PÓLYA ENUMERATION OF EXPANSIONS OF RESOLVABLE INCIDENCE STRUCTURES

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It is shown that the number of 'true-isomorphism' classes of expansion of a resolvable incidence structure to another incidence structure ('at infinity') can be expressed in terms of the cycle indices of the automorphism groups of the incidence structures. As an example all expansions of a small divisible design (with three parallelisms) to another divisible design are determined.

1. Introduction

Following the construction of a projective space from an affine space Bose, Shrikhande and Bhattacharya [2] introduced the 'method of extension' for the construction of divisible designs (cf. [1]) from two suitable divisible designs, one being resolvable (that is has a parallelism). Wolff [6] formulated this construction method for arbitrary incidence structures essentially as follows (throughout we use Dembowski's [4] notation):

Let \( \mathcal{I} = (\mathcal{P}, \mathcal{B}, I) \), \( \mathcal{I}' = (\mathcal{P}', \mathcal{B}', I') \) be incidence structures and \( I \) a parallelism of \( \mathcal{I}, \mathcal{I}' \) the set of \( I \)-classes, \( |I| = |\mathcal{P}| \), \( f \in \mathcal{B}(\mathcal{I}, \mathcal{P}') \), i.e. the set of bijections from \( \mathcal{I} \) onto \( \mathcal{P}' \), and w.l.o.g.

\[
\mathcal{P} \cap \mathcal{P}' = \mathcal{B} \cap \mathcal{B}' = (\mathcal{P} \cup \mathcal{P}') \cap (\mathcal{B} \cup \mathcal{B}') = \emptyset. 
\] (1.1)

Definition. The incidence structure

\[
E(\mathcal{I}, f, \mathcal{I}') = (\mathcal{P} \cup \mathcal{P}', \mathcal{B} \cup \mathcal{B}', I \cup I' \cup I_f)
\]

where \( I_f = \{(f(s), B) \mid B \in \mathcal{I}' \} \), is called the expansion of \( (\mathcal{I}, I) \) to \( \mathcal{I}' \) via the affix function \( f \).

Remarks. (i) The incidences of \( I_f \) "affix all blocks of each \( I \)-class \( s \) at the point \( f(s) \)."

(ii) We avoid the word 'extension', because this is used in design theory in a different meaning, cf. Dembowski [4, p. 76].

(iii) For several classes of incidence structures any expansion \( E(\mathcal{I}, f, \mathcal{I}') \) lies (for suitable parameters of \( \mathcal{I} \) and \( \mathcal{I}' \)) in the same class as \( \mathcal{I} \) and \( \mathcal{I}' \); e.g. 2-designs 0012-365X/83/0000-0000/$03.00 © 1983 North-Holland
with \( \lambda = 1 \), divisible designs with \( \lambda_2 = 1 \), near designs (Wolff [6], cf. also Section 6) and circle designs (Schäfer [5]).

**Notations.** If \( \| \) is a parallelism of \( \mathcal{I} = (\mathcal{P}, \mathcal{B}, I) \) and \( \|' \) a parallelism of \( \mathcal{I}' \), then \( \text{Iso}(\mathcal{I}, \|, \mathcal{I}', ||') \) denotes the set of isomorphism \( \varphi \) from \( \mathcal{I} \) to \( \mathcal{I}' \) satisfying \( B \| C \iff \varphi(B) ||' \varphi(C) \) for all \( B, C \in \mathcal{B} \).

The group \( \text{Aut}(\mathcal{I}, \|) = \text{Iso}(\mathcal{I}, \|, \mathcal{I}, \|) \) induces on the set \( \mathcal{T} \) of \( \| \)-classes of \( \mathcal{I} \) the group \( \text{Aut}(\mathcal{I}, \|)|\mathcal{T} = \{ \varphi | \varphi \in \text{Aut}(\mathcal{I}, \|) \} \), where \( \varphi(s) = \varphi s \) for all \( s \in \mathcal{T} \).

The group \( \text{Aut} \mathcal{I} \) induces on \( \mathcal{P} \) the group \( \text{Aut} \mathcal{I}|\mathcal{P} = \{ \varphi | \varphi \in \text{Aut} \mathcal{I} \} \).

**Unsolved Problem.** Given \( \mathcal{I}, \| \) and \( \mathcal{I}' \) as in (1.1) determine the number \( N(\mathcal{I}, \|, \mathcal{I}') \) of isomorphism classes in the set

\[ \mathcal{S}(\mathcal{I}, \|, \mathcal{I}') = \{ E(\mathcal{I}, f, \mathcal{I}') | f \in \mathcal{T}(\mathcal{I}, \mathcal{P}') \}. \]

We attack this problem in three steps:

**Step 1.** Introduction of true isomorphisms from \( E_1 = E(\mathcal{I}, f_1, \mathcal{I}') \) to \( E_2 = E(\mathcal{I}, f_2, \mathcal{I}') \) \((f_1, f_2 \in \mathcal{T}(\mathcal{I}, \mathcal{P}'))\), which will lead to an inequality \( N_1(\mathcal{I}, \|, \mathcal{I}') \geq N(\mathcal{I}, \|, \mathcal{I}') \), where \( N(\mathcal{I}, \|, \mathcal{I}') \) is the number of true-isomorphism classes in \( \mathcal{S}(\mathcal{I}, \|, \mathcal{I}') \).

**Step 2.** Characterization of true isomorphisms \( \varphi \) from \( E_1 \) to \( E_2 \) as a union

\[ \varphi = \varphi_1 \cup \varphi_2, \]

where \( \varphi_1 \in \text{Aut}(\mathcal{I}, \|) \), \( \varphi_2 \in \text{Aut} \mathcal{I}' \) such that \( \varphi_2 f_1 = f_2 \varphi_1 \) \((\varphi_1(s) = \varphi_1 s \) for all \( s \in \mathcal{T} \)).

**Step 3.** Application of Pólya’s counting method results in a formula for \( N(\mathcal{I}, \|, \mathcal{I}') \) in terms of the cycle indices of the groups \( \text{Aut}(\mathcal{I}, \|)|\mathcal{T} \) and \( \text{Aut} \mathcal{I}'|\mathcal{P}' \).

## 2. Isomorphisms between expansions

The following Theorem 1 leads to the definition of true isomorphisms and contains the characterization of true isomorphisms. Let

\[ E_1 = E(\mathcal{I}_{11}, f_1, \mathcal{I}_{12}), \quad E_2 = E(\mathcal{I}_{21}, f_2, \mathcal{I}_{22}), \]  

where for \( i, j \in \{1, 2\} \), \( \mathcal{I}_i = (\mathcal{P}_i, \mathcal{B}_i, I_i) \) is an incidence structure satisfying

\[ \mathcal{P}_{i1} \cap \mathcal{P}_{i2} = \mathcal{B}_{i1} \cap \mathcal{B}_{i2} = (\mathcal{P}_{i1} \cup \mathcal{P}_{i2}) \cap (\mathcal{B}_{i1} \cup \mathcal{B}_{i2}) = \emptyset, \]

and \( || \), a parallelism of \( \mathcal{I}_{11} \) with \( \mathcal{T}_i \) as the set of parallel-classes and \( f_i \in \mathcal{T}(\mathcal{T}_i, \mathcal{P}_{i2}) \).

Furthermore we assume

\[ [B]_{\mathcal{B}_i} > 0 \quad \text{for all} \quad B \in \mathcal{B}_i, \quad (i \in \{1, 2\}). \]  

**Theorem 1.** If (2.1) and (2.2) hold, then \( \varphi \in \text{Iso}(E_1, E_2) \) and \( \varphi \mathcal{P}_{11} = \mathcal{P}_{21} \) if and only if there exist

\[ \varphi_1 \in \text{Iso}(\mathcal{I}_{11}, ||, \mathcal{I}_{21}, ||), \quad \varphi_2 \in \text{Iso}(\mathcal{I}_{12}, ||), \]

such that \( \varphi = \varphi_1 \cup \varphi_2 \) and \( \varphi_2 f_1 = f_2 \varphi_1 \).
Proof. Choose \( \varphi_i = \varphi_i((\mathcal{P}_{11} \cup \mathcal{B}_{11})) \) (\( i \in \{1, 2\} \)). Using (2.1) and (2.2) it is easy to show that \( \varphi_1 \in \text{Iso}(\mathcal{S}_{11}, |1|, \mathcal{S}_{21}, |2|) \) and \( \varphi_2 \in \text{Iso}(\mathcal{S}_{12}, \mathcal{S}_{22}) \). To prove \( \varphi_2 \mathcal{f}_1 = f_2 \mathcal{f}_1 \) let \( E_i = (\mathcal{P}_i, \mathcal{B}_i, I_i) \) (\( i \in \{1, 2\} \)) and \( s \in \mathcal{I}_1, B \in s \). Then \( f_1(s) I_1 B, \) hence \( \varphi_2 f_1(s) \in \mathcal{P}_{22} \) and \( \varphi_2 f_1(s) I_2 \mathcal{f}_1(B) \in \mathcal{f}_1(s) \in \mathcal{I}_2, \) consequently \( \varphi_2 f_1(s) = f_2 \mathcal{f}_1(s) \).

The 'only if part' of Theorem 1 is easily obtained from the fact that for all \( p \in \mathcal{P}_{12}, B \in \mathcal{B}_{11} (\|_1 B = \text{the } \|_1 \text{-class containing } B) \)
\[
p I_1 B \Leftrightarrow p = f_1(I_1, B) = \varphi_2^{-1} f_2 \mathcal{f}_1(||, B) 
\]
\[
\Leftrightarrow \varphi_2(p) = f_2 \mathcal{f}_1(||, B) = f_2(||, \mathcal{f}_1(B)) 
\]
\[
\Leftrightarrow \varphi_2(p) I_2 \varphi_1(B) \Leftrightarrow \varphi(p) I_2 \varphi(B). 
\]

For later use we remark: If each \( \varphi \in \text{Iso}(E_1, E_2) \) satisfies \( \varphi \mathcal{P}_{11} = \mathcal{P}_{21} \), then
\[
(\mathcal{S}_{11}, |1|) \neq (\mathcal{S}_{21}, |2|) \Rightarrow E_1 \neq E_2. \quad (2.3)
\]

3. True isomorphisms

Specializing Theorem 1 to the situation \( \mathcal{S}_{11} = \mathcal{S}_{21} = \mathcal{S} = (\mathcal{P}, \mathcal{B}, I), \|_1 = \|_2 \), (\( \mathcal{I} \) the set of \|-classes), \( \mathcal{S}_{12} = \mathcal{S}_{22} = \mathcal{S}' = (\mathcal{P}', \mathcal{B}', I') \) we obtain

Theorem 2. Let \( \mathcal{S}, \mathcal{S}' \) satisfy (1.1), \( f_1, f_2 \in \mathbb{B}(\mathcal{I}, \mathcal{P}'), \) \( [B] > 0 \) for all \( B \in \mathcal{B}, \)
\( E_1 = E(\mathcal{S}, f_1, \mathcal{S}'), \) \( E_2 = E(\mathcal{S}, f_2, \mathcal{S}'). \)

Then \( \varphi \in \text{Iso}(E_1, E_2) \) and \( \varphi \mathcal{P} = \mathcal{P} \) if and only if there exist \( \varphi_1 \in \text{Aut}(\mathcal{S}, |\|), \)
\( \varphi_2 \in \text{Aut} \mathcal{S}' \) such that \( \varphi = \varphi_1 \cup \varphi_2 \) and \( \varphi f_1 = f_2 \mathcal{f}_1. \)

From now on we shall always use the hypothesis and the notations of Theorem 2.

Definition. Let \( E_1, \ E_2 \in \mathfrak{C}(\mathcal{S}, |\|, \mathcal{S}'). \) Then an isomorphism \( \varphi \in \text{Iso}(E_1, E_2) \) is called
true iff \( \varphi \mathcal{P} = \mathcal{P}. \)

\( E_1 \) is true-isomorphic to \( E_2 (E_1 \equiv E_2) \) if and only if there is a true isomorphism
from \( E_1 \) to \( E_2. \)

Clearly
\[
= \text{is an equivalence relation on } \mathfrak{C}(\mathcal{S}, |\|, \mathcal{S}'). \quad (3.1)
\]
and
\[
E_1 \equiv E_2 \Rightarrow E_1 \equiv E_2, \quad (3.2)
\]
hence the number \( N(\mathcal{S}, |\|, \mathcal{S}') \) of true-isomorphism classes in \( \mathfrak{C}(\mathcal{S}, |\|, \mathcal{S}') \) satisfies
\[
N(\mathcal{S}, |\|, \mathcal{S}') \Rightarrow N(\mathcal{S}, |\|, \mathcal{S}'), \quad (3.3)
\]
and the following conditions are equivalent:
\[
N(\mathcal{S}, |\|, \mathcal{S}') = N(\mathcal{S}, |\|, \mathcal{S}'), \quad (3.4.1)
\]
\[
E_1 \equiv E_2 \Rightarrow E_1 \equiv E_2 \quad \text{ (for all } E_1, E_2 \in \mathfrak{C}(\mathcal{S}, |\|, \mathcal{S}')). \quad (3.4.2)
\]
The next condition (3.5) implies (3.4.2) and hence (3.4.1):

\[
\text{each isomorphism from } E_1 \text{ to } E_2 \text{ is a true isomorphism}
\]

(for all \( E_1, E_2 \in \mathcal{E}(\mathcal{I}, \mathcal{J}) \)).

\[ (3.5) \]

Remark. We don't know an example for the case \( N_1(\mathcal{I}, \mathcal{J}, \mathcal{J}') \neq N(\mathcal{I}, \mathcal{J}, \mathcal{J}') \). If such an example exists, then by (3.4) there are \( E_1, E_2 \in \mathcal{E}(\mathcal{I}, \mathcal{J}) \) with \( E_1 \cong E_2 \), but \( E_1 \neq E_2 \).

4. Application of Pólya's counting method

Let \( \mathcal{I}, \mathcal{J}, \mathcal{J}' \) be given as in Theorem 2. We now determine the number \( N(\mathcal{I}, \mathcal{J}, \mathcal{J}') \) using a generalization of Pólya's theorem (cf. de Bruijn [3]).

Let \( \Gamma = \text{Aut}(\mathcal{I}, \mathcal{J}) \), \( \Delta = \text{Aut}(\mathcal{J}, \mathcal{J}') \). On \( B(\mathcal{I}, \mathcal{J}') \) we define an equivalence relation (see [3, p. 161]) by

\[
f_1 \sim f_2 \iff \text{there exist } \gamma \in \Gamma, \delta \in \Delta \text{ such that } \delta f_1 = f_2 \gamma.
\]

From Theorem 2 we obtain for all \( f_1, f_2 \in B(\mathcal{I}, \mathcal{J}') \)

\[
f_1 \sim f_2 \iff E(\mathcal{I}, f_1, \mathcal{J}') = E(\mathcal{I}, f_2, \mathcal{J}'). \tag{4.2}
\]

Therefore and since the mapping \( E: B(\mathcal{I}, \mathcal{J}') \to \mathcal{E}(\mathcal{I}, \mathcal{J}, \mathcal{J}') \), \( f \mapsto E(\mathcal{I}, f, \mathcal{J}') \) is a bijection the number \( N(\mathcal{I}, \mathcal{J}, \mathcal{J}') \) equals the number of equivalence classes (patterns) in \( B(\mathcal{I}, \mathcal{J}') \), which is given in [3, Theorem 5.3].

Let \( n = |\mathcal{I}| = |\mathcal{J}'| \) and \( P_1(X_1, \ldots, X_n), P_\Delta(X_1, \ldots, X_n) \) be the cycle index of \( \Gamma \), resp. \( \Delta \). From Theorem 5.3 in [3] we now obtain

**Theorem 3.** Let \( \mathcal{I}, \mathcal{J}, \mathcal{J}' \) satisfy (1.1), \( |B| > 0 \) for all \( B \in \mathcal{B} \), \( n = |\mathcal{I}| = |\mathcal{J}'| \). Then

\[
N(\mathcal{I}, \mathcal{J}, \mathcal{J}') = P_1(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n})P_\Delta(z_1, 2z_2, \ldots, nz_n)|_{z_1 = \ldots = z_n = 0}. \tag{4.3}
\]

5. The graphs of an expansion

To define the graphs of an expansion we first recall the definition of the graphs of a parallelism (cf. Wolff [6]).

Let \( \mathcal{I} = (\mathcal{P}, \mathcal{B}, I) \) be an incidence structure, \( |\mathcal{B}| \geq 2, \| \) a parallelism of \( \mathcal{I} \) with \( \mathcal{J} \) as the set of \( \| \)-classes,

\[
I(\mathcal{I}) = \max\{[B, C] \mid B, C \in \mathcal{B}, B \neq C\} \geq 1. \tag{5.1}
\]

For each \( n \in \mathbb{N} \), \( 1 \leq n \leq L(\mathcal{I}) \) we define an incidence structure \( G_n(\|) = (\mathcal{I}, \mathcal{K}, I_n) \) by

\[
\mathcal{K}_n = \{[B, C] \mid B, C \in \mathcal{B}, B \neq C, [B, C] = n\},
\]

\[
I_n = \{((s, K) \mid s \in \mathcal{I}, K \in \mathcal{K}_n, s \cap K \neq \emptyset\}. 
\]
$G_n = G_n(||)$ is a graph since $[B, C]_n = ||B, ||C|| = 2$ for each edge $\{B, C\} \in \mathcal{K}_n$.

The number of edges of $\mathcal{K}_n$ connecting two vertices $s, t \in \mathcal{I}$ is

$$[s, t]_n = |\{(B, C) | B \in s, C \in t, [B, C] = n\}| \quad (5.2)$$

and (see Wolff [6, p. 27])

$$\sum_{n=1}^{L(\mathcal{I})} n[s, t]_n = |\mathcal{P}|. \quad (5.3)$$

Under the hypothesis (5.1) let $\mathcal{Y}' = (\mathcal{P}', \mathcal{B}', I')$ be an incidence structure with $|\mathcal{I}| = |\mathcal{P}'|$, $f \in \mathcal{B}(\mathcal{I}, \mathcal{P})$, $1 \leq n \leq L(\mathcal{I})$, $0 \leq m \leq M(\mathcal{Y}') = \max\{|p, q| | p, q \in \mathcal{P}'', p \neq q\}$.

We define a graph $G(f, m) = (\mathcal{I}, \mathcal{K}(f, m), I_m)$ by

$$\mathcal{K}(f, m) = \{(s, t) | s, t \in \mathcal{I}, s \neq t, [f(s), f(t)] = m\}$$

and $u \in I_m\{s, t\} \iff u \in \{s, t\} \ (u \in \mathcal{I}, \{s, t\} \in \mathcal{K}(f, m))$.

Clearly

$$\bigcup_{m=0}^{M(\mathcal{Y}')} G(f, m)$$

is the complete graph on $\mathcal{I}$. \quad (5.4)

**Definition.** The graphs $G(f, m, n)$ of the expansion $E(\mathcal{A}, f, \mathcal{Y}')$ $(0 \leq m \leq M(\mathcal{Y}'))$, $1 \leq n \leq L(\mathcal{I})$ are defined by

$$G(f, m, n) = G(f, m) \cup G_n(||) = (\mathcal{I}, \mathcal{K}(f, m) \cup \mathcal{K}_n, I_m \cup I_n),$$

(note that $\mathcal{K}(f, m) \cap \mathcal{K}_n = \emptyset$).

The following Proposition 1 will be used in Section 6 to show the non-isomorphy of some expansions.

**Proposition 1.** Under the hypotheses (1.1) and (5.1) let $f_1, f_2 \in \mathcal{B}(\mathcal{I}, \mathcal{P}')$. If $\varphi$ is a true isomorphism from $E(\mathcal{A}, f_1, \mathcal{Y}')$ to $E(\mathcal{A}, f_2, \mathcal{Y}')$ and $\eta = f_2^{-1} \varphi f_1$, then for $1 \leq n \leq L(\mathcal{I})$, $0 \leq m \leq M(\mathcal{Y}')$ we have:

(a) $\eta \in \text{Aut} G_n|\mathcal{I}$,

(b) $\eta \in \text{Iso}(G(f_1, m), G(f_2, m))|\mathcal{I}$,

(c) $\eta \in \text{Iso}(G(f_1, m, n), G(f_2, m, n))|\mathcal{I}$.

**Proof.** (a) was proved by Wolff [6, Satz 16c].

To prove (b) let $s, t \in \mathcal{I}, s \neq t$, then

$$\{s, t\} \in \mathcal{K}(f_1, m) \iff m = [f_1(s), f_1(t)] = [\varphi f_1(s), \varphi f_1(t)] = [f_2 \eta(s), f_2 \eta(t)]$$

$$\iff \{\eta(s), \eta(t)\} \in \mathcal{K}(f_2, m).$$

(c) follows from (a) and (b).
6. An example: The expansions of FB(2, 6) to FB(3, 3)

We choose the same example as Bose, Shrikhande, Bhattacharya [2].

The first incidence structure (3) is the complete graph $K_6$ 'with double diagonals'. This is the only near design (Fast-Blockplan) with $k = 2, r = 6$, called FB(2, 6). (A near design $(\mathcal{P}, B, I)$ is a 1-(v, k, r)-design with $[p, q] \geq 1$ for all $p, q \in \mathcal{P}, p \neq q$ and $r(k - 1) = v$; cf. Wolff [6].)

The second incidence structure (3') is the only near design with $k = 3 = r$, called FB(3, 3), and is generated by 0, 1, 3 (mod 6), hence has the blocks (columns):

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 & 0 \\
3 & 4 & 5 & 0 & 1 & 2 \\
\end{array}
\]

Any expansion of FB(2, 6) to FB(3, 3) is a near design with $k = 3, r = 6$. The results of the preceding sections now enable us to count and even to construct all expansions of FB(2, 6) to FB(3, 3).

6.1. The parallelisms of FB(2, 6)

It was shown by Wolff [6] that FB(2, 6) has (up to isomorphisms) exactly 3 parallelisms $l_1, l_2, l_3$, whose parallel classes (named $r, s, t, u, v, w$ in each case as indicated) are the rows of the following list:

\[
\begin{array}{cccccccc}
03 & 14 & 25 & 03 & 14 & 25 & 01 & 25 & 34 & r \\
03 & 14 & 25 & 23 & 14 & 05 & 02 & 14 & 35 & s \\
12 & 34 & 05 & 24 & 03 & 15 & 03 & 12 & 45 & t \\
04 & 23 & 15 & 04 & 12 & 35 & 03 & 15 & 24 & u \\
13 & 02 & 45 & 01 & 34 & 25 & 04 & 13 & 25 & v \\
24 & 01 & 35 & 13 & 02 & 45 & 05 & 14 & 23 & w \\
\end{array}
\]

From (5.3) and $L(FB(2, 6)) = 2$ we see that we need only the graphs $G_{2i} = G_2(l_i)$ shown in Fig. 1.

6.2. Cycle indices

6.2.1. The group $\Gamma_1 = \text{Aut}(FB(2, 6), l_1)|\mathcal{F}_1$ equals $\langle (tu), (tv), (tw), (uvw), (rs) \rangle$ and has cycle index

\[
P_1(X_1, \ldots, X_6) = \frac{1}{48}(X_1^6 + 7X_1^4X_2 + 8X_1^3X_3 + 9X_1^2X_2^2 + 6X_3^2X_4 \\
+ 8X_1X_2X_3 + 3X_2^3 + 6X_2X_4).
\]

6.2.2. The group $\Gamma_2 = \text{Aut}(FB(2, 6), l_2)|\mathcal{F}_2$ equals $\langle (uw), (st), (stv) \rangle$ and has cycle index

\[
P_2(X_1, \ldots, X_6) = \frac{1}{12}(X_1^6 + 4X_1^4X_2 + 2X_1^3X_3 + 3X_1^2X_2^2 + 2X_1X_2X_3).
\]
6.2.3. The group \( \Gamma_3 = \text{Aut}(\mathcal{F}_B(2,6)) \) equals \( \langle (ru)(tu), (rst)(uw), (rsuw)(tu) \rangle \) and has cycle index

\[
P_3(X_1, \ldots, X_6) = \frac{1}{24}(X_1^6 + 9X_1^2X_2^2 + 6X_2X_4 + 8X_3^2).
\]

6.2.4. The group \( \Delta = \text{Aut}(\mathcal{F}_B(3,3)) \) equals \( \langle (14, (25), (012345)) \rangle \) and has cycle index

\[
P_\Delta(X_1, \ldots, X_6) = \frac{1}{24}(X_1^6 + 3X_1^4X_2 + 3X_1^2X_2^2 + X_2^3 + 8X_3^2 + 8X_6).
\]

Remarks. (i) For the determination of \( \Gamma_1 \) and \( \Gamma_2 \) we used the fact, that (under the hypothesis (5.1))

\[
\text{Aut}(\mathcal{F}_B(2), |\mathcal{F}\rangle) \leq \text{Aut}(G_{n,\mathcal{F}}) \quad (1 \leq n \leq L(\mathcal{F})).
\]

(ii) For the determination of \( \Gamma_3 \) we used that \( \text{Aut}(\mathcal{F}_B(2), |\mathcal{F}\rangle) \) is a homomorphic image of \( \text{Aut}(\mathcal{F}_B(2), |\mathcal{F}\rangle) \) if \( (\mathcal{F}, \mathcal{F}) \) has no double classes (i.e. \( \{(B) | B \in s\} \neq \{(C) | C \in t\} \)) for all \( s, t \in \mathcal{F} \) with \( s \neq t \), where \( (B) \) is the set of points incident with \( B \).

6.3. The number of isomorphism classes of expansions of \( \mathcal{F}_B(2,6) \) to \( \mathcal{F}_B(3,3) \)

Wolff [6] proved that condition (3.5) and therefore (3.4.1) is satisfied, if \( \mathcal{F} = \mathbb{F}(3,3) \). Hence we obtain the number of isomorphism classes of expansions of \( \mathcal{F}_B(2,6) \) to \( \mathcal{F}_B(3,3) \) by Theorem 3:

\[
N(\mathcal{F}_B(2,6), |\mathcal{F}_B(3,3)) = \frac{1}{48} \left[ \left( \frac{\partial}{\partial z_1} \right)^6 + 7 \left( \frac{\partial}{\partial z_1} \right)^4 \left( \frac{\partial}{\partial z_2} \right) + 8 \left( \frac{\partial}{\partial z_1} \right)^3 \left( \frac{\partial}{\partial z_3} \right) + 9 \left( \frac{\partial}{\partial z_1} \right)^2 \left( \frac{\partial}{\partial z_2} \right)^2 + 6 \left( \frac{\partial}{\partial z_1} \right)^2 \left( \frac{\partial}{\partial z_4} \right) + 8 \frac{\partial}{\partial z_1} \frac{\partial}{\partial z_2} \frac{\partial}{\partial z_3} + 3 \left( \frac{\partial}{\partial z_2} \right)^3 + 6 \frac{\partial}{\partial z_2} \frac{\partial}{\partial z_4} \right] \frac{1}{24} \left[ z_1^6 + 3z_1^4z_2^2 + 3z_1^3(2z_2)^2 + (2z_2)^3 + 8(3z_2)^2 + 8(6z_3)^2 \right]_{z_1=\ldots=z_6=0} = \frac{1}{48 \cdot 24} (6! + 7 \cdot 3 \cdot 2 \cdot 4! + 9 \cdot 3 \cdot 2 \cdot 2^2 \cdot 2 + 3 \cdot 2^3 \cdot 3!) = 2.
\]

Analogously we obtain from 6.2.2, 6.2.3, 6.2.4

\[
N(\mathcal{F}_B(2,6), |\mathcal{F}_B(3,3)) = 5, \quad N(\mathcal{F}_B(2,6), |\mathcal{F}_B(3,3)) = 4.
\]
Hence by Theorem 1 and (3.5) the total number of isomorphism classes of expansions of FB(2, 6) to FB(3, 3) is 11.

6.4. The expansions of FB(2, 6) to FB(3, 3)

For each parallelism \( \parallel \) of \( \text{FB}(2, 6) \) we now construct a system of distinct representatives (resp. \( \equiv \)) of expansions of \( (\text{FB}(2, 6), \parallel) \) to \( \text{FB}(3, 3) \) (given by their affix functions).

**Notation.** A *spread* of an incidence structure \( (\mathcal{P}, \mathcal{B}, I) \) is a subset \( s \) of \( \mathcal{B} \) such that for each point \( p \in \mathcal{P} \) there is exactly one block \( B \in s \) with \( p \in B \).

First we mention that for each affix function \( f \mathcal{K}(f, 2) \) is a spread since \( \mathcal{X}' = \text{FB}(3, 3) \). (By (5.4) \( G(f, 1) \) is determined by \( G(f, 2) \) since \( M(\text{FB}(3, 3)) = 2 \) and \( \mathcal{X}(f, 0) = \emptyset \).) Searching for 'representative' affix functions \( f \) we construct 'candidates' for \( G(f, 2, 2) \) as the unions of \( G_{2i} \) and a spread of 3 edges \( (i \in \{1, 2, 3\}) \). Up to isomorphism there are exactly 8 such graphs. For each of these graphs \( G \) one can immediately find an affix function \( f \) with \( G \equiv G(f, 2, 2) \). Three of these graphs possess two affix functions giving non-isomorphic expansions. The following list presents 11 affix functions:

1. \( \parallel_1 \):

\[
\begin{array}{cccccc}
11 & 0 & 1 & 2 & 3 & 4 & 5 \\
12 & 0 & 3 & 1 & 2 & 4 & 5 \\
\end{array}
\]

(isomorphic to the example in [2])

2. \( \parallel_2 \):

\[
\begin{array}{cccccc}
21 & 0 & 1 & 2 & 3 & 4 & 5 \\
22 & 0 & 2 & 4 & 1 & 5 & 3 \\
23 & 0 & 2 & 3 & 1 & 5 & 4 \\
24 & 0 & 1 & 3 & 2 & 4 & 5 \\
25 & 0 & 1 & 3 & 2 & 5 & 4 \\
\end{array}
\]

\( G(f_{21}, 2, 2) \equiv G(f_{22}, 2, 2) \)

3. \( \parallel_3 \):

\[
\begin{array}{cccccc}
31 & 0 & 1 & 2 & 5 & 3 & 4 \\
32 & 0 & 1 & 2 & 4 & 3 & 5 \\
33 & 0 & 1 & 2 & 3 & 4 & 5 \\
34 & 1 & 5 & 4 & 0 & 2 & 3 \\
\end{array}
\]

\( G(f_{33}, 2, 2) \equiv G(f_{34}, 2, 2) \)

To check that these 11 functions yield pairwise non-isomorphic expansions we first remember that by (2.3) and (3.5) expansions on non-isomorphic parallelisms are non-isomorphic. Secondly affix functions \( f \) with non-isomorphic graphs \( G(f, 2, 2) \) yield non-isomorphic expansions by Proposition 1(c) and (3.5). Let \( E_{ij} \) be the expansion via \( f_{ij} \). To check that \( E_{21} \not\equiv E_{22}, E_{23} \not\equiv E_{24}, E_{33} \not\equiv E_{34} \) we count the number of spreads in these expansions. These numbers are respectively

\[
n_{21} = 6 \neq 3 = n_{22}, \quad n_{23} = 8 \neq 4 = n_{24}, \quad n_{33} = 3 \neq 9 = n_{34}.
\]
Hence we have constructed (up to isomorphism) all expansion of \( \text{FB}(2, 6) \) to \( \text{FB}(3, 3) \).

References