COMMUNICATION

THE CARTESIAN PRODUCT OF THREE TRIANGLES CAN BE EMBEDDED INTO A SURFACE OF GENUS 7

Bojan MOHAR* and Tomaž PISANSKI*
University of Ljubljana, Department of Mathematics, 61111 Ljubljana, Yugoslavia

Martin ŠKOVIERA
Comenius University, Department of Theoretical Cybernetics, 84215 Bratislava, Czechoslovakia

Arthur WHITE†
University of Oxford, Mathematical Institute, Oxford OX1 3LB, United Kingdom

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The smallest group with unknown genus is $\mathbb{Z}_3^3$. It is shown that it is embeddable into an orientable surface of genus 7 and into a nonorientable surface with the same Euler characteristic.

It has been known for quite a long time that the group $\mathbb{Z}_3^3 = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ can be embedded into an orientable surface of genus 10. It is also known that $\gamma(\mathbb{Z}_3^3) \geq 5$, see for instance [4, 2, 1]. Recall that $\gamma$ denotes the genus and $\tilde{\gamma}$ denotes the nonorientable genus of a graph. Recently, White [6] announced that $\gamma(\mathbb{Z}_3^3) \leq 9$. In this note we first exhibit an embedding of $\mathbb{Z}_3^3$ into a nonorientable surface of Euler characteristic $-12$ and then an orientable embedding into a surface with the same Euler characteristic, thereby we obtain new upper bounds: $\gamma(\mathbb{Z}_3^3) \leq 7$ and $\tilde{\gamma}(\mathbb{Z}_3^3) \leq 14$.

Consider the graph $G = C_3 \times C_3 \times C_3$, the Cartesian product of three triangles. It is clearly a Cayley graph for the group $\mathbb{Z}_3^3$. An embedding of $G$ is constructed as follows. Let $H$ denote the product $C_3 \times C_3$. First take three copies, $H_0$, $H_1$ and $H_2$, of the graph $H$, each embedded into torus as shown by Fig. 1 (except that the embedding of $H_1$ is the mirror-image of the other two). To form $G$ we have to add the remaining 27 edges, 9 between each pair of copies of $H$.

(a) Between $H_0$ and $H_1$ we add 6 edges passing from the vertices on the boundary of the hexagon $A_0$ (i.e., the copy of the hexagon $A$ in $H_0$) to the vertices on the boundary of $A_1$. By removing the two hexagons and adding a tube

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† This work was done while the author was on stabbatical leave from Western Michigan University, Kalamazoo, Michigan, MI, 49008, U.S.A.

on which the 6 added edges form lateral sides of 6 quadrilaterals we connect two tori and form a double torus (see [3] for a similar construction).

(b) In a similar way we add six more edges and a tube between $H_1$ and $H_0$ using the hexagons $B$ this time. This yields an orientable surface of genus $g = 3$.

(c) When the same construction is used between $H_2$ and $H_0$ with hexagons $B_0, C_1, A_2$ we obtain a nonorientable surface of Euler characteristic $-6$ is obtained, leaving 9 edges to be added in the last phase of the construction.

(d) Let $F$ be the graph that we have constructed so far. Notice that we have embedded into the surface of nonorientable genus 8 with 18 triangles, 27 quadrilaterals (6 on each tube) and the 3 hexagons $B_0, C_1, A_2$ that were not for tubes. If we add the 9 edges of $G$ which do not lie in $F$ to the boundaries of the hexagons $B_0, C_1$ and $A_2$, we obtain a cubic graph on 18 vertices with girth 6. We denote this graph by $M$. Figure 2 represents its genus embedding on a torus. Since hexagons $B_0, C_1$ and $A_2$ on Fig. 2 simultaneously form 2-cells of an embedding of $F$, we may remove them and glue the two surfaces appropriately along the boundaries of these hexagons thereby obtaining a nonorientable embedding of $G$ with 18 triangles, 18 quadrilaterals and 6 hexagons. Since $G$ has $p = 18$ vertices, $q = 81$ edges and $r = 42$ 2-cells, it is routine to verify by Euler's formula...
Table 1. Encoding of a rotation system of $G$

<table>
<thead>
<tr>
<th>Encoding</th>
<th>Rotation System</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>(+A, +B, +C)</td>
<td>000, 111, 222</td>
<td></td>
</tr>
<tr>
<td>(-A, -B, -C)</td>
<td>012, 120, 201</td>
<td></td>
</tr>
<tr>
<td>(+A, -C, +B)</td>
<td>010, 122</td>
<td></td>
</tr>
<tr>
<td>(-A, +B, -C)</td>
<td>020, 112</td>
<td></td>
</tr>
<tr>
<td>(+A, +C, -B)</td>
<td>022, 110</td>
<td></td>
</tr>
<tr>
<td>(-A, +C, +B)</td>
<td>001, 002, 021, 220, 221</td>
<td></td>
</tr>
<tr>
<td>(+A, -C, -B)</td>
<td>011, 100, 200, 210, 211</td>
<td></td>
</tr>
<tr>
<td>(-A, -C, +B)</td>
<td>101, 102, 121, 202, 212</td>
<td></td>
</tr>
</tbody>
</table>

that the nonorientable genus of the surface, into which $G$ is embedded, is indeed 14.

Table 1 represents an encoding of the rotation system for an ad hoc orientable embedding of $G$. For a discussion of rotation systems see, for instance, Ringel [5].

The vertices of the graph $G$ are denoted by triples 000, 001, . . . , 222, which are in turn regarded as vectors over $\mathbb{Z}_3$. Let $A = 100$, $B = 010$, and $C = 001$ be generators of $\mathbb{Z}_3^3$.

The vertices in the same line of Table 1 have the same type of rotation. Let $X$ be an arbitrary vertex and let $E$ be any of $A$, $B$, or $C$. If $E$ appears with the plus sign in the encoding of the rotation of $X$ it should be substituted by the ordered pair $X-E, X+E$, whereas if it appears with the minus sign it should be substituted by the sequence $X+E, X-E$. The rotation of any vertex $X$ which has the encoding, say $(+A, -C, +B)$, should be read as: $(X-A, X+A, X+C, X-C, X-B, X+B)$; in particular the cyclic permutation of the neighbours of the vertex 010 is (210, 110, 011, 012, 000, 020).

It is possible to verify that the embedding has 27 triangles, 9 quadrilaterals, 5 hexagons and a face of length 15. This means that the Cartesian product of three triangles can be embedded into an orientable surface of genus 7.

Acknowledgement

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References