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# A NOTE ON THE STRONG MAXIMUM PRINCIPLE FOR ELLIPTIC DIFFERENTIAL INEQUALITIES

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ABSTRACT. – We consider the strong maximum principle and the compact support principle for quasilinear elliptic differential inequalities, under generally weak assumptions on the quasilinear operators and the nonlinearities involved. This allows us to give necessary and sufficient conditions for the validity of both principles. © 2000 Éditions scientifiques et médicales Elsevier SAS

## 1. Introduction

We are interested in the strong maximum principle and the compact support principle for quasilinear elliptic differential inequalities, under generally weak assumptions on the quasilinear operators in question. We consider in particular the canonical divergence structure differential inequalities:

$$\operatorname{div}\{A(|Du|)Du\} - f(u) \leqslant 0, \quad u \geqslant 0, \tag{1.1}$$

and

$$\operatorname{div}\{A(|Du|)Du\} - f(u) \geqslant 0, \quad u \geqslant 0, \tag{1.2}$$

in a domain $D$, possibly unbounded, of $\mathbb{R}^n$, $n \geqslant 2$. Here we assume throughout the paper the following conditions on the operator $A = A(t)$ and the nonlinearity $f = f(u)$,

(A1) $A \in C(0, \infty)$,
(A2) $t \mapsto tA(t)$ is strictly increasing in $(0, \infty)$ and $tA(t) \to 0$ as $t \to 0$,
(F1) $f \in C[0, \infty)$,
(F2) $f(0) = 0$ and $f$ is non-decreasing on some interval $[0, \delta)$, $\delta > 0$.

Condition (A2) is a minimal requirement for ellipticity of (1.1)–(1.2). Furthermore, it allows singular and degenerate behavior of the operator $A$ at $t = 0$, that is at critical points of $u$. We emphasize that no assumptions of differentiability are made on either $A$ or $f$ when dealing with the canonical models (1.1) and (1.2).

By a *solution* of (1.1) or (1.2) in $D$ we mean a non-negative function $u \in C^1(D)$ which satisfies (1.1) or (1.2) in the distribution sense.

With the notation $\Omega(t) = tA(t)$ when $t > 0$, and $\Omega(0) = 0$, we introduce the function

$$H(t) = t\Omega(t) - \int_0^t \Omega(s)\,\mathrm{d}s, \quad t \geqslant 0. \tag{1.3}$$

Letting $\Omega^{-1}(\omega)$ be the inverse of the strictly increasing function $\Omega(t)$, then from Stieltjes integration it is easy to see that

$$H(t) = \int_0^{\Omega(t)} \Omega^{-1}(\omega)\,\mathrm{d}\omega, \quad t \geqslant 0. \tag{1.4}$$

Therefore $H$ is strictly increasing on $[0, \infty)$.

For the Laplace operator, that is when (1.1) takes the classical form

$$\Delta u - f(u) \leqslant 0, \quad u \geqslant 0,$$

we have $A(t) \equiv 1$ and $H(t) = \frac{1}{2}t^2$. Similarly, for the degenerate $m$-Laplace operator, $m > 1$, we have $A(t) = t^{m-2}$ and $H(t) = (m-1)t^m/m$, while for the mean curvature operator, $A(t) = 1/\sqrt{1+t^2}$ and $H(t) = 1 - 1/\sqrt{1+t^2}$.

It is also worth observing that (1.1), when equality holds, is precisely the Euler–Lagrange equation for the variational integral

$$I[u] = \int_D \bigl\{G\bigl(|Du|\bigr) + F(u)\bigr\}\,\mathrm{d}x, \quad F(u) = \int_0^u f(s)\,\mathrm{d}s,$$

where $G$ and $A$ are related by $A(t) = G'(t)/t$, $t > 0$. In this case $H(t) = tG'(t) - G(t)$, the pre-Legendre transform of $G$. Further comments and other examples of operators satisfying (A1), (A2) are given in [6].

By the strong maximum principle for (1.1) we mean the statement that *if $u$ is a solution of* (1.1) *with $u(x_0) = 0$ for some $x_0 \in D$, then $u \equiv 0$ in $D$.*

We can now state our main results.

THEOREM 1. – *In order for the strong maximum principle to hold for* (1.1) *it is necessary and sufficient either that $f(s) \equiv 0$ for $s \in [0, \mu)$, $\mu > 0$, or that $f(s) > 0$ for $s \in (0, \delta)$ and*

$$\int_0^\delta \frac{\mathrm{d}s}{H^{-1}(F(s))} = \infty. \tag{1.5}$$

The background and literature for Theorem 1 is fairly complicated and deserves a number of comments:

*Necessity.* For the case of the Laplace operator the necessity of (1.5) is due to Benilan, Brezis and Crandall [1], while for the $m$-Laplacian it is due to Diaz and Herrero, see [2]. In these cases we observe that (1.5) reduces respectively to

$$\int_0^\delta \frac{\mathrm{d}s}{\sqrt{F(s)}} = \infty \quad \text{and} \quad \int_0^\delta \frac{\mathrm{d}s}{[F(s)]^{1/m}} = \infty.$$

For general operators satisfying (A1), (A2), necessity is due to Diaz [2, Theorem 1.4], see also [6, Corollary 1].

*Sufficiency.* For the case of the Laplace operator and also for the $m$-Laplacian, the result is due to Vazquez [8], see also [2]. For general operators satisfying (A1), (A2), sufficiency was proved in [6, Theorem 1] under an additional technical assumption, see (2.5) in [6], hereafter referred to as condition I.

Diaz, Saa and Thiel stated a slightly weaker version of Theorem 1, see [3, Theorem 6], but for completeness, their proof requires a further not trivial argument at the final step, together with the additional condition I. It turns out that a rigorous treatment of the full sufficiency result of Theorem 1, avoiding use of the technical assumption I, is fairly tricky, involving a new method for the solution of differential inequalities whose structure includes driving and amplifying terms which reinforce each other. At the same time, the new proof uses only standard calculus, requiring neither fixed point theory (as [6]) nor monotone operator theory (as [8,2,3]). In this sense, it is closer to the original method of E. Hopf than other more recent proofs. We hope that this technique could have further applications as well.

In the next result we consider the situation when the integral in (1.5) is convergent. Here the appropriate hypotheses are that $u$ satisfies the converse inequality (1.2) and also “vanishes” at $\infty$, rather than at some finite point $x_0 \in D$.

More precisely, by the compact support principle for (1.2) we mean the statement that *if $u$ is a solution of* (1.2) *in an exterior domain $D$, with $u(x) \to 0$ as $|x| \to \infty$, then $u$ has compact support in $D$.*

THEOREM 2. – *In order for the compact support principle to hold for* (1.2) *it is necessary and sufficient that $f(s) > 0$ for $s \in (0, \delta)$ and*

$$\int_0^\delta \frac{\mathrm{d}s}{H^{-1}(F(s))} < \infty. \tag{1.6}$$

As in the case of the strong maximum principle it is worth commenting on the background and literature for Theorem 2.

*Necessity.* This was first shown in [6, Corollary 1] under the additional condition I. It should be noted that the proof is not at all easy.

*Sufficiency.* This is due to [6, Theorem 2], but see also [7] and the remarks following the statement of Theorem 2 in [6]. For radially symmetric solutions of (1.2) sufficiency was proved in [4] under the weaker assumption that $F(s) > 0$ for $s \in (0, \delta)$, see [4, Proposition 1.3.1].

If Theorem 2 were an exact analogue of Theorem 1, the conclusion of the compact support principle would be that $u \equiv 0$ in $D$, but this would be incorrect since (1.2) admits non-trivial compact support solutions under assumption (1.6), see [6, Theorem 3].

The results described above can be extended to a wider class of differential inequalities by replacing $\operatorname{div}\{A(|Du|)Du\}$ by the more general operator $D_i\{a^{ij}(x)A(|Du|)D_ju\}$ and $f(u)$ by $B(x, u, Du)$, where $a^{ij}(x)$ is a positive definite symmetric matrix on $D$ and where $B$ satisfies a condition of the form

$$B(x,u,p) \leqslant \text{Const.}|p|A\big(|p|\big) + f(u), \tag{1.7}$$

for $x \in D,\ u \geqslant 0$ and all $p \in \mathbb{R}^n$ with $|p|$ sufficiently small (reverse the inequality sign for the compact support principle). These extensions are the second purpose of the paper.

In the next section we prove our main results for the canonical models (1.1) and (1.2), while in Section 3 we consider the case of fully quasilinear inequalities

$$D_i\big\{a^{ij}(x)A\big(|Du|\big)D_j u\big\} - B(x,u,Du) \leqslant 0\ (\geqslant 0) \tag{1.8}$$

(where the obvious summation convention is used).

Finally, in Section 4 we treat several special cases where the main proof reduces to a simpler form. As a byproduct of this discussion we obtain a polynomial comparison function for linear inequalities alternative to the classical exponential function of E. Hopf.

## 2. Proofs of Theorems 1 and 2

We require several preliminary lemmas.

LEMMA 1. – (i) *For any constant $\sigma \in [0,1]$ there holds*

$$F(\sigma u) \leqslant \sigma F(u), \quad u \in [0,\delta).$$

(ii) *Let $w = w(r)$ be of class $C^1(r_0,r_1)$ with $w'(r) > 0$. Then $\Omega \circ w'$ is of class $C^1(r_0,r_1)$ if and only if $H \circ w'$ is of class $C^1(r_0,r_1)$, and in this case*

$$\big\{H\big(w'(r)\big)\big\}' = w'(r)\big\{\Omega\big(w'(r)\big)\big\}' \quad \text{in } (r_0,r_1).$$

To obtain (i), observe that $\sigma f(\sigma u) \leqslant \sigma f(u)$ for $u \in [0,\delta)$, since $f$ is non-decreasing. Integrating this relation from 0 to $u$ yields the result.

On the other hand, (ii) is an immediate consequence of (1.4).

LEMMA 2. – *Suppose $f(s) > 0$ for $0 < s < \delta$ and*

$$\int_0^\delta \frac{\mathrm{d}s}{H^{-1}(F(s))} = \infty. \tag{2.1}$$

*Then for any positive numbers $k,\ \ell,\ R$, and for $\varepsilon \in (0,\delta)$, the ordinary differential inequality*

$$\big[\Omega\big(|v'|\big)\big]' + \frac{k}{r}\Omega\big(|v'|\big) + \ell f(v) \leqslant 0 \tag{2.2}$$

*has a $C^1$ solution $v = v(r)$ in the interval $[R/2,R]$, with*

$$v(R) = 0, \qquad v'(R) = -\alpha < 0 \tag{2.3}$$

*and*

$$0 < v < \varepsilon \quad in\ [R/2,R), \tag{2.4}$$

*provided $\alpha$ is sufficiently small.*

In stating condition (2.1) we assume, without loss of generality, that the value $\delta$ is so small that $F(\delta) \in H[0,\infty)$. This is automatic for the Laplace and $m$-Laplace operators, since for these cases $H[0,\infty) = [0,\infty)$, but for the mean curvature operator there holds $H[0,\infty) = [0,1)$, which gives the restriction $F(\delta) < 1$. The same remarks of course apply to conditions (1.5) and (1.6).

*Proof of Lemma 2.* – It is enough to treat the case $\ell = 1$, since by Lemma 1(i) the integral

$$\int_0^\delta \frac{\mathrm{d}s}{H^{-1}(\ell F(s))}$$

diverges if and only if (2.1) is satisfied.

The strategy for obtaining the required solution will be, first, to construct for each $\alpha > 0$ a *candidate* $v(r)$ for the solution of (2.2), having one of the following three properties:

(i) $v(r)$ is defined on $[R/2, R]$ and satisfies (2.3), (2.4),

(ii) $v(r)$ is defined on some interval $[\overline{R}, R]$, $\overline{R} \in [R/2, R)$, satisfies (2.3),

$$0 < v < \varepsilon \quad \text{in } (\overline{R}, R) \tag{2.5}$$

and also $v(\overline{R}) = \varepsilon$, or

(iii) $v(r)$ is defined on some interval $(\overline{R}, R]$, $\overline{R} \in [R/2, R)$, satisfies (2.3), (2.5) and also

$$\lim_{r\to\overline{R}} v'(r) = -\infty.$$

**Step 1. Construction of $v(r)$.** We use a recursive continuation procedure, backwards from the point $r = R$. The starting point will be the construction of an "initial" function $v = v_1$, defined to be the $C^1$ solution of

$$\big[\Omega\big(|v'|\big)\big]' + \frac{2k}{r}\Omega\big(|v'|\big) = 0, \tag{2.6}$$

$$v(R) = 0, \qquad v'(R) = -\alpha \tag*{$(2.7)_1$}$$

on the *maximal interval* $I_1 = (R_1, R]$, $R_1 \in [R/2, R)$, for which both the inequalities

$$0 \leqslant v < \varepsilon \tag{2.8}$$

and

$$\frac{k}{r}\Omega\big(|v'|\big) > f(v) \tag{2.9}$$

are satisfied.

To show that such a function $v = v_1$ and maximal interval $I_1$ exist, we observe by direct integration that (2.6), $(2.7)_1$ imply

$$\Omega\big(|v'|\big) = Cr^{-2k}, \quad v' < 0, \tag{2.10}$$

where $C = C_1 = R^{2k}\Omega(\alpha)$. Hence

$$v'(r) = -\Omega^{-1}\big(Cr^{-2k}\big), \tag{2.11}$$

from which $v(r)$ is immediately obtained by quadrature (and the fact that $v(R)=0$). Also, either by (2.6) or (2.11), one sees that $v$ is convex, whence $v \geqslant 0,\ v' < 0$ on $I_1$. The existence of $I_1$ is now obvious, and moreover both $v(R_1) > 0$ and $v'(R_1) < 0$ exist (finite).

If $R_1 = R/2$ or $v(R_1) = \varepsilon$ we stop, having obtained a solution of either type (i) or (ii), in the latter case with $\overline{R} = R_1$.

The remaining possibility is that $R_1 > R/2$, that (2.8) holds on the full interval $[R_1, R]$, while (2.9) fails at $r = R_1$, namely

$$\frac{k}{R_1}\Omega\big(\big|v'(R_1)\big|\big) = f\big(v(R_1)\big).$$

In this case, the continuation switches to a new maximal interval $J_2 = (R_2, R_1]$, on which the function $v = v_2$ is defined as the solution of the problem

$$\big[\Omega\big(|v'|\big)\big]' + 3f(v) = 0, \tag{2.12}$$

$$v(R_1) = v_1(R_1), \qquad v'(R_1) = -\alpha_1 = v_1'(R_1) < 0, \tag*{(2.13)$_2$}$$

subject to the conditions (2.8) and

$$\frac{k}{r}\Omega\big(|v'|\big) < 2f(v). \tag{2.14}$$

Here it is important to note that both (2.8) and (2.14) are satisfied at the "initial" point $R_1$ of $J_2$.

Again it must be shown that the solution $v = v_2$ and the maximal interval $J_2$ exist. However, as in the case of (2.6), $(2.7)_1$, the problem (2.12), $(2.13)_2$ allows direct integration. Indeed, with the help of Lemma 1, (2.12) implies

$$\big[H\big(|v'|\big)\big]' = 3f(v)v' = 3\big[F(v)\big]'$$

and in turn

$$H\big(|v'|\big) = 3F(v) + \text{const.}$$

One then finds

$$\int^{v(r)} \frac{\mathrm{d}s}{H^{-1}(3F(s) + \text{const.})} = r + \text{const.},$$

implicitly defining the solution $v = v_2(r)$. By (2.12) and the monotonicity of $\Omega$ it is easy to see that $v$ is convex as long as (2.8) holds, and even more, by using the "initial" conditions $(2.13)_2$, that $v > 0$ and $v' < 0$. Thus again the existence of the maximal interval $J_2$ follows at once. Moreover it is clear that the end values $v(R_2) > 0$ and $v'(R_2) < 0$ exist.

If $R_2 = R/2$ or $v(R_2) = \varepsilon$ we stop, again having attained (i) or (ii).

Otherwise, (2.14) fails at the left endpoint $r = R_2 > R/2$ of $J_2$, and the continuation switches to a new maximal interval $I_3 = (R_3, R_2]$ on which the function $v = v_3$ is defined as the solution of the problem (2.6), (2.8), (2.9), with "initial" data:

$$v(R_2) = v_2(R_2), \qquad v'(R_2) = -\alpha_2 = v_2'(R_2) < 0 \tag*{(2.7)$_3$}$$

given at the right endpoint $R_2$ of $I_3$. Note again that (2.8) and (2.9) are satisfied at $R_2$.

Thus, as before, the existence of the solution $v = v_3$ and the maximal interval $I_3$ are easily established, along with the convexity of $v$ and the existence of the endvalues $v(R_3) > 0$ and $v'(R_3) < 0$.

We continue this recursive switching procedure to successive maximal intervals $J_4,\ I_5,\ J_6, \ldots$ such that the respective conditions (2.12), $(2.13)_i$, (2.8) and (2.14) hold on $J_i$, $i$ even, and (2.6), $(2.7)_i$, (2.8) and (2.9) hold on $I_i$, $i$ odd.

The functions $v_i$, $i \geqslant 1$, are convex on the respective intervals $I_1,\ J_2,\ I_3,\ J_4, \ldots$. Since the continued function $v$ is of class $C^1$ in view of the matching conditions $(2.7)_i$, $(2.13)_i$ at the endpoints $R_i$, it follows that $v$ *is convex on the entire continuation, and so also* $v > 0$, $v' < 0$ *along the continuation.*

The continuation stops if $R_i = R/2$ or $v(R_i) = \varepsilon$ for some $i \geqslant 1$, in which case either (i) or (ii) is satisfied. Otherwise it continues indefinitely. In the latter case, the endpoints $R_i$ clearly have an accumulation point

$$R_0 = \lim_{i\to\infty} R_i \geqslant R/2.$$

At the same time, $v$ is convex on $(R_0, R]$, so that

$$v(R_i) \to v(R_0), \qquad v'(R_i) \to v'(R_0)$$

with $v(R_0) \leqslant \varepsilon$ and $v'(R_0) < 0$ or possibly $v'(R_0) = -\infty$. We assert that in fact $v'(R_0) = -\infty$. Assuming this so, it follows that $v$ is of one of the types (i), (ii) or (iii).

Thus suppose $v'(R_0)$ is finite. Using (2.9) and (2.14) then gives (in the limit as $i$ tends to $\infty$)

$$\Omega\big(\big|v'(R_0)\big|\big) = \frac{k}{R_0} f\big(v(R_0)\big) = \frac{2k}{R_0} f\big(v(R_0)\big).$$

It follows that $f(v(R_0)) = 0$. But then by monotonicity of $f$ we get $f(u) = 0$ for $0 \leqslant u \leqslant v(R_0)$. This is a contradiction since it implies that $I_1$ is the only interval in the continuation (see (2.9)).

This completes the construction of the function $v(r)$. Note however that it has not yet been shown that inequality (2.2) is satisfied. We shall do this in the final Step 3.

**Step 2.** We show next that if $\alpha$ is sufficiently small, then cases (ii), (iii) cannot occur. To this end, observe that (2.12) holds on any interval $J_i$, $i \geqslant 2$ even, while on any interval $I_i = (R_i,\ R_{i-1}]$, $i \geqslant 3$ odd,

$$-\big[\Omega\big(|v'|\big)\big]' = \frac{2k}{r}\Omega\big(|v'|\big) = \frac{2k}{r} \cdot Cr^{-2k} \quad \text{(by (2.10))},$$

where

$$C = C_i = R_{i-1}^{2k}\Omega\big(\big|v'(R_{i-1})\big|\big) = R_{i-1}^{2k} \cdot \frac{2R_{i-1}}{k} f\big(v(R_{i-1})\big)$$

since (2.14) fails at the left endpoint $R_{i-1}$ of $J_{i-1}$. Consequently,

$$-\big[\Omega\big(|v'|\big)\big]' = 4\left(\frac{R_{i-1}}{r}\right)^{2k+1} f\big(v(R_{i-1})\big) \leqslant 2^{2k+3} f\big(v(R_{i-1})\big) \leqslant 2^{2k+3} f(v)$$

since $f$ is non-decreasing on $[0, \varepsilon]$ and $v(r) > v(R_{i-1})$ on $I_i$. Hence at any point $r \in (\overline{R}, R_1]$, whether in an interval of type $I$ or of type $J$, $i \geqslant 2$, we have

$$-\big[\Omega\big(|v'|\big)\big]' \leqslant (\mu - 1) f(v), \quad \mu = 2^{2k+3} + 1 > 9,$$

that is, using Lemma 1,

$$\big[H\big(|v'|\big)\big]' \geqslant (\mu-1)\big[F(v)\big]', \qquad \overline{R} < r \leqslant R_1. \tag{2.15}$$

Integrating (2.15) on $[r, R_1]$, we get

$$H\big(\big|v'(r)\big|\big) \leqslant (\mu-1)F\big(v(r)\big) + H(\alpha_1),$$

or, inverting $H$,

$$\big|v'(r)\big| \leqslant H^{-1}\big((\mu-1)F\big(v(r)\big) + H(\alpha_1)\big), \qquad \overline{R} < r \leqslant R_1. \tag{2.16}$$

It follows that $|v'|$ is bounded on the continuation (recall $v \leqslant \varepsilon$), so that case (iii) cannot occur.

Consider next case (ii), that is (2.5) holds and $v(\overline{R}) = \varepsilon$. Let $\alpha_0 > 0$ be such that

$$\alpha_1 < 2\varepsilon/R \tag{2.17}$$

whenever $\alpha \leqslant \alpha_0$. This can be done since (recalling that $\alpha_1$ is defined in condition $(2.13)_2$)

$$\begin{aligned} \alpha_1 &= \big|v_1'(R_1)\big| = \Omega^{-1}\big(C_1 R_1^{-2k}\big) \quad \text{by (2.11)} \\ &= \Omega^{-1}\big((R/R_1)^{2k}\Omega(\alpha)\big) \leqslant \Omega^{-1}\big(2^{2k}\Omega(\alpha)\big) \\ &\leqslant 2^{2k}\alpha, \end{aligned} \tag{2.18}$$

by Lemma 1 with $F$ replaced by $\Omega$ and with $\sigma = 2^{-2k}$. Then for $\alpha \leqslant \alpha_0$ we have by the convexity of $v$ and by (2.17)

$$v(R_1) = \int_{R_1}^{R} |v'|\,\mathrm{d}s \leqslant \int_{R_1}^{R} \alpha_1\,\mathrm{d}s \leqslant \alpha_1 R/2 < \varepsilon = v(\overline{R}). \tag{2.19}$$

In turn, necessarily $\overline{R} < R_1$.

We now divide into two cases, first, when $f(u) \equiv 0$ for $0 < u < \tau$, for some $\tau \in (0, \delta]$, and second, when $f(u) > 0$ for $0 < u < \delta$. In the first instance, assuming without loss of generality that $\delta = \tau$, it is clear that (2.9) necessarily holds on the entire continuation, so that $R_1 = \overline{R}$, a contradiction.

Thus we assume from here on (the main case) that $f(u) > 0$ for $0 < u < \delta$. Consequently $F(u)$ is strictly increasing on $0 \leqslant u < \delta$ and so has a strictly increasing inverse $F^{-1}$ on the interval $[0, F(\delta))$. This being the case, we may add a further condition on $\alpha_0$, namely that

$$H(\alpha_1) < F(\varepsilon) \quad \text{when } \alpha_1 < \alpha_0. \tag{2.20}$$

For simplicity in what follows, put $v_1 = v(R_1)$, $F_1 = F(v_1)$, $H_1 = H(\alpha_1)$. We now define (the purpose will appear later)

$$\gamma = \begin{cases} v_1 & \text{if } F_1 \geqslant H_1, \\ F^{-1}(H_1) & \text{if } F_1 < H_1, \end{cases} \tag{2.21}$$

(in the second line, recall that $H_1 < F(\varepsilon) < F(\delta)$ so that $F^{-1}(H_1)$ is well defined).

We claim that

$$v_1 \leqslant \gamma < \varepsilon. \tag{2.22}$$

When $\gamma = v_1$ this is obvious since $v_1 < \varepsilon$ by (2.19). On the other hand, in case $\gamma = F^{-1}(H_1)$ we have $F(\gamma) = H_1 > F_1$ so $v_1 < \gamma$, while also $F(\gamma) = H_1 < F(\varepsilon)$ by (2.20), thus yielding $\gamma < \varepsilon$. Note also that $F(\gamma) \geqslant H_1$ in both cases of (2.21).

Now let $\rho$ be defined by $\gamma = v(\rho)$, this being possible because of (2.22) and the facts that $v(\overline{R}) = \varepsilon$ and $v' < 0$. Clearly $\overline{R} < \rho \leqslant R_1$.

Then for $\overline{R} < r \leqslant \rho$ we have

$$F\big(v(r)\big) \geqslant F\big(v(\rho)\big) = F(\gamma) \geqslant H_1. \tag{2.23}$$

It follows from (2.16) and (2.23) that

$$|v'| < H^{-1}\big(\mu F(v)\big), \qquad \overline{R} < r \leqslant \rho.$$

In turn, since $\mu > 1$, we see by the first part of Lemma 1, with $\sigma = 1/\mu$, that

$$|v'| < H^{-1}\big(F(\mu v)\big), \qquad \overline{R} < r \leqslant \rho. \tag{2.24}$$

(Here, one can assume without loss of generality that $\varepsilon < \delta/\mu$.) Integrating from $\overline{R}$ to $\rho$ then yields

$$\int_\gamma^\varepsilon \frac{\mathrm{d}v}{H^{-1}(F(\mu v))} \leqslant \frac{R}{2}$$

after changing to the natural variable $v$ of integration. A further change of variables $s = \mu v$ gives finally

$$\int_{\mu\gamma}^{\mu\varepsilon} \frac{\mathrm{d}s}{H^{-1}(F(s))} \leqslant \frac{\mu R}{2}. \tag{2.25}$$

We assert that $\gamma \to 0$ as $\alpha \to 0$. Indeed if $\gamma = v_1$ then by (2.19) we have $\gamma \leqslant \alpha_1 R/2$. But from (2.18) one obtains $\alpha_1 \to 0$ as $\alpha \to 0$, giving the required result. Next, if $\gamma = F^{-1}(H_1)$ the assertion is obvious, since $H_1 = H(\alpha_1) \to 0$ as $\alpha \to 0$.

This being shown, the principal divergence condition (2.1) applied to (2.25) yields an immediate contradiction as $\alpha$, and so $\gamma$, tends to 0. Thus case (ii) also cannot happen, and the constructed function $v$ is therefore of type (i).

**Step 3.** It remains to show that $v$ satisfies the inequality (2.2) in $[R/2, R]$. First, on any interval of type $I$ we have from (2.6)

$$\big[\Omega\big(|v'|\big)\big]' + \frac{k}{r}\Omega\big(|v'|\big) + f(v) = -\frac{k}{r}\Omega\big(|v'|\big) + f(v) < 0,$$

where at the last step we have used (2.9). On the other hand, on any interval of type $J$ we see from (2.12) that

$$\big[\Omega\big(|v'|\big)\big]' + \frac{k}{r}\Omega\big(|v'|\big) + f(v) = \frac{k}{r}\Omega\big(|v'|\big) - 2f(v) < 0$$

by (2.14). This completes the proof of the lemma. □

LEMMA 3 (Weak comparison principle). – *Let $u$ and $v$ be respective solutions of* (1.1) *and* (1.2) *in a bounded domain $D$. Suppose also that $u$ and $v$ are continuous in $\overline{D}$, with $v < \delta$ in $D$ and $u \geqslant v$ on $\partial D$. Then $u \geqslant v$ in $D$.*

For proof, see [6, Lemma 3].

Now we are ready to prove Theorem 1. We first show that the function $v(x) = v(r)$, $r = |x|$, where $v$ is given by Lemma 2, satisfies the differential inequality (1.2) in

$$E_R = \big\{x \in \mathbb{R}^n\colon\ R/2 \leqslant |x| \leqslant R\big\}.$$

This is a consequence of the calculation:

$$\begin{aligned}\operatorname{div}\big\{A\big(|Dv|\big)Dv\big\} - f(v) &= \operatorname{div}\big\{A\big(|v'|\big)v'x/r\big\} - f(v)\\ &= -\big\{\Omega\big(|v'|\big)\big\}' - \frac{(n-1)}{r}\Omega\big(|v'|\big) - f(v) \geqslant 0,\end{aligned} \tag{2.26}$$

where we recall that $v' < 0$ and use Lemma 2 with $k = n-1$, $\ell = 1$.

This being shown, the proof of sufficiency is now exactly the same as in the standard demonstration of the strong maximum principle (see [5, proof of Theorem 3.5 on page 35]), since the comparison function $v$ satisfies the conditions, see [5, proof of Lemma 3.4 on page 34]:

(i) $v > 0$ in $E_R$,
(ii) $v = 0$ when $|x| = R$,
(iii) $\partial v/\partial \nu = v' < 0$ when $|x| = R$, where $\nu$ is the outer normal to $\partial E_R$,
(iv) $v < \varepsilon$ when $|x| = R/2$,

where $\varepsilon,\ R > 0$ can be taken arbitrarily small and the origin of coordinates can be chosen arbitrarily in $D$. Note that the use of the weak maximum principle (Corollary 3.2 of [5]) is here replaced by application of Lemma 3. This completes the proof of the sufficiency part of Theorem 1.

As remarked in the introduction, the necessity is due to Diaz. Hence Theorem 1 is proved.

*Remark.* – The necessity of condition (1.5) can be obtained under weaker hypothesis than (F2). In fact, it is enough to replace (F2) by

$$f(0) = 0 \quad \text{and} \quad F(s) > 0 \text{ for } s \in (0, \delta). \tag{F2$'$}$$

This is because the principal construction required for Diaz' proof uses only condition (F2)$'$; see also [6, construction of the function $w = w(r)$, $r = x_1$, in the proof of Theorem 2].

*Proof of Theorem 2.* – Sufficiency was shown in [6, Theorem 2].

To prove necessity, suppose (1.6) fails. We let $u$ be a non-trivial solution of (2.1) in the domain $D_R = \{x \in \mathbb{R}^n\colon\ |x| > R\}$ such that $u(x) \to 0$ as $|x| \to \infty$. (The existence of such a solution $u$, indeed with equality in (2.1), is guaranteed by Theorem 3 of [6].) By Theorem 1, since (1.5) must hold it is clear that $u > 0$ in $D_R$, but this violates the compact support principle. Hence (1.6) is necessary, completing the proof of Theorem 2. □

*Remark.* – The simplicity of this proof is deceptive. Theorem 3 of [6] in particular is quite difficult to prove: one would wish a simpler demonstration in which the required solution of (2.1) in $D_R$ is obtained by some more elementary means.

## 3. Fully quasilinear case

Let $D$ be a domain in $\mathbb{R}^n$. Let $\{a^{ij}(x)\}$, $i, j = 1, \ldots, n$, be a continuously differentiable, symmetric coefficient matrix on $D$, which is uniformly elliptic in the sense that

$$a^{ij}(x)\xi_i\xi_j \geqslant \lambda|\xi|^2, \quad x \in D,\ \xi \in \mathbb{R}^n,$$

for some positive number $\lambda$. Moreover, let $B(x, u, p)$ be a continuous function on $D \times \mathbb{R}_0^+ \times \mathbb{R}^n$.

Consider the differential inequality

$$D_i\left\{a^{ij}(x)A\big(|Du|\big)D_j u\right\} - B(x, u, Du) \leqslant 0, \quad u \geqslant 0,\ x \in D. \tag{3.1}$$

We shall suppose that the operator $A = A(t)$ satisfies the following strengthened versions of (A1), (A2), namely:

(A1)′ $A \in C^1(0, \infty)$,

(A2)′ $\Omega'(t) > 0$ for $t > 0$, and $\Omega(t) \to 0$ as $t \to 0$;

we also continue to assume that the nonlinearity $f$ obeys (F1) and (F2).

THEOREM 3 (Strong maximum principle). – *Assume that there exists a constant $\kappa > 0$ such that*

$$B(x, u, p) \leqslant \kappa\,\Omega\big(|p|\big) + f(u) \tag{3.2}$$

*for $x \in D$, $u \geqslant 0$, and all $p \in \mathbb{R}^n$ with $|p| < 1$. Suppose finally that either $f(s) \equiv 0$ for $s \in [0, \tau)$, $\tau > 0$, or else* (1.5) *holds.*

*If $u$ is a $C^1$ solution of* (3.1) *with $u(x_0) = 0$ for some $x_0 \in D$, then $u \equiv 0$ in $D$.*

This result was obtained in [6, Theorem 1′] under the additional technical assumption [6, (2.5)]. For comments on earlier work, see [6, Section 4].

To obtain Theorem 3 we require a slightly strengthened version of Lemma 2.

LEMMA 4. – *Lemma* 2 *holds with* (2.4) *replaced by*

$$0 < v < \varepsilon, \qquad -1 < v' < 0 \quad in\ [R/2, R). \tag{3.3}$$

*Proof.* – Without loss of generality we can assume that $\varepsilon > 0$ is so small that

$$F(\varepsilon) < 2^{-2k}H(1). \tag{3.4}$$

We assert that (3.3) then holds, provided that $\alpha$ is made even smaller if necessary, so that $\alpha < 2^{-2k}$.

There are two cases: first, when $f(u) \equiv 0$ for $0 < u < \tau$, for some $\tau \in (0, \delta]$, and second, when $f(u) > 0$ for $0 < u < \delta$. In the first instance, assuming without loss of generality that $\delta = \tau$, it is clear that (2.9) necessarily holds on the entire continuation, so that $R_1 = R/2$. Then, as in (2.18), we get

$$\left|v'(R/2)\right| \leqslant 2^{2k}\alpha < 1.$$

Hence $(3.3)_2$ follows since $v' < 0$ and $v$ is convex.

In the second case, we find as in (2.24)

$$\left|v'(R/2)\right| \leqslant H^{-1}\big(F(\mu\varepsilon)\big) < 1$$

by (3.4). This completes the proof of the lemma. □

*Proof of Theorem 3.* – Let $O$ be an arbitrary origin in $D$. Put $E_R = \{x \in \mathbb{R}^n\colon\ R/2 \leqslant |x| \leqslant R\}$ and define

$$\Lambda = \text{ max eigenvalue of } \{a^{ij}(x)\} \text{ in } E_R, \qquad a = \max\left|D_i\, a^{ij}(x)\right| \text{ in } E_R.$$

It is easy to see that

$$D_i\left(a^{ij}(x)\frac{x_j}{r}\right) = \left(D_i\, a^{ij}(x)\right)\frac{x_j}{r} + \frac{a^{ij}}{r}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right),$$

so

$$\max_{E_R}\left|D_i\left(a^{ij}(x)\frac{x_j}{r}\right)\right| \leqslant a + \frac{n-1}{r}\Lambda.$$

Let $v = v(r)$, $r = |x|$, be the function given by Lemmas 2, 4. Then we have $-1 < v' < 0$ in $E_R$, and in turn

$$\begin{aligned}
&D_i\{a^{ij}(x)A(|Dv|)D_j v\} - \kappa\Omega(|Dv|) - f(v)\\
&\quad = -D_i\left\{a^{ij}(x)\frac{x_j}{r}\right\}\Omega(|v'|) - a^{ij}(x)\frac{x_i x_j}{r^2}\{\Omega(|v'|)\}' - \kappa\Omega(|v'|) - f(v)\\
&\quad \geqslant -a^{ij}(x)\frac{x_i x_j}{r^2}\{\Omega(|v'|)\}' - \left(a + \frac{n-1}{r}\Lambda + \kappa\right)\Omega(|v'|) - f(v)\\
&\quad \geqslant -a^{ij}(x)\frac{x_i x_j}{r^2}\left\{[\Omega(|v'|)]' + \frac{k}{r}\Omega(|v'|) + \frac{f(v)}{\lambda}\right\},
\end{aligned}$$

where $k = [(n-1)\Lambda + (a+\kappa)R]/\lambda$. Now from Lemma 4 with $\ell = 1/\lambda$, together with the main assumption (1.5), we obtain

$$[\Omega(|v'|)]' + \frac{k}{r}\Omega(|v'|) + \frac{f(v)}{\lambda} \leqslant 0, \tag{3.5}$$

so in turn

$$D_i\{a^{ij}(x)A(|Dv|)D_j v\} - \kappa\Omega(|Dv|) - f(v) \geqslant 0, \quad v \geqslant 0, \tag{3.6}$$

in $E_R$.

We next require a comparison lemma corresponding to Lemma 3, but applying to the more general inequality (3.1).

LEMMA 5 (Comparison principle). – *Let $u$ and $v$ be respectively solutions of* (3.1) *and* (3.6) *in a bounded domain $D$. Suppose that $|Du| + |Dv| > 0$ in $D$; that $u$ and $v$ are continuous in $\overline{D}$; and that*

$$v < \delta \quad \text{in } D, \qquad u \geqslant v \quad \text{on } \partial D.$$

*Then $u \geqslant v$ in $D$.*

The main point of Lemma 5 is that if $|Du| + |Dv| > 0$ in $D$, then just as for Lemma 3 *it is not necessary to have ellipticity at the value $p = 0$*. For proof of Lemma 5, see [6, Section 5].

The rest of the proof of Theorem 3 is now the same as the sufficiency part of Theorem 1, the only change being that at the last step we rely on Lemma 5 instead of Lemma 3. □

There is a corresponding compact support principle for the inequality

$$D_i\{a^{ij}(x)A(|Du|)D_j u\} - B(x,u,Du) \geqslant 0, \quad u \geqslant 0,$$

where for some constant $\kappa > 0$,

$$B(x,u,p) \geqslant -\kappa\Omega(|p|) + f(u),$$

for $x \in D,\ u \geqslant 0$, and all $p \in \mathbb{R}^n$ with $|p| < 1$. For the statement and proof of this principle, see [6, Theorem 2′].

## 4. Special cases

**4.1.** Consider the linear inequality

$$D_i\{a^{ij}(x)D_j u\} + b^i(x)D_i u + c(x)u \leqslant 0, \quad u \geqslant 0, \tag{4.1}$$

for $x \in D$. This is the special case of (3.1) with $A(t) \equiv 1,\ B(x,u,p) = -b^i(x)p_i - c(x)u$. Here we can apply the result Theorem 3, assuming

$$\kappa = \sup_D \sum_{i=1}^n |b^i(x)| < \infty, \qquad c = -\inf_D \{c(x),\ 0\} < \infty,$$

and defining $f(u) = cu$. Then $\Omega(t) = t,\ H^{-1}(t) = \sqrt{2t}$ and $F(u) = \frac{1}{2}cu^2$, so that (3.2) and (1.5) hold as required. This gives the strong maximum principle for (4.1), essentially the classical theorem of E. Hopf, and moreover leads us to expect that the main proof can be simplified for the special linear case.

In fact, the construction of the function $v = v(r)$ in Step 1 of Lemma 2 suggests that $v$ can be obtained directly by solving Eq. (2.6). This gives at once

$$v(r) = \frac{\alpha R}{\theta - 1}\left[\left(\frac{R}{r}\right)^{\theta-1} - 1\right], \tag{4.2}$$

where $\theta = 2k$; here we assume $k > 1/2$ so that $\theta > 1$. Then $\Omega(|v'|) = |v'| = \alpha(R/r)^{2k}$ and so

$$\begin{aligned}
&[\Omega(|v'|)]' + \frac{k}{r}\Omega(|v'|) + \frac{1}{\lambda}f(v) \\
&= -\frac{2k\alpha}{R}\left(\frac{R}{r}\right)^{\theta+1} + \frac{k\alpha}{R}\left(\frac{R}{r}\right)^{\theta+1} + \frac{c\alpha R}{\lambda(\theta-1)}\left[\left(\frac{R}{r}\right)^{\theta-1} - 1\right] \\
&\leqslant \frac{\alpha}{r}\left(\frac{R}{r}\right)^{2k}\left[\frac{cr^2}{\lambda(\theta-1)} - k\right] \leqslant 0,
\end{aligned}$$

provided that

$$R^2 \leqslant \frac{k(2k-1)}{\lambda c}, \tag{4.3}$$

that is, (3.5)–(3.6) hold under the conditions (4.3) and

$$k = (n-1)\frac{\Lambda}{\lambda} + (a+\kappa)\frac{R}{\lambda}. \tag{4.4}$$

Thus the polynomial comparison function (4.2) can be used for the linear inequality (4.1), alternative to the standard exponential function, see [5, page 34],

$$v(r) = \varepsilon\left(\mathrm{e}^{-\alpha r^2} - \mathrm{e}^{-\alpha R^2}\right).$$

**4.2.** A similar simplification can be used for the canonical inequality

$$\Delta_m u - f(u) \leqslant 0, \quad u \geqslant 0, \tag{4.5}$$

for the $m$-Laplace operator, $m > 1$. For our present purpose, we assume also that

$$f(u) \leqslant c u^{m-1}, \tag{4.6}$$

the borderline case for (1.5).

The comparison function $v = v(r)$ again can be taken in the form (4.2), with now

$$\theta = \frac{2k}{m-1}, \quad k > \frac{m-1}{2}.$$

Then $\Omega(|v'|) = |v'|^{m-1} = \alpha^{m-1}(R/r)^{2k}$, so as above we get

$$\left[\Omega\left(|v'|\right)\right]' + \frac{k}{r}\Omega\left(|v'|\right) + \frac{1}{\lambda} f(v) \leqslant \frac{\alpha^{m-1}}{r}\left(\frac{R}{r}\right)^{2k}\left[\frac{c r^m}{\lambda(\theta-1)^{m-1}} - k\right] \leqslant 0$$

provided that

$$R \leqslant (k/\lambda c)^{1/m}(\theta - 1)^{1/m'}. \tag{4.7}$$

Thus to obtain (2.26) it is enough to have (4.7) and $k = \max\{n-1,\ m-1\}$.

In summary, for the borderline case (4.6) of inequality (4.5), we get an elementary proof of Vazquez' strong maximum principle, avoiding the delicate arguments of Section 2 or of [8].

*Remark.* – It is easy to see that the simple comparison function (4.2) does not suffice for general operators or for more complicated nonlinearities. This observation indicates the need for the new construction of $v = v(r)$ used in the proof of Lemma 2.

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