ and the formula is valid. Therefore we will assume $s > 1$.

We begin by analyzing coefficients at expressions of the type $T^{k_i} A(J, s-2-|J|-i)$ for various sets J and $i = 0, \dots, s-2$.

If $i=0$, then $|J| \leqslant s-2$, and this term appears once with coefficient $(-1)^{|J|}$ as a special term in the second sum which corresponds to the choice $\mathcal{I}=J$, and a second time with coefficient $(-1)^{|J|+1}$ corresponding to the choice $\mathcal{I}=J \cup 1$ (for which $|\mathcal{I}|=s-1$) and both terms cancel out.

If $0<i \leqslant s-2-|J|$, then this term appears twice. The first time it appears with coefficient $(-1)^{|J|+i}$ corresponding to $\mathcal{I}=J \cup i$ (for which $|\mathcal{I}| \leqslant s-2$), and the second time with coefficient $(-1)^{|J|+i+1}$ corresponding to $\mathcal{I}=J \cup i+1$ (for which $|\mathcal{I}| \leqslant s-1$) and both terms cancel out.

Therefore all terms of type $T^{k_i} A(J, s-2-|J|-i)$ will cancel out.

Similar argument can be applied to expressions of the type $T^{k_i} C(J, l, l-1-|J|-i)$ for any fixed l. In this case there will be two terms, first with coefficient $(-1)^{s+l+|J|+i}$, and second with coefficient $(-1)^{s+l+|J|+i+1}$ and they will cancel out as well.

Finally, we analyze terms of type $T^{s(p-k)} B(\mathcal{I}, s-|J|-1)$. We have $l-1-|J|<s-1-|J|$ and assume that $0<l-1-|J|$. In this first case there are three terms, two of them with coefficients $(s-l)(-1)^{|J|+s+l}$ and $(s-(l-1))(-1)^{|J|+s+(l-1)}$ corresponding to $\mathcal{I}=J$ and the third term with coefficient $(-1)^{|J|+s-l}$ corresponding to the choice $\mathcal{I}=J \cup s-l$ (for which $|\mathcal{I}|<s-1$). Note that in this case $l>1$ and $l-1 \geqslant 1$ is within our range of summation. All these three terms will cancel out.

If $0=l-1-|J|$ and $|J|>0$, then the same argument remains valid since $l>1$.

The only remaining case is when $J=\emptyset$ and $l=1$. The corresponding term $T^{p-k} B(\emptyset, 0)$ appears twice. The first time with coefficient $(s-1)(-1)^{s-1}$ corresponding to $\mathcal{I}=J$, the second time with coefficient $(-1)^{s-1}$ corresponding to $\mathcal{I}=J \cup s-1$. The sum of these two terms equals $(-1)^{s+1} s T^{p-k} B(\emptyset, 0)$. Therefore

$$\begin{aligned} \psi(w) &= (-1)^{s+1} s T^{p-k} B(\emptyset, 0) \\ &= (-1)^{s+1} s T^{p-k} Sym_{x,m-1}\left(k^{m-1}\right) Sym_{y,n-1}\left((p-k)^{n-1}\right). \qquad \square \end{aligned}$$

We can now easily prove Proposition 3.1.

Proof of Proposition 3.1. Since $s<p$, we can take

$$v_k=\frac{(-1)^s}{s} w+Sym_{x,m}\left(k^{m-1}\right) Sym_{y,n}\left((p-k)^n\right).$$

Then

$$\begin{aligned} \psi(v_k)= & -T^{p-k} Sym_{x,m-1}\left(k^{m-1}\right) Sym_{y,n-1}\left((p-k)^{n-1}\right)+\left((1-\delta_{m,1}) T^k Sym_{x,m-1}\left(k^{m-2}\right)\right. \\ & \left.+Sym_{x,m-1}\left(k^{m-1}\right)\right) T^{p-k} Sym_{y,n-1}\left((p-k)^{n-1}\right) \equiv 0 \end{aligned}$$

meaning that $v_k \in A_s(m|n)$.

Observe that $w|_{x_m=0}=0$ because all numbers k, $l(p-k)$ and each k_i are positive. Therefore $v_k|_{x_m=0}=Sym_{x,m-1}(k^{m-1}) Sym_{y,n}((p-k)^n)=u_k(m-1|n)$. It remains to observe that v_k is homogeneous of degree $(m-1)k+(p-k)n$. $\square$

4. Concluding remarks

Let us comment that if the characteristic of K is positive, then the condition that $f|_{x_m=y_n=T}$ does not depend on T is stronger than the condition that $\frac{d}{dT} f|_{x_m=y_n=T}=0$.

Proposition 3.1 of [3] states that, in the case of characteristic zero, the algebra A_s is infinitely generated. In the case of positive characteristic we have the following.

Proposition 4.1. *The algebra A_s is finitely generated.*

Proof. The algebra A_s is contained in $B = K[\sigma_i(x|m), \sigma_j(y|n) \mid 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n]$. The algebra B is finitely generated over its subalgebra $B' = K[\sigma_i(x|m)^p, \sigma_j(y|n)^p \mid 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n]$, hence B a Noetherian B'-module. However, A_s contains B' and is therefore a finitely generated B'-module. Since B' is finitely generated, so is A_s. □

Acknowledgment

Authors would like to thank the referee for careful reading and helpful suggestions that improved the presentation of this article.

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