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Edge-transitive regular Z_n-covers of the Heawood graph

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ABSTRACT

A regular cover of a graph is said to be an edge-transitive cover if the fibre-preserving automorphism subgroup acts edge-transitively on the covering graph. In this paper we classify edge-transitive regular Z_n-covers of the Heawood graph, and obtain a new infinite family of one-regular cubic graphs. Also, as an application of the classification of edge-transitive regular Z_n-covers of the Heawood graph, we prove that any bipartite edge-transitive cubic graph of order $14p$ is isomorphic to a normal Cayley graph of dihedral group if the prime $p > 13$.

1. Introduction

Throughout this paper, graphs are assumed to be finite, and unless specified otherwise, simple, undirected and connected. The group theoretic notations are standard (see [8] and [35]). For a graph X, we denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its automorphism group, respectively. An s-arc in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \le i \le s$ and $v_{i-1} \ne v_{i+1}$ for $1 \le i \le s-1$; in other words, a directed walk of length s which never includes the reverse of an arc just crossed.

A graph X is said to be s-arc-transitive if $\mathrm{Aut}(X)$ is transitive on the set of all s-arcs in X. In particular, 0-arc-transitive means vertex-transitive, and 1-arc-transitive means arc-transitive (or symmetric). Let $G \le \mathrm{Aut}(X)$. The graph is said to be G-edge-transitive (resp. G-vertex-transitive) if G acts transitively on its edge set (resp. its vertex set), and to be G-half-transitive if G is vertex-transitive and edge-transitive but not arc-transitive, and to be G-semisymmetric if it is regular and G-edge-transitive but not G-vertex-transitive. In particular, if $G = \mathrm{Aut}(X)$, we simply call the graph X edge-transitive, vertex-transitive, half-transitive, and semisymmetric, respectively. An arc-transitive graph X is said to be s-regular if $\mathrm{Aut}(X)$ acts regularly on the set of all s-arcs of X. Clearly, a 1-regular graph must be connected and a graph of valency 2 is 1-regular if and only if it is a cycle.

There is a connection between the edge-transitivity and the vertex-transitivity of the graphs. Many people investigated the automorphism groups of symmetric and semisymmetric cubic graphs. Several different types of infinite families of s-regular graphs were constructed (see [3,6,7,11,13,19,31,33,36]). Marušič and Xu [29] showed a way to construct a 1-regular cubic graph Y from a half-transitive graph X of valency 4 with girth 3. Also, Marušič and Pisanski classified s-regular cubic graphs of dihedral groups [28], and Feng, Kwak and Wang classified cubic symmetric graphs of orders $8p$ and $8p^2$ [12]. The study of semisymmetric graphs originated by Folkman who also posed several problems in [14]. With the solving of the problems, several families of semisymmetric graphs were constructed [1,2,4,10,18,20,21,23–25,30]. Du and Xu classified semisymmetric graphs of order $2pq$, where p and q are primes [9]. As for the classifications of semisymmetric cubic graphs, Infinova and Ivanov classified biprimitive cubic graphs [17], Malnič, Marušič and Wang classified those of order $2p^3$ [26], Lu, Wang and Xu classified those of order $6p^2$ [22].

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This paper aims to find the influence of edge-transitivity of a regular Z_n-cover of the Heawood graph on its vertex-transitivity. We shall classify edge-transitive regular Z_n-covers of the Heawood graph, and obtain a new infinite family of one-regular cubic graphs. Also, we prove that if $p > 13$, any bipartite edge-transitive cubic graph of order $14p$ is a normal Cayley graph of D_{14p} and one-regular, and hence known by [28].

Given a positive integer n, we shall use the symbol Z_n to denote the ring of residues modulo n as well as the cyclic group of order n.

The main result of this paper is as follows.

Theorem 1.1. *Let X be a connected edge-transitive Z_n-cover of the Heawood graph $\mathcal{H}$. Then $n = 3^k p_1^{e_1} \cdots p_t^{e_t}$, $k = 0$ or 1, $t \geq 1$, primes p_i, $i = 1, \ldots, t$, are different primes with $p_i = 1 (\mathrm{mod}\ 3)$, and X is symmetric and isomorphic to a normal Cayley graph $\mathrm{Cay}(K, T)$ for some group K with respect to a generating set T. Furthermore, one of the following holds:*

(1) *If 7 is coprime to n, then $K = \langle x, z | x^{7n} = z^2 = 1, zxz = x^{-1}\rangle \cong D_{14n}$, $T = \{z, xz, x^{s+1}z\}$, $s^2 + s + 1 = 0 (\mathrm{mod}\ 7n)$, and X is one-regular;*
(2) *If $n = 7$, then X is either one-regular and $K = \langle x, z | x^{49} = z^2 = 1, zxz = x^{-1}\rangle \cong D_{98}$ and $T = \{z, xz, x^{19}z\}$, or 2-regular and isomorphic to $K = \langle x, y, z | x^7 = y^7 = z^2 = 1, xy = yx, zxz = x^{-1}, zyz = y^{-1}\rangle \cong (Z_7 \times Z_7) : Z_2$, $T = \{z, x^4yz, x^5yz\}$.*
(3) *If $n = 21$, then X is either one-regular and $K = \langle x, z | x^{147} = z^2 = 1, zxz = x^{-1}\rangle \cong D_{294}$ and $T = \{z, xz, x^{68}z\}$, or 2-regular and $K = \langle x, y, z | x^{21} = y^7 = z^2 = 1, xy = yx, zxz = x^{-1}, zyz = y^{-1}\rangle \cong (Z_{21} \times Z_7) : Z_2$, $T = \{z, x^4yz, x^5yz\}$.*
(4) *If $7|n$ and $n \notin \{7, 21\}$, then X is one-regular, and either $K = \langle x, y, z | x^n = y^7 = z^2 = 1, xy = yx, zxz = x^{-1}, zyz = y^{-1}\rangle \cong (Z_n \times Z_7) : Z_2$, $T = \{z, x^{-2}yz, x^{-3}yz\}$, or $K = \langle x, z | x^{7n} = z^2 = 1, zxz = x^{-1}\rangle \cong D_{14n}$, $T = \{z, xz, x^{s+1}z\}$, $s^2 + s + 1 = 0 (\mathrm{mod}\ 7n)$.*

The Cayley graphs of dihedral groups in Theorem 1.1 are one-regular Cayley graphs of dihedral groups D_{14n} and have been found in [19]. The one-regular Cayley graphs of $(Z_n \times Z_7) : Z_2$ in (4) of Theorem 1.1 are new, because they have different Sylow 7-subgroups of their automorphism groups from the one-regular Cayley graphs in (1) of Theorem 1.1. Therefore we obtain a new infinite family of one-regular cubic graphs.

Theorem 1.2. *Let X be a connected edge-transitive cubic graph of order $14p$, where p is a prime greater than 13. If X is bipartite, then $p = 1 (\mathrm{mod}\ 3)$, and X is one-regular and isomorphic to a normal Cayley graph of D_{14p} (as stated in (1) of Theorem 1.1).*

For $p = 13$ there is a semisymmetric cubic graph of order $14p$ by [19]. Also, the case $p = 7$ has already been investigated in Theorem 1.1.

2. Preliminaries

A Cayley graph $X = \mathrm{Cay}(G, S)$ is defined to be a graph with vertex set G and edge set $\{(g, sg) | g \in G, s \in S\}$, where S is a generating set of a group G with $1 \notin S$ and $S^{-1} = S$. Clearly, the group $R(G)$ of right translations is a subgroup of $\mathrm{Aut}(X)$. If $R(G)$ is normal in $\mathrm{Aut}(X)$, we call the Cayley graph X normal. Denote $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}G | S^\alpha = S\}$. Then $\mathrm{Aut}(G, S)$ is a subgroup of A_1, and $\mathrm{Cay}(G, S)$ is normal if and only if $\mathrm{Aut}(G, S) = A_1$. Determining the normality of a Cayley graph play an important role in the study of the Cayley graph.

A graph X is called a covering graph of Y with projection $\mathcal{P} : X \longrightarrow Y$ if there is a surjection $\mathcal{P} : V(X) \longrightarrow V(Y)$ such that $\mathcal{P}|_{N(u)} : N(u) \longrightarrow N(v)$ is a bijection for any vertex $v \in V(Y)$ and $u \in \mathcal{P}^{-1}(v) \subset V(X)$, where $N(u)$ is the neighborhood of u. The covering graph is said to be regular (or K-covering) if there is a subgroup K of $\mathrm{Aut}(X)$ which is semiregular on both $V(X)$ and $E(X)$ such that the graph Y is isomorphic to the quotient graph X/K. If the regular covering is connected, then K is called a covering transformation group. The fibre of an edge or a vertex is its preimage under $\mathcal{P}$. The graph X is called the covering graph and Y is the base graph. The group of automorphisms of which maps fibres to fibres is called the fibre -preserving subgroup of $\mathrm{Aut}(X)$.

Every edge of a graph X gives rise to a pair of arcs of opposite direction. Let K be a finite group and denote by $A(Y)$ the arc set of Y. An ordinary voltage assignment (or, K-voltage assignment) of Y is a function $\phi : A(Y) \longrightarrow K$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in A(Y)$. We call the value $\phi(e)$ the voltage of the arc e, and K the voltage group. The ordinary derived graph $Y \times_\phi K$ derived from an ordinary voltage assignment $\phi : A(Y) \longrightarrow K$ has vertex set $V(Y) \times K$ and edge set $A(Y) \times K$, so that an edge (e, g) of $Y \times_\phi K$ joins a vertex (u, g) to $(v, \phi(e)g)$ for $e = (u, v) \in A(Y)$ and $g \in K$.

Let $\mathcal{P} : \widetilde{Y} \longrightarrow Y$ be a K-covering. We call $\widetilde{Y}$ an edge-transitive cover of Y if the fibre-preserving automorphism subgroup acts edge-transitively on $\widetilde{Y}$. If $\alpha \in \mathrm{Aut}(Y)$ and $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{Y})$ satisfy $\widetilde{\alpha}\mathcal{P} = \mathcal{P}\alpha$, we call $\widetilde{\alpha}$ a lift of α, and α the projection of $\widetilde{\alpha}$. Concepts such as a lift of a subgroup of $\mathrm{Aut}(Y)$ and the projection of a subgroup of $\mathrm{Aut}(\widetilde{Y})$ are self-explanatory. The lifts and the projections of such subgroups are of course subgroups in $\mathrm{Aut}(\widetilde{Y})$ and $\mathrm{Aut}(Y)$, respectively. Note that the covering transformation group K is the lift of the identity group.

Proposition 2.1 ([32]). *Each connected K-covering of a graph Y can be derived from a K-voltage assignment which assigns the identity voltage 1 to the arcs on an arbitrary fixed spanning tree of Y.*

Let $Y \times_\phi K$ be a K-covering of the connected base graph Y, where $\phi = 1$ on the arcs of a spanning tree T of Y. Such ϕ is called a T-reduced voltage assignment.

Proposition 2.2 ([16]). *The covering graph $Y \times_\phi K$ is connected if and only if the voltage on the cotree arcs generates the voltage group K.*

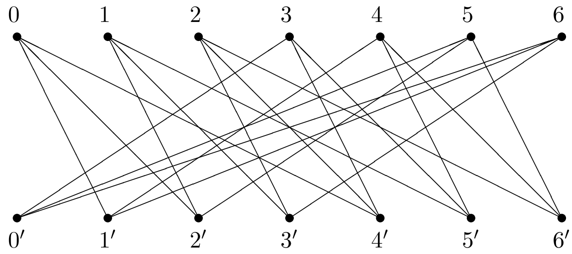

Fig. 1. The Heawood graph $\mathcal{H}$ with voltage assignment ϕ.

The problem whether an automorphism α of Y lifts is solved in [27].

Proposition 2.3 ([27]). *Let $Y \times_{\phi} K \longrightarrow Y$ be a connected K-covering. Then an automorphism α of Y lifts if and only if $\alpha^{\#}$ extends to an automorphism of K.*

Proposition 2.4 ([32]). *Leaving the voltages of a spanning tree trivial and replacing the voltage assignments on the cotree arcs by their images under an automorphism of the voltage group results in an equivalent covering projection.*

The following proposition gives a criterion for the existence of one-regular Cayley graphs of dihedral groups. We shall need it later to classify edge-transitive Z_n-covers of Heawood graph.

Proposition 2.5 ([24]). *There is a subgroup $\langle s\rangle$ of order 3 in Z_n^* such that $s^2+s+1=0(\text{mod } n)$ if and only if $n=3^k p_1^{e_1}\cdots p_t^{e_t}$, where $k=0$ or 1, $t\geq 1$, p_i's are different primes, and $p_i=1(\text{mod } 3)$.*

Proposition 2.6 ([34]). *Let X be a connected cubic G-semisymmetric graph for some subgroup G of $\mathrm{Aut}\,X$, $\{u,v\}\in E(X)$, and $N\lhd G$. Then*

(1) *If 3 divides neither $|N_u|$ nor $|N_v|$, in particular, if 3 does not divide $|N|$, then N is semiregular on each part of the bipartite graph X, and X is a regular N-cover of X/N.*
(2) *If 3 does not divide $|G/N|$, then N is semisymmetric on X.*
(3) *If N is not semiregular on each partition set of X, nor semisymmetric on X, then N is transitive on one partition set of the bipartite graph X but intransitive on the other.*

3. The Proof of Theorem 1.1

First we investigate the edge-transitive Z_n-covers of the Heawood graph. Recall its definition. Let $U=\{0,1,2,3,4,5,6\}$ and $U'=\{0',1',2',3',4',5',6'\}$ be two copies of Z_7. The Heawood graph $\mathcal{H}$ has $V=U\bigcup U'$ as its vertex set, with $i\in U$ being adjacent to $j'\in U'$ if and only if $j-i\in\{1,2,4\}$. It is well-known that $\mathrm{Aut}(\mathcal{H})\cong PGL(2,7)$. Note that the unique minimal arc-transitive subgroup of $\mathrm{Aut}(\mathcal{H})$ (up to conjugation in $\mathrm{Aut}(\mathcal{H})$) is generated by the following permutations:

$$\rho: i\longrightarrow i+1,\qquad i'\longrightarrow (i+1)'$$
$$\sigma: i\longrightarrow 2i,\qquad i'\longrightarrow (2i)'$$
$$\tau: i\longrightarrow (-i)',\qquad i'\longrightarrow (-i).$$

Moreover, the unique minimal edge-transitive subgroup of $\mathrm{Aut}(\mathcal{H})$, up to conjugation in $\mathrm{Aut}(\mathcal{H})$, is the group $S=\langle\rho,\sigma\rangle\cong Z_7:Z_3$.

Lemma 3.1. *Let X be a connected edge-transitive regular Z_n-cover of the Heawood graph $\mathcal{H}$ such that the automorphisms ρ and σ lift. Then X is arc-transitive.*

Proof. We shall show that τ also lifts. Without loss of generality we can assume that the arcs on the Hamiltonian path $(5,0',6,1',0,2',1,3',2,4',3,5',4,6')$ have the trivial voltage. Let e be the voltage of the arc $(5,6')$ and x_i the voltage of the arc $(i,(i+4)')$, for $i\in Z_7$.

We can check that the cycles $(5,6',4,5',3,4',2,3',1,2',0,1',6,0',5)$, $(0,4',2,3',1,2',0)$, $(1,5',3,4',2,3',1)$, $(2,6',4,5',3,4',2)$, $(3,0',6,1',0,2',1,3',2,4',3)$, $(4,1',0,2',1,3',2,4',3,5',4)$, $(5,2',0,1',6,0',5)$, $(6,3',1,2',0,1',6)$ are the fundamental cycles of the graph $\mathcal{H}$ (see Fig. 1). It follows that $\rho^{\#}$, $\sigma^{\#}$ and $\tau^{\#}$ map the voltages $x_0,x_1,x_2,x_3,x_4,x_5,x_6$ and e as follows (we use the additive notation for the operation in the abelian group $G=Z_n$) (see Table 1).

It is well-known that, for any divisor m of n, $G=Z_n$ has exactly one subgroup of order m, which is invariant under any automorphism of G. Since ρ and σ lift, by Proposition 2.3, $\rho^{\#}$ and $\sigma^{\#}$ extend to automorphisms of G. So the voltages of the images of the fundamental cycles under ρ and σ generate the same subgroups as the voltages of the fundamental cycles. It follows from the row related to $\rho^{\#}$ in Table 1 that

$$\langle x_0\rangle=\langle x_1\rangle=\langle x_2\rangle=\langle x_3+e\rangle=\langle x_4+e\rangle=\langle x_5\rangle=\langle x_6\rangle.$$

Also, by the row related to $\sigma^{\#}$ in Table 1 we get that $\langle e\rangle=\langle x_0+x_1+x_2+x_3+x_4+x_5+x_6\rangle$, and $\langle x_5\rangle=\langle x_0+x_3+x_5\rangle$. This implies that $\langle x_3+x_0\rangle\leq\langle x_5\rangle=\langle x_0\rangle$. Therefore, we have $\langle e,x_0,x_1,x_2,x_3,x_4,x_5,x_6\rangle=\langle x_0\rangle$. Since X is connected, the voltages $\langle e,x_0,x_1,x_2,x_3,x_4,x_5,x_6\rangle$ generates the voltage group Z_n.

Table 1
Voltages of the images of fundamental cycles.

Vol	e	x_0	x_1	x_2
$\rho^\#$	e	x_1	x_2	x_3+e
$\sigma^\#$	$\sum_{i=0}^{i=6}(-x_i)$	$-x_0-x_2-x_4$	$-x_2-x_4-x_6$	$-x_1-x_4-x_6$
$\tau^\#$	$-e$	$-x_3-e$	$-x_2$	$-x_1$
Vol	x_3	x_4	x_5	x_6
$\rho^\#$	x_4	x_5-e	x_6	x_0
$\sigma^\#$	$x_0+x_2+x_4+x_5$	$x_0+x_2+x_4+x_6$	$-x_3-x_5-x_0$	$e-x_0-x_2-x_5$
$\tau^\#$	$-x_0+e$	$-x_6+e$	$-x_5$	$-x_4-e$

Table 2
Voltages of the images of fundamental cycles.

Vol	$e=s+3$	$x_0=1$	$x_1=1$	$x_2=1$
$\rho^\#$	$s+3$	1	1	1
$\sigma^\#$	$2s-1$	s	s	s
$\tau^\#$	$-s-3$	-1	-1	-1
Vol	$x_3=-s-2$	$x_4=-s-2$	$x_5=1$	$x_6=1$
$\rho^\#$	$-s-2$	$-s-2$	1	1
$\sigma^\#$	$1-s$	$1-s$	s	s
$\tau^\#$	$s+2$	$s+2$	-1	-1

Since ρ and σ lift, there are λ and μ in Z_n^* such that $\rho^\#(x)=\lambda x$ and $\sigma^\#(x)=\mu x$ for $x\in\{e,x_0,x_1,x_2,x_3,x_4,x_5,x_6\}$. Since ρ is of order 7, so is $\rho^\#$. Similarly, $\sigma^\#$ has order 3. It follows that $\lambda^7=\mu^3=1$.

By Proposition 2.4, we may assume that $x_0=1$. Then we obtain from the row of $\rho^\#$ in Table 1 that $e=\lambda e, x_1=\lambda x_0=\lambda$, $x_2=\lambda x_1=\lambda^2, x_3=\lambda^3-e, x_4=\lambda^4-e, x_5=\lambda^5$, and $x_6=\lambda^6$. Since $\sigma^\#(x_0)=-x_0-x_2-x_4$ we have $\mu=-x_0-x_2-x_4$. It follows that $\mu=-\lambda^4+e-\lambda^2-1$. Similarly, $-x_1-x_4-x_6=\sigma^\#(x_2)=\mu x_2$. Combining with $e=\lambda e$, we have that $\lambda^2=\lambda$. Consequently,

$$x_1=x_2=x_5=x_6=\lambda,\qquad x_3=x_4=\lambda-e,\qquad \mu=-1-2\lambda+e.$$

From $e-x_0-x_2-x_5=\sigma^\#(x_6)=\mu x_6$, we get that $e-1-2\lambda=-\lambda-2\lambda^2+e\lambda=e-3\lambda$, and hence

$$\lambda=1,\qquad x_1=x_2=x_5=x_6=1,\qquad x_3=x_4=1-e,\qquad \mu=e-3.$$

Since $\sum_{i=0}^{i=6}(-x_i)=\mu e$, we conclude that $e^2-5e+7=0(\text{mod } n)$. Set $s=e-3$. Then $s^2+s+1=0(\text{mod } n)$, and $s\in Z_n^*$. By Proposition 2.5 it follows that $n=3^k p_1^{e_1}\cdots p_t^{e_t}$, where $k=0$ or 1, $t\geq 1$, p_i's are different primes, and $p_i=1(\text{mod } 3)$.

So Table 1 can be showed as follows (see Table 2).

Observe from Table 2 that $\rho^\#=\text{id}$, $\sigma^\#: x\longrightarrow sx$, and $\tau^\#=-\text{id}$. By Proposition 2.3 τ lifts. This implies that the covering graph is arc-transitive. ■

Next we determine the lifted group. Note that the Heawood graph has a full automorphism group isomorphic to $PGL(2,7)$, and its arc-stabilizer is isomorphic to D_8. Denote

$$a=(36)(1'5')(3'4')(01),$$
$$b=(1'3')(4'5')(01)(24),$$
$$c=(36)(24)(5'3')(1'4'),$$
$$d=(0'2')(24)(3061)(3'5'4'1').$$

Then a,b,c,d fix the arc $(5,6')$. It is easy to see that $c^2=d^4=1, d^2=a, d^c=d^{-1}$, and $b=ac$. So the arc-stabilizer of $(5,6')$ is $\langle a,b,c,d\rangle=\langle c,d\rangle\cong D_8$.

Lemma 3.2. *Let X be a connected edge-transitive regular Z_n-cover of H. If ρ,σ and τ lift, then the lifted group acts regularly on the arc set.*

Proof. By checking the fundamental cycles we know that $a^\#, b^\#, c^\#, d^\#, (bd)^\#$ and $(cd)^\#$ map the voltages x_0,x_1,x_2,x_3,x_4,x_5, x_6 and e as follows (see Table 3).

Since

$$a^\#(x_1)=x_6-x_0=0,\qquad b^\#(x_5)=x_5-x_6=0,\qquad c^\#(x_6)=-x_1+x_0=0,$$
$$d^\#(x_5)=x_6-x_5=0,\qquad (cd)^\#(x_2)=x_2-x_1=0,\qquad (bd)^\#(x_1)=-x_6+x_0=0,$$

it follows by Proposition 2.3 that a,b,c,d,cd and bd do not lift. ■

Table 3
Voltages of the images of fundamental cycles.

Vol	e	x_0	x_1	x_2	x_3	x_4	x_5	x_6
$a^\#$	$e+x_4+x_6-x_0+x_1+x_3$	$-x_0$	x_6-x_0	$x_2+x_4+x_6$	$-x_3-x_1+x_0-x_6$	$-x_1+x_0-x_4-x_6$	$x_1+x_3+x_5$	$-x_0+x_1$
$b^\#$	$e-x_2+x_4-x_6$	x_1+x_4	x_0+x_4	$-x_2$	$x_3+x_6-x_4$	$-x_4$	x_5-x_6	$-x_6$
$c^\#$	$e-x_2-x_6-x_4-x_1+x_0+x_3$	$-x_1-x_4$	$-x_6-x_4-x_1$	$-x_2-x_4-x_6$	$-x_3-x_0+x_1+x_4$	$-x_0+x_1+x_4+x_6$	$x_0+x_3+x_5$	$-x_1+x_0$
$d^\#$	$e-x_2-x_0-x_4+x_3+x_6-x_5$	$-x_4+x_3$	$-x_0-x_4$	$-x_0-x_2-x_4$	$-x_3-x_6+x_4$	$-x_3-x_6+x_4+x_0$	x_6-x_5	$x_1+x_3+x_6$
$(cd)^\#$	$e-x_1+x_3-x_5$	x_3+x_6	$-x_1$	x_2-x_1	$-x_3$	$x_4-x_3+x_1$	$-x_5$	x_0+x_3
$(bd)^\#$	$e+x_4+x_0-x_6-x_1-x_3-x_5$	$-x_6-x_3$	$-x_6+x_0$	$x_2+x_4+x_0+x_6$	$x_3+x_1+x_6-x_0$	$x_3+x_6-x_0-x_4$	$-x_5-x_1-x_3$	$-x_6-x_1-x_3$

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $X = X(n, s)$ be the edge-transitive Z_n-cover of the Heawood graph $\mathcal{H}$ determined in Lemma 3.1. By Proposition 2.5 we may suppose that $n = 3^k p_1^{e_1} \cdots p_t^{e_t}$, where $k = 0$ or 1, $t \geq 1$, p_i's are different primes, and $p_i = 1 (\mathrm{mod}\ 3)$, and that $s \in Z_n$ satisfy $s^2 + s + 1 = 0 (\mathrm{mod}\ n)$. It is easy to see that the edge-transitive Z_n-cover $X = X(n, s)$ of the Heawood graph is defined by

$$V(X) = \{i_x, i'_x \mid i \in Z_7, x \in Z_n\},$$

$$E(X) = \{i_x(i+1)'_x \mid i \in \{0, 1, 2, 3, 4, 6\}, x \in Z_n\} \bigcup \{i_x(i+2)'_x \mid i \in Z_7, x \in Z_n\}$$

$$\bigcup \{0_x 4'_{x+1}, 1_x 5'_{x+1}, 2_x 6'_{x+1}, 3_x 0'_{x-s-2}, 4_x 1'_{x-s-2}, 5_x 6'_{x+s+3}, 5_x 2'_{x+1}, 6_x 3'_{x+1} \mid x \in Z_n\}.$$

Set

$\alpha : i_x \longrightarrow i_{x+1}, i'_x \longrightarrow i'_{x+1}$;

$\beta_1 : 0_x \longrightarrow 0_{sx}, 1_x \longrightarrow 2_{sx+1}, 2_x \longrightarrow 4_{sx+2}, 3_x \longrightarrow 6_{sx-s}, 4_x \longrightarrow 1_{sx-s+1}, 5_x \longrightarrow 3_{sx+s+1}, 6_x \longrightarrow 5_{sx-1}, 0'_x \longrightarrow 0'_{sx-1}, 1'_x \longrightarrow 2'_{sx}, 2'_x \longrightarrow 4'_{sx+1}, 3'_x \longrightarrow 6'_{sx+2}, 4'_x \longrightarrow 1'_{sx-s}, 5'_x \longrightarrow 3'_{sx-s+1}, 6'_x \longrightarrow 5'_{sx-s+2}$;

$\beta_2 : 0_x \longrightarrow 6_{sx}, 1_x \longrightarrow 1_{sx+1}, 2_x \longrightarrow 3_{sx+2}, 3_x \longrightarrow 5_{sx-s}, 4_x \longrightarrow 0_{sx-s+1}, 5_x \longrightarrow 2_{sx+s+1}, 6_x \longrightarrow 4_{sx+s+2}, 0'_x \longrightarrow 6'_{sx+s+2}, 1'_x \longrightarrow 1'_{sx}, 2'_x \longrightarrow 3'_{sx+1}, 3'_x \longrightarrow 5'_{sx+2}, 4'_x \longrightarrow 0'_{sx-s}, 5'_x \longrightarrow 2'_{sx-s+1}, 6'_x \longrightarrow 4'_{sx-s+2}$;

$\gamma : 0_x \longleftrightarrow 4'_{-x+1}, 1_x \longleftrightarrow 3'_{-x+1}, 2_x \longleftrightarrow 2'_{-x+1}, 3_x \longleftrightarrow 1'_{-x+1}, 4_x \longleftrightarrow 0'_{-x+1}, 5_x \longleftrightarrow 6'_{-x+1}, 6_x \longleftrightarrow 5'_{-x+1}$.

Then we can check that $\alpha, \beta_1, \beta_2, \gamma \in \mathrm{Aut}(X)$, $\alpha^n = \beta_1^3 = \beta_2^3 = \gamma^2 = 1$, $\beta_1^{-1}\alpha\beta_1 = \beta_2^{-1}\alpha\beta_2 = \alpha^s$, and $\gamma^{-1}\alpha\gamma = \alpha^{-1}$. Put $G = \langle \alpha, \beta_1, \beta_2, \gamma \rangle$ and $\varphi = \beta_1^{-1}\beta_2$. Then $\varphi\alpha = \alpha\varphi$, and

$\varphi : 0_x \longrightarrow 6_x, 1_x \longrightarrow 0_x, 2_x \longrightarrow 1_x, 3_x \longrightarrow 2_x, 4_x \longrightarrow 3_x, 5_x \longrightarrow 4_{x+s+3}, 6_x \longrightarrow 5_x, 0'_x \longrightarrow 6'_{x+s+3}, 1'_x \longrightarrow 0'_x, 2'_x \longrightarrow 1'_x, 3'_x \longrightarrow 2'_x, 4'_x \longrightarrow 3'_x, 5'_x \longrightarrow 4'_x, 6'_x \longrightarrow 5'_x$. So G is transitive on $V(X)$ and $E(X)$, and X is G-arc-transitive. It is easy to see that $\alpha\varphi = \varphi\alpha$ and $\varphi^7 = \alpha^{s+3}$, and

$\gamma^{-1}\varphi\gamma : 0_x \longrightarrow 1_x, 1_x \longrightarrow 2_x, 2_x \longrightarrow 3_x, 3_x \longrightarrow 4_x, 4_x \longrightarrow 5_{x-s-3}, 5_x \longrightarrow 6_x, 6_x \longrightarrow 0_x, 0'_x \longrightarrow 1'_x, 1'_x \longrightarrow 2'_x, 2'_x \longrightarrow 3'_x, 3'_x \longrightarrow 4'_x, 4'_x \longrightarrow 5'_x, 5'_x \longrightarrow 6'_x, 6'_x \longrightarrow 0'_{x-s-3}$.

So $\gamma^{-1}\varphi\gamma = \varphi^{-1}$. Since

$[\beta_1, \gamma] = \beta_1^{-1}\gamma^{-1}\beta_1\gamma : 0_x \longrightarrow 3_{x+1}, 1_x \longrightarrow 4_{x+1}, 2_x \longrightarrow 5_{x-s-2}, 3_x \longrightarrow 6_{x-s-2}, 4_x \longrightarrow 0_{x-s-2}, 5_x \longrightarrow 1_{x+1}, 6_x \longrightarrow 2_{x+1}, 0'_x \longrightarrow 3'_{x+1}, 1'_x \longrightarrow 4'_{x+1}, 2'_x \longrightarrow 5'_{x+1}, 3'_x \longrightarrow 6'_{x+1}, 4'_x \longrightarrow 0'_{x-s-2}, 5'_x \longrightarrow 1'_{x-s-2}, 6'_x \longrightarrow 2'_{x-s-2}$,

It follows that $[\beta_1, \gamma] = \varphi^4\alpha^{-s-2}$. Also,

$\beta_1^{-1}\varphi\beta_1 : 0_x \longrightarrow 5_{x-1}, 1_x \longrightarrow 6_{x-1}, 2_x \longrightarrow 0_{x-1}, 3_x \longrightarrow 1_{x-1}, 4_x \longrightarrow 2_{x-1}, 5_x \longrightarrow 3_{x+s+2}, 6_x \longrightarrow 4_{x+s+2}, 0'_x \longrightarrow 5'_{x+s+2}, 1'_x \longrightarrow 6'_{x+s+2}, 2'_x \longrightarrow 0'_{x-1}, 3'_x \longrightarrow 1'_{x-1}, 4'_x \longrightarrow 2'_{x-1}, 5'_x \longrightarrow 3'_{x-1}, 6'_x \longrightarrow 4'_{x-1}$.

Now we can check that $\beta_1^{-1}\varphi\beta_1 = \varphi^2\alpha^{-1}$.

Let $A = \mathrm{Aut}(X)$. First we show that A_v is faithful on $N(v)$. Assume that $\psi \in A$ fixes 0_0, $1'_0$, $2'_0$ and $4'_1$. There are three cycles of length 6 passing through 0_0, namely:

$$\mathcal{C}_1 : (0_0, 2'_0, 5_{-1}, 6'_{s+2}, 4_{s+2}, 1'_0, 0_0),$$

$\mathcal{C}_2 : (0_0, 4'_1, 2_1, 3'_1, 6_0, 1'_0, 0_0)$,
$\mathcal{C}_3 : (0_0, 2'_0, 1_0, 5'_3, 3_1, 4'_1, 0_0)$.

Note that there is a unique 6-cycles passing through the vertices 0_0, $1'_0$, and $2'_0$. Since ψ fixes the vertices 0_0, $1'_0$, and $2'_0$, it fixes every vertex on $\mathcal{C}_1$. Therefore ψ fixes 5_{-1} and 4_{s+2}. Similarly, ψ also fixes every vertex on $\mathcal{C}_2$ and $\mathcal{C}_3$. So ψ fixes every vertex in on $N_2(v)$. By induction, we know that ψ fixes all vertices of X. So $\psi = 1$. Therefore A_v is faithful on $N(v)$ for any $v \in V(X)$, and hence it is isomorphic to a subgroup of S_3. It follows that X is either one-regular or 2-regular.

Set $H = \langle \varphi, \alpha \rangle$. We divide the proof into two cases according to weather 7 divides n or not.

Case 1. 7 is coprime to n

Then $7 \in Z_n^*$. By $s^2 + s + 1 = 0 (\text{mod } n)$ we have that $(s + 3)(s - 2) = -7 \in Z_n^*$. This implies that $s + 3 \in Z_n^*$, and thus $H = \langle \varphi, \alpha \rangle = \langle \varphi \rangle \cong Z_{7n}$, $G = \langle \varphi, \beta_1, \gamma \rangle$. Then $\langle \varphi, \gamma \rangle \cong D_{14n}$, and $G/\langle \varphi \rangle$ is abelian. Hence $G/\langle \varphi \rangle \cong Z_6$. Since $\beta_1^{-1} \varphi \beta_1 = \varphi^2 \alpha^{-1}$, and $\gamma^{-1} \varphi \gamma = \varphi^{-1}$, it follows that H is normal in G.

We claim that H is the unique cyclic subgroup of G of order $7n$. Suppose that H_1 is a cyclic subgroup of G of order $7n$ with $H_1 \neq H$. Then by the normality of H in G we have HH_1 is a subgroup of G. Note that n is odd. This implies that HH_1 has odd order, too. It follows that $|HH_1| = 21n$. Since G/H is isomorphic to a subgroup of Z_6, $\langle \varphi, \beta_1 \rangle$ is the unique subgroup of G of order $21n$. Hence $HH_1 = \langle \varphi, \beta_1 \rangle \cong Z_{7n} : Z_3$. Then $|H \bigcap H_1| = \frac{7n}{3}$, and 3 dives n. Therefore, $H \bigcap H_1 = \langle \varphi^3 \rangle$ is contained in $Z(HH_1)$. However, it is obvious that $\varphi^3 \beta_1 : 0_x \longrightarrow 1_{sx+s}$, and $\beta_1 \varphi^3 : 0_x \longrightarrow 4_{sx+s+3}$. So $\varphi^3 \beta_1 \neq \beta_1 \varphi^3$, a contradiction. This proves our claim.

Now H is characteristic in G. Since G has index at most 2 in A, G is normal in A. It follows that H is normal in A. Hence $\langle \alpha \rangle = \langle \varphi^7 \rangle$ is normal in A, and A projects to $\text{Aut}(\mathcal{H})$. This implies that X is one-regular, and $A = G$. Then by the classification of cubic one-regular Cayley graphs of dihedral groups [11], we conclude that $X \cong \text{Cay}(H, \{\gamma, \varphi\gamma, \varphi^{s+1}\gamma\})$, where $s^2 + s + 1 = 0 (\text{mod } n)$.

Case 2. 7 divides n

Let $n = 7m$. Then $(s + 3)(s - 2) = -7 (\text{mod } n)$, and $\alpha^7 \in \langle \alpha^{s+3} \rangle = \langle \varphi^7 \rangle$. This implies that $\langle \alpha^{s+3} \rangle = \langle \alpha \rangle$ or $\langle \alpha^7 \rangle$. If $\langle \alpha^{s+3} \rangle = \langle \alpha \rangle$, we have also $\langle \varphi, \alpha \rangle = \langle \varphi \rangle$, and we can prove similarly that X is one-regular, and X is a normal Cayley graph of dihedral group $\langle \varphi, \gamma \rangle$. Therefore we have again that $X \cong \text{Cay}(H, \{\gamma, \varphi\gamma, \varphi^{s+1}\gamma\})$, where $s^2 + s + 1 = 0 (\text{mod } n)$.

Now let $\langle \alpha^{s+3} \rangle = \langle \alpha^7 \rangle$. Since $7|n$, and $s^2 + s + 1 = 0 (\text{mod } n)$, it follows that $s = 4$ or $2 (\text{mod } 7)$. Now $\langle \alpha^{s+3} \rangle = \langle \alpha^7 \rangle$ implies that $s = 4 (\text{mod } 7)$, and $(s - 2, 7) = 1$. In particular, $7 \nmid (s - 2)$. It follows by the definition of φ that $\varphi^{s-2} \notin \langle \alpha \rangle$.

Set $\delta = \varphi^{s-2}\alpha$. Then $\delta \neq 1$, and $\delta^7 = \varphi^{7(s-2)}\alpha^7 = \alpha^{(s+3)(s-2)+7} = 1$ for $\varphi^7 = \alpha^{s+3}$. Put $H = \langle \varphi, \alpha \rangle$, and $K = \langle H, \gamma \rangle$. Then $H = \langle \varphi \rangle \times \langle \delta \rangle \cong Z_n \times Z_7$, and $K = H : \langle \gamma \rangle$. For any $h \in H$, we have $\gamma^{-1} h \gamma = h^{-1}$. Moreover, K is regular on $V(X)$. So X is a Cayley graph of K with respect to $T = \{\gamma, \varphi^{-2}\alpha\gamma = \varphi^{-s}\delta\gamma, \varphi^{-3}\alpha\gamma = \varphi^{-s-1}\delta\gamma\}$. It can be checked that $\psi : \varphi \longrightarrow \varphi^{-s-1}\delta$, $\delta \longrightarrow \delta^s$, $\gamma \longrightarrow \varphi^{-s}\delta\gamma$ is in $\text{Aut}(K, T)$ of order 3. Hence $\text{Cay}(K, T)$ is arc-transitive, indeed.

Set $G = \langle \varphi, \alpha, \beta_1, \gamma \rangle$. Then H and K are normal in G, and $G = \langle \varphi, \alpha \rangle : \langle \beta_1, \gamma \rangle$, and $G/H \cong Z_6$.

We show that K is normal in $\text{Aut}(X)$, and hence $X = \text{Cay}(K, T)$ is a normal Cayley graph of K with respect to T. Suppose first that 7 divides n but 3 does not. Then $(|H|, |G/H|) = 1$, and so H is characteristic in G. Since K/H is characteristic in G/H, it follows that K is characteristic in G, and hence, is normal in A. Next we suppose that $7|n$ and $3|n$. Then by Proposition 2.5, 9 does not divide n. Note that $n = 3p_1^{e_1} \cdots p_t^{e_t}$, where p_i's are different primes, and $p_i = 1 (\text{mod } 3)$, and $H = \langle \varphi, \alpha \rangle = \langle \varphi \rangle \times \langle \delta \rangle \cong Z_n \times Z_7$. So H have a normal Sylow p_i-subgroup P_i. Then P_i is characteristic in H, and hence is also normal in G. Similarly, $\langle \alpha^{\frac{n}{3}} \rangle$ is normal in G of order 3. If G has a normal Sylow 3-subgroup P of order 9, then $P \cong Z_9$ or Z_3^2, and $\text{Aut}(P)$ is a $\{2, 3\}$-group. So each P_i centralises P, and hence G is abelian. This contradiction implies that $O_3(G)$ has order 3, and so $\langle \alpha^{\frac{n}{3}} \rangle = O_3(G)$. It follows that $H = F(G)$, the Fitting subgroup of G generated by all normal p-subgroups for all prime divisors p of $|G|$. Therefore, H is characteristic in G. Similarly we have that K is normal in A, and X is a normal Cayley graph of K with respect to T.

Suppose that $X = X(n, s)$ is not one-regular. Then there is some $\theta \in \text{Aut}(K, T)$ which fixes γ and interchanges $\varphi^{-s}\delta\gamma$ with $\varphi^{-s-1}\delta\gamma$. It follows that θ interchanges $\varphi^{-s}\delta$ with $\varphi^{-s-1}\delta$. So $\theta : \varphi \longrightarrow \varphi^{-1}$, $\delta \longrightarrow \varphi^{-2s-1}\delta$. Note that both δ and $\varphi^{-2s-1}\delta$ have order 7. Since δ commute with φ, it follows that $7(2s + 1) = 0$. Set $n = 7m$. Then $s(s - 1) = 0 (\text{mod } m)$. Since $s \in Z_n^*$, we have $s = 1 (\text{mod } m)$, and so $3 = 0 (\text{mod } m)$ by $s^2 + s + 1 = 0 (\text{mod } n)$. Consequently, $m = 1$ or 3, and $n = 7$ or 21, respectively.

If $n = 7$, then $K \cong Z_7^2 : Z_2$, $s = 4$, $T = \{\gamma, \varphi^3\delta\gamma, \varphi^2\delta\gamma\}$. It is easy to see that $\theta_1 : \gamma \longrightarrow \gamma$, $\varphi \longrightarrow \varphi^{-1}$, $\delta \longrightarrow \varphi^5\delta$ is in $\text{Aut}(K, T) \leq \text{Aut}(X)$. So X is 2-regular.

If $n = 21$, then $K \cong (Z_{21} \times Z_7) : Z_2$, $s = 4$ or 16, and $T = \{\gamma, \varphi^{-s}\delta\gamma, \varphi^{-s-1}\delta\gamma\}$. It is easy to see that $\theta_2 : \gamma \longrightarrow \gamma$, $\varphi \longrightarrow \varphi^{-1}$, $\delta \longrightarrow \varphi^{s(s-1)}\delta$ is in $\text{Aut}(K, T) \leq \text{Aut}(X)$. So X is 2-regular. Since the mapping $\theta : \gamma \longrightarrow \gamma$, $\varphi \longrightarrow \varphi^{-1}$, $\delta \longrightarrow \delta$ is a group automorphism of K, it follows that $\text{Cay}(K, \{\gamma, \varphi^{-4}\delta\gamma, \varphi^{-5}\delta\gamma\}) \cong \text{Cay}(K, \{\gamma, \varphi^4\delta\gamma, \varphi^5\delta\gamma\})$.

This completes the proof of Theorem 1.1. ■

Finally, we prove Theorem 1.2.

Proof of Theorem 1.2. Let $A = \text{Aut}(X)$, and G be the subgroup of A keeping the bipartition sets. Then $[A : G] = 2$ or $G = A$ depending on whether X is vertex-transitive or not. Moreover, the cubic graph X is G-semisymmetric. By Proposition 2.6 in [26], G is faithful on each bipartition set. By the result of Goldschmidt on the vertex-stabilizer of edge-transitive graphs [15], we have $|G| = 2^r \cdot 3 \cdot 7 \cdot p$, where $r \leq 7$.

We show that G has a non-trivial normal Sylow p-subgroup. Otherwise, as the intersection of all Sylow p-subgroups of G, $O_p(G) = 1$. Observe that $O_2(G)$ and $O_3(G)$ are in a vertex-stabilizer. This implies that $O_2(G) = O_3(G) = 1$. Now we consider $O_7(G)$. If $O_7(G) = 1$, then G has an unsolvable minimal normal subgroup N which is a direct product of isomorphic simple groups. If 3 does not divides $|N|$ then by Proposition 2.6 N is semiregular on $V(X)$, forcing that $|N|$ divides $7p$. This is impossible. So 3 divides $|N|$, and again by Proposition 2.6 N is semisymmetric on X. So $|N| = 2^k \cdot 3 \cdot 7 \cdot p$. It is easy to see that N is a simple group by Burnside's $p^a q^b$-theorem. Since $p > 13$, by the classification theorem of finite simple groups we know that there is no simple group of this order. Therefore, we have that $Q = O_7(G) \cong Z_7$. Obviously, Q is semiregular on $V(X)$ and $E(X)$ and is normal in A. So the quotient graph X/Q is a G/Q-semisymmetric cubic graph of order $2p$. By [14] there is no semisymmetric graph of order $2p$. So X is vertex-transitive, and hence is arc-transitive. Since $p > 13$, by [5], we know that X/Q is one-regular, $p = 1 (\text{mod } 3)$ and $\text{Aut}(X/Q) \cong Z_p : Z_6$. This implies that A/Q has a normal Sylow p-subgroup M/Q. Since $|M| = 7p$ and $p > 13$, M has a normal Sylow p-subgroup P. It follows that $M = Q \times P$. Therefore P is characteristic in M, and hence is normal in A.

Thus, $P = O_p(G)$ is a Sylow p-subgroup of G, and it is semiregular on both $V(X)$ and $E(X)$. So X is an edge-transitive regular Z_p-cover of X/P. It is well-known that the Heawood graph is the only edge-transitive cubic graph of order 14. So X/P is the Heawood graph. Now the conclusion follows immediately from Theorem 1.1.

This completes the proof of Theorem 1.2. ■

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