Non-differentiable minimax fractional programming with generalized $\alpha$-univexity

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Received 24 January 2007

Abstract

In this paper, we study a non-differentiable minimax fractional programming problem under the assumption of generalized $\alpha$-univex function. In this paper we extend the concept of $\alpha$-invexity [M.A. Noor, On generalized preinvex functions and monotonicities, J. Inequalities Pure Appl. Math. 5 (2004) 1–9] and pseudo $\alpha$-invexity [S.K. Mishra, M.A. Noor, On vector variational-like inequality problems, J. Math. Anal. Appl. 311 (2005) 69–75] to $\alpha$-univexity and pseudo $\alpha$-univexity from a view point of generalized convexity. We also introduce the concept of strict pseudo $\alpha$-univex and quasi $\alpha$-univex functions. We derive Karush–Kuhn–Tucker-type sufficient optimality conditions and establish weak, strong and converse duality theorems for the problem and its three different form of dual problems. The results in this paper extend a few known results in the literature.

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MSC: 29A51; 49J35; 90C32

Keywords: Non-differentiable minimax fractional problem; Duality; Sufficient optimality conditions; Univexity

1. Introduction

Recently, several authors have been interested in the optimality conditions and duality results for minimax programming problems. Yadav and Mukherjee [13] established the optimality conditions to construct the two dual problems and derived duality theorems for differentiable fractional minimax programming. Chandra and Kumar [3] pointed out that the formulation of Yadav and Mukherjee [13] has some omissions and inconsistencies and they constructed two modified dual problems and proved duality theorems for (convex) differentiable fractional minimax programming. To relax convexity assumptions involved in sufficient optimality conditions and duality theorems, various generalized convexity notions have been proposed. Yang and Hou [14] paid the much attention on minimax fractional programming problem and established the sufficient optimality conditions and derived a number of duality results.

Schmitendorf [12] introduced necessary and sufficient optimality conditions for generalized minimax programming, much attention has been paid to optimality conditions and duality theorems for generalized minimax programming problems, for example, see, [1,3–5,10]. Bector and Bhatia [1] relaxed the convexity assumptions in the sufficient optimality condition in [12] and also employed the optimality conditions to construct several dual models which involve pseudo-convex and quasi-convex functions, and derived weak and strong duality theorems.

Bector et al. [2] introduced some classes of univex functions by relaxing the definition of an invex function. Optimality and duality results are also obtained for a non-linear multiobjective programming problem in [2]. In this paper, we introduce the concept of strict pseudo \( z \)-univex and quasi \( z \)-univex functions and extend the results of Lai and Lee [6] and Lai et al. [7] to the classes of functions introduced in Section 2. This paper is organized as follows. Some definitions and notations are given in Section 2. In Section 3, we derive the sufficient optimality conditions for non-differentiable minimax fractional programming problems under the assumption of generalized \( z \)-univexity. We discuss duality between the primal problem and different types of dual models in Sections 4–6. This work extends several existing results on fractional minimax problems.

2. Preliminaries

Let \( X \) be a non-empty subset of \( R^n \), \( \eta : X \times X \to R^n \) is an \( n \)-dimensional vector valued function and \( \alpha(x, u) : X \times X \to R_+ \setminus \{0\} \) be a bifunction. First, we recall the following definition.

**Definition 2.1 (Noor [11]).** A subset \( X \) is said to be \( z \)-invex set, if there exist \( \eta : X \times X \to R^n \), \( \alpha(x, a) : X \times X \to R_+ \) such that for all \( x \in X \)

\[
u + k \alpha(x, a) \eta(x, a) \in X \quad \forall x, a \in X, \ k \in [0, 1].
\]

Note that \( z \)-invex set need not be a convex set, see Noor [11].

From now onward we assume that the set \( X \) is a non-empty \( z \)-invex set with respect to \( \alpha(., .) \) and \( \eta(., .) \) unless otherwise specified.

Let \( f, g : R^n \times R^m \to R \) be \( C^1 \)-functions and \( h : R^n \to R^p \) a vector valued \( C^1 \)-mapping. Let \( A \) and \( B \) be \( n \times n \) positive semi-definite matrices. Suppose that \( Y \), an \( x \)-invex set, is a compact subset of \( R^m \). Consider the following non-differentiable minimax fractional problem:

\[
(P) \quad \inf_{x \in X} \sup_{y \in Y} \frac{f(x, y) + \langle x, Ax \rangle^{1/2}}{g(x, y) - \langle x, Bx \rangle^{1/2}}
\]

s.t. \( h(x) \leq 0 \),

where \( \langle \cdot, \cdot \rangle \) denotes the inner product in Euclidean space. This problem is non-differentiable programming problem if either \( A \) or \( B \) is non-zero. If \( A \) and \( B \) are null matrices, the problem (P) is a minimax fractional programming problem.

We denote by \( X_P \) the set of all feasible solutions of (P) and by \( R^+_n \) the positive orthant of \( R^n \). For each \( (x, y) \in X \times X \), define

\[
\phi(x, y) = \frac{f(x, y) + \langle x, Ax \rangle^{1/2}}{g(x, y) - \langle x, Bx \rangle^{1/2}}.
\]

Assume that for each \( (x, y) \in R^n \times X \), \( f(x, y) + \langle x, Ax \rangle \geq 0 \) and \( g(x, y) - \langle x, Bx \rangle > 0 \).

Denote

\[
\tilde{Y}(x) = \left\{ \tilde{y} \in Y : \frac{f(x, \tilde{y}) + \langle x, Ax \rangle^{1/2}}{g(x, \tilde{y}) - \langle x, Bx \rangle^{1/2}} = \sup_{y \in Y} \frac{f(x, y) + \langle x, Ax \rangle^{1/2}}{g(x, y) - \langle x, Bx \rangle^{1/2}} \right\},
\]

\( J = \{1, 2, \ldots, p\}, \quad J(x) = \{j \in J : h_j(x) = 0\}. \)
Let $K$ be a triplet such that

$$K(x) = \left\{ (s, t, \tilde{y}) \in N \times R_+^s \times R^{ms} : 1 \leq s \leq n + 1, t = (t_1, t_2, \ldots, t_s) \in R_+^s, \sum_{i=1}^s t_i = 1 \text{ and } \tilde{y} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_s) \text{ and } \tilde{y}_i \in \tilde{Y}(x), \forall i = 1, 2, \ldots, s \right\}.$$ 

Since $f$ and $g$ are continuous differentiable, and $Y$ is a compact subset of $R^m$, it follows that for each $x_0 \in \mathcal{X}$, $\tilde{Y}(x_0) \neq \phi$. Thus for any $\tilde{y}_i \in \tilde{Y}(x_0)$, we have a positive constant $k_0 = \phi(x_0, \tilde{y}_i)$. We shall need the following generalized Schwarz inequality in our discussions:

$$\langle x, Av \rangle \leq \langle x, Ax \rangle^{1/2} \langle v, Av \rangle^{1/2} \quad \text{for some } x, v \in R^n$$

(1)

the equality holds when $Ax = \lambda Av$ for some $\lambda \geq 0$. Hence, if $\langle v, Av \rangle \leq 1$, we have

$$\langle x, Av \rangle \leq \langle x, Ax \rangle^{1/2}.$$ 

(2)

In order to relax the convexity assumption in the above lemma, we impose the following definitions. Assume that $b_0, b_1 : X \times X \times [0, 1] \to R^+$, $b(x, a) = \lim_{\lambda \to 0} b(x, a, \lambda) \geq 0$, and $b$ does not depend on $\lambda$ if the function is differentiable, $\phi_0, \phi_1 : R \to R$.

**Definition 2.2.** $f$ is said to be $\alpha$-univex at $a \in X$ with respect to $b_0, \phi_0, \alpha$ and $\eta$ if there exist functions $b_0, \phi_0, \alpha$ and $\eta$ such that, for every $x \in X$, we have $b_0(x, a)\phi_0(f(x) - f(a)) \geq \langle \alpha(x, a)\nabla f(a), \eta(x, a) \rangle$.

**Remark 1.** Note that any $\alpha$-univex function is $\alpha$-univex if we define $\phi : R \to R$ with $\phi(V) = V$ and $b_0(x, a) = 1$. But the converse does not necessarily hold. It can be seen in the following example.

**Example 2.1.** Let $f : R \to R$ be defined by $f(x) = x^3$ and

$$b(x, a) = \begin{cases} \frac{x^2}{s-a}, & x > a, \\ 0, & x \leq a \end{cases}$$

and

$$\eta(x, a) = \begin{cases} x^2 + a^2 + xa, & x > a, \\ x - a, & x \leq a. \end{cases}$$

Let $\alpha(x, a) = 1, \phi : R \to R$ be defined by $\phi(V) = 3V$. The function $f$ is $\alpha$-univex but not $\alpha$-univex, because for $x = -3, a = 1, f(x) - f(a) < \eta(x, a)\nabla f(a)$.

**Definition 2.3.** $f$ is said to be pseudo $\alpha$-univex at $a \in X$ with respect to $b_0, \phi_0, \alpha$ and $\eta$ if there exist functions $b_0, \phi_0, \alpha$ and $\eta$ such that, for every $x \in X$, we have

$$\langle \alpha(x, a)\nabla f(a), \eta(x, a) \rangle \geq 0 \Rightarrow b_0(x, a)\phi_0(f(x) - f(a)) \geq 0,$$

equivalently,

$$b_0(x, a)\phi_0(f(x) - f(a)) < 0 \Rightarrow \langle \alpha(x, a)\nabla f(a), \eta(x, a) \rangle < 0.$$

**Definition 2.4.** $f$ is said to be strict pseudo $\alpha$-univex at $a \in X$ with respect to $b_0, \phi_0, \alpha$ and $\eta$ if there exist functions $b_0, \phi_0, \alpha$ and $\eta$ such that, for every $x \in X$, we have

$$\langle \alpha(x, a)\nabla f(a), \eta(x, a) \rangle \geq 0 \Rightarrow b_0(x, a)\phi_0(f(x) - f(a)) > 0,$$
equivalently,
\[ b_0(x, a)\phi_0[f(x) - f(a)] \leq 0 \Rightarrow \langle \alpha(x, a)\nabla f(a), \eta(x, a) \rangle < 0. \]

**Example 2.2.** The function \( f : R \rightarrow R \) defined by \( f(x) = (x - 1)^3 \) is strict pseudo \( \alpha \)-univex at \( a = 0 \) with respect to \( b_0(x, a) = 1 = \alpha(x, a), \eta(x, a) = \{(x - 1)/2\} \) and \( \phi \) is an identity function on \( R \) but \( f(x) \) is not \( \alpha \)-univex with respect same \( b_0(x, a), \alpha(x, a), \eta(x, a) \) and \( \phi \) as can be seen by taking \( x = -1 \).

**Definition 2.5.** \( f \) is said to be quasi \( \alpha \)-univex at \( a \in X \) with respect to \( b_0, \phi_0, \alpha \) and \( \eta \) if there exist functions \( b_0, \phi_0, \alpha \) and \( \eta \) such that, for every \( x \in X \), we have
\[ \langle \alpha(x, a)\nabla f(a), \eta(x, a) \rangle > 0 \Rightarrow b_0(x, a)\phi_0[f(x) - f(a)] > 0, \]
equivalently,
\[ b_0(x, a)\phi_0[f(x) - f(a)] \leq 0 \Rightarrow \langle \alpha(x, a)\nabla f(a), \eta(x, a) \rangle \leq 0. \]

The following example shows that quasi \( \alpha \)-univex function exists.

**Example 2.3.** The function \( f : R \rightarrow R \) defined by \( f(x) = (2x - 1)^3 \) is quasi \( \alpha \)-univex at \( a = 0 \) with respect to \( b_0(x, a) = 1 = \alpha(x, a), \eta(x, a) = (x) \) and \( \phi \) is an identity function on \( R \) but \( f(x) \) is neither \( \alpha \)-univex with respect same \( b_0(x, a), \alpha(x, a), \eta(x, a) \) and \( \phi \) as can be seen by taking \( x = 1 \) nor strict pseudo \( \alpha \)-univex with respect same \( b_0(x, a), \alpha(x, a), \eta(x, a) \) and \( \phi \) as can be seen by taking \( x = 0 \).

The following result from [6] is needed in the sequel.

**Lemma 1.** Let \( x_0 \) be an optimal solution for (P) satisfying \( \langle x_0, A x_0 \rangle > 0, \langle x_0, B x_0 \rangle > 0 \) and \( \nabla h_j(x_0), j \in J(x_0) \) are linearly independent. Then there exist \( (s, t^*, \bar{y}) \in K(x_0), u, v \in R^n \) and \( \mu^* \in R^n_+ \) such that
\[
\sum_{i=1}^{s} t_i^* (\nabla f(x_0, \bar{y}_i) + Au - k_0(\nabla g(x_0, \bar{y}_i) - Bv)) + \nabla \langle \mu^*, h(x_0) \rangle = 0,
\]
\[
f(x_0, \bar{y}_i) + \langle x_0, A x_0 \rangle^{1/2} - k_0(g(x_0, \bar{y}_i) - \langle x_0, B x_0 \rangle^{1/2}) = 0, \quad i = 1, 2, \ldots, s,
\]
\[
\langle \mu^*, h(x_0) \rangle = 0,
\]
\[
t_i^* \in R_+^s \quad \text{with} \quad \sum_{i=1}^{s} t_i^* = 1,
\]
\[
\langle u, Au \rangle \leq 1, \quad \langle v, Bv \rangle \leq 1,
\]
\[
\langle x_0, Au \rangle = \langle x_0, A x_0 \rangle^{1/2}, \quad \langle x_0, Bv \rangle = \langle x_0, B x_0 \rangle^{1/2}.
\]

It should be noted that both the matrices \( A \) and \( B \) are positive definite at the solution \( x_0 \) in the above lemma. If one of \( \langle A x_0, x_0 \rangle \) and \( \langle B x_0, x_0 \rangle \) is zero, or both \( A \) and \( B \) are singular at \( x_0 \), then for \( (s, t^*, \bar{y}) \in K(x_0) \), we can take \( Z_f(x_0) \) defined in [6] by
\[
Z_f(x_0) = \{ z \in R^n : \langle \nabla h_j(x_0), z \rangle \leq 0, j \in J(x_0) \} \text{ with any one of the following (i)–(iii) holds}
\]

(i)
\[
\langle A x_0, x_0 \rangle > 0, \quad \langle B x_0, x_0 \rangle = 0
\]
\[
\Rightarrow \left( \sum_{i=1}^{s} t_i^* \nabla f(x_0, \bar{y}_i) + \frac{A x_0}{\langle A x_0, x_0 \rangle^{1/2}} - k_0 \nabla g(x_0, \bar{y}_i), z \right) + \langle (k_0^2 B)z, z \rangle^{1/2} < 0,
\]
Theorem 3.1. \(\langle A x_0, x_0 \rangle = 0, \quad \langle B x_0, x_0 \rangle > 0\)

\[
\Rightarrow \left\{ \sum_{i=1}^{s} t^*_i \left( \nabla f(x_0, \bar{y}_i) - k_0 \left( \nabla g(x_0, \bar{y}_i) - \frac{B x_0}{\langle B x_0, x_0 \rangle^{1/2}} \right) \right) , z \right\} + \langle B z, z \rangle^{1/2} < 0.
\]

Proof. Suppose that \(x^* \in X\) is a feasible solution for (P). Then there exists \(k_0 \in \mathbb{R}_+\), \((s, t^*, \bar{y}) \in K(x_0), u, v \in \mathbb{R}^n\) and \(\mu^* \in \mathbb{R}_+^n\) satisfying (3)–(7). Assume that one of the following conditions holds:

(a) \(\varphi(.) = \sum_{i=1}^{s} t^*_i ((f \circ \bar{y}_i) + \langle ., Au \rangle) - k_0 (g \circ \bar{y}_i) - \langle ., B v \rangle)\) and \((\mu^*, h(.)\) are z-univex with respect to \(b_0, b_1, \phi_0, \phi_1, x_0\) and \(\eta\) with \(\phi(V) \geq 0 \Rightarrow V \geq 0\) and \(\phi_1(V) \geq V\);

(b) \(\varphi(.) = \sum_{i=1}^{s} t^*_i ((f \circ \bar{y}_i) + \langle ., Au \rangle) - k_0 (g \circ \bar{y}_i) - \langle ., B v \rangle)\) is pseudo z-univex with respect to \(b_0, \phi_0, x_0\) and \(\eta\) with \(V < 0 \Rightarrow \phi(V) < 0\) and \(\mu^*, h(.)\) is quasi z-univex with respect to \(b_1, \phi_1, x_1\) and \(\eta\) with \(V \leq 0 \Rightarrow \phi_1(V) \leq 0\);

(c) \(\varphi(.) = \sum_{i=1}^{s} t^*_i ((f \circ \bar{y}_i) + \langle ., Au \rangle) - k_0 (g \circ \bar{y}_i) - \langle ., B v \rangle)\) is quasi z-univex with respect to \(b_0, \phi_0, x_0\) and \(\eta\) and \(\mu^*, h(.)\) is strictly pseudo z-univex with respect to \(b_1, \phi_1, x_1\) and \(\eta\) with \(V \leq 0 \Rightarrow \phi_0(V) \leq 0\) and \(\phi_0(V) \geq 0 \Rightarrow V \geq 0\).

Then \(x^*\) is an optimal solution of (P).

3. Optimality condition

In this section, we shall establish a sufficient optimality condition.

Theorem 3.1 (Sufficient optimality conditions). Suppose that \(x_0 \in X_P\) be a feasible solution for (P). Suppose that there exist \(k_0 \in \mathbb{R}_+, (s, t^*, \bar{y}) \in K(x_0), u, v \in \mathbb{R}^n\) and \(\mu^* \in \mathbb{R}_+^n\) satisfying (3)–(7). Assume that one of the following conditions holds:

(a) \(\varphi(.) = \sum_{i=1}^{s} t^*_i ((f \circ \bar{y}_i) + \langle ., Au \rangle) - k_0 (g \circ \bar{y}_i) - \langle ., B v \rangle)\) and \((\mu^*, h(.)\) are z-univex with respect to \(b_0, b_1, \phi_0, \phi_1, x_0\) and \(\eta\) with \(\phi(V) \geq 0 \Rightarrow V \geq 0\) and \(\phi_1(V) \geq V\);

(b) \(\varphi(.) = \sum_{i=1}^{s} t^*_i ((f \circ \bar{y}_i) + \langle ., Au \rangle) - k_0 (g \circ \bar{y}_i) - \langle ., B v \rangle)\) is pseudo z-univex with respect to \(b_0, \phi_0, x_0\) and \(\eta\) with \(V < 0 \Rightarrow \phi(V) < 0\) and \(\mu^*, h(.)\) is quasi z-univex with respect to \(b_1, \phi_1, x_1\) and \(\eta\) with \(V \leq 0 \Rightarrow \phi_1(V) \leq 0\);

(c) \(\varphi(.) = \sum_{i=1}^{s} t^*_i ((f \circ \bar{y}_i) + \langle ., Au \rangle) - k_0 (g \circ \bar{y}_i) - \langle ., B v \rangle)\) is quasi z-univex with respect to \(b_0, \phi_0, x_0\) and \(\eta\) and \(\mu^*, h(.)\) is strictly pseudo z-univex with respect to \(b_1, \phi_1, x_1\) and \(\eta\) with \(V \leq 0 \Rightarrow \phi_0(V) \leq 0\) and \(\phi_0(V) \geq 0 \Rightarrow V \geq 0\).

Then \(x^*\) is an optimal solution of (P).
From (2), (4), (6), (7) and (8), we get
\[
\varphi(x_1) = \sum_{i=1}^{s} t_i^s((f(x_1, \bar{y}_i) + \langle x_1, Au \rangle) - k_0(g(x_1, \bar{y}_i) - \langle x_1, Bv \rangle))
\]
\[
\leq \sum_{i=1}^{s} t_i^s((f(x_1, \bar{y}_i) + \langle x_1, Ax_1 \rangle^{1/2}) - k_0(g(x_1, \bar{y}_i) - \langle x_1, Bx_1 \rangle^{1/2})) < 0
\]
\[
= \sum_{i=1}^{s} t_i^s((f(x_0, \bar{y}_i) + \langle x_0, Ax_0 \rangle^{1/2}) - k_0(g(x_0, \bar{y}_i) - \langle x_0, Bx_0 \rangle^{1/2}))
\]
\[
= \sum_{i=1}^{s} t_i^s((f(x_0, \bar{y}_i) + \langle x_0, Au \rangle) - k_0(g(x_0, \bar{y}_i) - \langle x_0, Bv \rangle)) = \varphi(x_0).
\]  
(9)

That is, \( \varphi(x_1) < \varphi(x_0) \).

If condition (a) holds, then
\[
b_0(x_1, x_0)\phi_0 \left[ \sum_{i=1}^{s} t_i^s((f(x_1, \bar{y}_i) + \langle x_1, Au \rangle) - k_0(g(x_1, \bar{y}_i) - \langle x_1, Bv \rangle)) \right]
\]
\[
- \sum_{i=1}^{s} t_i^s((f(x_0, \bar{y}_i) + \langle x_0, Au \rangle) - k_0(g(x_0, \bar{y}_i) - \langle x_0, Bv \rangle)) \]
\[
\geq \langle z_0(x_1, x_0)\nabla \varphi(x_0), \eta(x_1, x_0) \rangle
\]
\[
= \langle z_0(x_1, x_0)\{-\nabla \mu^s(h(x_0))\}, \eta(x_1, x_0) \rangle
\]
\[
\geq - b_1(x_1, x_0)\phi_1[\langle \mu^s, h(x_1) \rangle - \langle \mu^s, h(x_0) \rangle] \quad \text{(by the \( z \)-univexity of \( \langle \mu^s, h(.) \rangle \))}
\]
\[
\geq [\langle \mu^s, h(x_0) \rangle - \langle \mu^s, h(x_1) \rangle] \quad \text{(by the positivity of \( b_1 \) and \( \phi_1(V) \geq V \))}
\]
\[
\geq 0 \quad \text{(by the feasibility and (5)).}
\]

Since \( \phi_0(V) \geq 0 \Rightarrow V \geq 0 \) and \( b_1 \geq 0 \), we get
\[
\varphi(x_1) \geq \varphi(x_0),
\]
which contradicts (9).

If condition (b) holds, by the positivity of \( b_0 \), \( V < 0 \Rightarrow \phi_0(V) < 0 \) and from the inequality (9), we get
\[
b_0(x_1, x_0)[\varphi(x_1) - \varphi(x_0)] < 0.
\]  
(10)

By the pseudo \( z \)-univexity of \( \varphi \), the above inequality gives
\[
\langle z_0(x_1, x_0)\nabla \varphi(x_0), \eta(x_1, x_0) \rangle < 0.
\]

By (10) and (3), we get
\[
\langle z_0(x_1, x_0)\{-\nabla \langle \mu^s, h(x_0) \rangle\}, \eta(x_1, x_0) \rangle < 0,
\]
by the positivity of \( z_0 \), we get
\[
\langle \nabla \langle \mu^s, h(x_0) \rangle, \eta(x_1, x_0) \rangle > 0.
\]  
(11)

Since \( x_1 \in \mathcal{S}_P \), \( \mu^s \in R^P_+ \), from (5), we get
\[
[\langle \mu^s, h(x_1) \rangle - \langle \mu^s, h(x_0) \rangle] \leq 0.
\]  
(12)

By the condition \( V \leq 0 \Rightarrow \phi_1(V) \leq 0 \) and the positivity of \( b_1 \), (12) gives
\[
b_1(x_1, x_0)\phi_1[\langle \mu^s, h(x_1) \rangle - \langle \mu^s, h(x_0) \rangle] \leq 0.
\]
By the quasi z-univexity of \( \sum_{j=1}^{P} \langle \mu^*, h(.) \rangle \) and the above inequality, we get
\[
\langle x_1(x_1, x_0) \nabla \langle \mu^*, h(x_0) \rangle, \eta(x_1, x_0) \rangle \leq 0.
\]

By the positivity of \( x_1 \), we get
\[
\langle \nabla \langle \mu^*, h(x_0) \rangle, \eta(x_1, x_0) \rangle \leq 0,
\]
which contradicts \( (11) \).

For condition (c) the proof is similar as the proof of condition (b). This completes the proof. \( \square \)

**Remark 2.** If we take \( \phi_0, \phi_1 \) as the identity maps, and \( b_0 = 1 = b_1 \) in the above Theorem 3.1, we get Theorem 3.1 in [9].

**4. First duality model**

In this section, we consider the following dual to (P):

\[
\begin{align*}
\text{(DI)} & \quad \max_{(s, t, \bar{y}) \in K} \sup_{(z, t, \bar{y}) \in H_1(s, t, \bar{y})} \sup_{k} \sum_{i=1}^{s} t_i \left\{ \frac{1}{2} k (\nabla g(z, y_i) + \langle v, Bv \rangle) + \nabla \langle \mu, h(z) \rangle \right\} = 0, \\
& \quad \text{s.t.} \quad \sum_{i=1}^{s} t_i \left\{ \frac{1}{2} k (\nabla g(z, y_i) + \langle v, Bv \rangle) + \nabla \langle \mu, h(z) \rangle \right\} = 0, \\
& \quad \sum_{i=1}^{s} t_i \left\{ \nabla g(z, y_i) + \langle z, Au \rangle - k (\nabla g(z, y_i) + \langle z, Bv \rangle) \right\} \geq 0, \\
& \quad \langle \mu, h(z) \rangle \geq 0, \\
& \quad \langle z, Az \rangle \leq 1, \quad \langle z, Bz \rangle \leq 1,
\end{align*}
\]

where \( H_1(s, t, \bar{y}) \) denotes the set of all triplets \( (z, \mu, v) \in R^a \times R^p_+ \times R_+ \) satisfying \( (13) \)--\( (15) \) and \( (s, t, \bar{y}) \in K(z) \).

For a triplet \( (s, t, \bar{y}) \in K \), if the set \( H_1(s, t, \bar{y}) \) is empty, then we define the supremum over it to be \( -\infty \). In this section we denote
\[
\psi(.) = \sum_{i=1}^{s} t_i \left\{ \nabla g(z, y_i) + \langle z, Au \rangle - k \langle v, Bv \rangle \right\}.
\]

**Theorem 4.1.** (Weak duality). Let \( x \in S_p \) be a feasible solution for (P) and let \( (z, \mu, u, v, s, t, \bar{y}) \) be a feasible solution for (DI). Assume that one of the following conditions holds:

(a) \( \psi(.) \) and \( \langle \mu, h(.) \rangle \) are z-univex with respect to \( b_0, b_1, \phi_0, \phi_1, x_0 \) and \( \eta \) with \( \phi_0(V) \geq 0 \Rightarrow V \geq 0 \) and \( \phi_1(V) \geq V \);
(b) \( \psi(.) \) is pseudo z-univex with respect to \( b_0, \phi_0, x_0 \) and \( \eta \) with \( V < 0 \Rightarrow \phi_0(V) < 0 \) and \( \langle \mu, h(.) \rangle \) is quasi z-univex with respect to \( b_1, \phi_1, x_1 \) and \( \eta \) with \( V \leq 0 \Rightarrow \phi_1(V) \leq 0 \);
(c) \( \psi(.) \) is quasi z-univex with respect to \( b_0, \phi_0, x_0 \) and \( \eta \) with \( V < 0 \Rightarrow \phi_0(V) < 0 \) and \( \langle \mu, h(.) \rangle \) is strictly pseudo z-univex with respect to \( b_1, \phi_1, x_1 \) and \( \eta \) with \( V \leq 0 \Rightarrow \phi_1(V) \leq 0 \).

Then
\[
\sup_{y \in Y} \frac{f(x, y) + \langle x, Ax \rangle^{1/2}}{g(x, y) - \langle x, Bx \rangle^{1/2}} \geq k.
\]

**Proof.** Suppose contrary to the result, that is
\[
\sup_{y \in Y} \frac{f(x, y) + \langle x, Ax \rangle^{1/2}}{g(x, y) - \langle x, Bx \rangle^{1/2}} < k.
\]
Then we get
\[
f(x_1, \bar{y}_i) + \langle x_1, A x_1 \rangle^{1/2} - k_0(g(x_1, \bar{y}_i) - \langle x_1, B x_1 \rangle^{1/2}) < 0 \quad \text{for all } y \in Y.
\]
That is,
\[
t_i[f(x_1, \bar{y}_i) + \langle x_1, A x_1 \rangle^{1/2} - k_0(g(x_1, \bar{y}_i) - \langle x_1, B x_1 \rangle^{1/2})] \leq 0, \quad i = 1, 2, \ldots, s.
\]
From (2), (14) and (16) and the above inequality, we get
\[
\sum_{i=1}^{s} t_i [(f(x, y_i) + \langle x, A u \rangle) - k_0(g(x, y_i) - \langle x, B v \rangle)] 
\leq \sum_{i=1}^{s} t_i [(f(x, y_i) + \langle x, A x \rangle^{1/2} - k_0(g(x, y_i) - \langle x, B x \rangle^{1/2})] < 0
\]
\[
\leq \sum_{i=1}^{s} t_i [(f(z, y_i) + \langle z, A u \rangle) - k_0(g(z, y_i) - \langle z, B v \rangle)] = \psi(z).
\]
That is,
\[
\psi(x) < \psi(z).
\]
If condition (a) holds, then
\[
\phi_0(x, z) \phi_0 \left[ \sum_{i=1}^{s} t_i^a [(f(x, y_i) + \langle x, A u \rangle) - k_0(g(x, y_i) - \langle x, B v \rangle)]
\right.
\]
\[
- \sum_{i=1}^{s} t_i^a [(f(z, y_i) + \langle z, A u \rangle) - k_0(g(z, y_i) - \langle z, B v \rangle)]
\]
\[
\geq \langle z_0(x, z) \nabla \psi(z), \eta(x, z) \rangle
\]
\[
= \langle z_0(x, z) [\nabla \langle \mu, h(z) \rangle], \eta(x, z) \rangle
\]
\[
\geq - b_1(x, z) [\mu(h(z)) - \langle \mu, h(z) \rangle] \quad \text{(by the \textit{z}-univexity of } \langle \mu, h(.), \rangle) \]
\[
\geq [\langle \mu, h(z) \rangle - \langle \mu, h(x) \rangle] \quad \text{(by the positivity of } b_1 \text{ and } \phi_1(V) \geq V) \]
\[
\geq 0 \quad \text{(by the feasibility and (15)).}
\]
Since \( \phi_0(V) \geq 0 \Rightarrow V \geq 0 \) and \( b_0 \geq 0 \), we get
\[
\psi(x) \geq \psi(z)
\]
which contradicts (17).
If condition (b) holds, by the positivity of \( b_0, V < 0 \Rightarrow \phi_0(V) < 0 \) and from the inequality (17), we get
\[
b_0(x, z) \phi_0[\psi(x) - \psi(z)] < 0.
\]
By the pseudo \textit{z}-univexity of \( \psi \), the above inequality gives
\[
\langle z_0(x, z) \nabla \psi(z), \eta(x, z) \rangle < 0.
\]
By (18) and (13), we get
\[
\langle z_0(x, z) [\nabla \langle \mu, h(z) \rangle], \eta(x, z) \rangle < 0,
\]
by the positivity of \( z_0 \), we get
\[
\langle \nabla \langle \mu, h(z) \rangle, \eta(x, z) \rangle > 0.
\]
Theorem 4.2. The optimality of this feasible solution for (DI) follows from Theorem 4.1. The remaining part of the proof is similar to one of the Theorem 4.1 by replacing $x$ by $x^*$.

Proof. By Lemma 1, there exist $v, \bar{v}, x, \bar{x}, s, \bar{s}, t, \bar{t}, \tilde{y}$ with $V < f(u, v)$, and from the above inequality, we get

$$\langle z_1(x, \bar{z}) \rangle \leq 0.$$  

By the positivity of $z_1$, we get

$$\langle \nabla (\mu, h(z)), \eta(x, \bar{z}) \rangle \leq 0,$$

which contradicts (19).

For condition (c) the proof is similar to that of the proof given above for condition (b). □

Theorem 4.3. (Strict converse duality). Let $x^*$ and $(\bar{z}, \bar{k}, \bar{u}, \bar{v}, \bar{s}, \bar{t}, \bar{y})$ be optimal for (P) and (DI), respectively. Assume that the hypothesis of Theorem 4.2 is fulfilled. Assume that any one of the following conditions holds:

(a) $\sum_{i=1}^s \tilde{i}_i ((f \cdot \bar{y}_i) + (\cdot, A\tilde{u})) - k_0 (g \cdot \bar{y}_i) - (\cdot, B\bar{v}))$ and $(\bar{y}, h(\cdot))$ are strictly $\bar{u}$-uniex with respect to $b_0, b_1, \phi_0, \phi_1, 0$ and $\eta$ with $\phi(V) \geq 0 \Rightarrow V \geq 0$ and $\phi(V) \geq V$;

(b) $\sum_{i=1}^s \tilde{i}_i ((f \cdot \bar{y}_i) + (\cdot, A\tilde{u})) - k_0 (g \cdot \bar{y}_i) - (\cdot, B\bar{v}))$ is strictly pseudo $\bar{u}$-uniex with respect to $b_0, \phi_0, 0$ and $\eta$ with $V < 0 \Rightarrow \phi(V) < 0$ and $(\bar{y}, h(\cdot))$ is quasi $\bar{u}$-uniex with respect to $b_1, \phi, 0$ and $\eta$ with $V \leq 0 \Rightarrow \phi(V) \leq 0$.

Then $x^* = \bar{z}$; that is, $\bar{z}$ is an optimal solution for (P) and

$$\sup_{y \in Y} \frac{f(\bar{z}, y) + (\bar{z}, A\bar{z})^{1/2}}{g(\bar{z}, y) - (\bar{z}, B\bar{z})^{1/2}} = \bar{k}.$$

Proof. Suppose on the contrary that $x^* \neq \bar{z}$. From Theorem 4.2, we know that there exist $(s^*, t^*, \bar{y}^*) \in K(x^*)$ and $(x^*, \mu^*, k^*, u^*, v^*) \in H_1(s^*, t^*, \bar{y}^*)$ such that $(x^*, \mu^*, k^*, u^*, v^*, s^*, t^*, \bar{y}^*)$ is optimal for (DI) with the optimal value

$$\sup_{y \in Y} \frac{f(x^*, y) + (x^*, A\bar{x})^{1/2}}{g(x^*, y) - (x^*, B\bar{x})^{1/2}} = k^*.$$

The remaining part of the proof is similar to one of the Theorem 4.1 by replacing $x$ by $x^*$ and $(x, \mu, k, u, v, s, t, y)$ by $(\tilde{x}, \tilde{\mu}, \tilde{k}, \tilde{u}, \tilde{v}, \tilde{s}, \tilde{t}, \tilde{y})$, we get

$$\sup_{y \in Y} \frac{f(x^*, y) + (x^*, A\bar{x})^{1/2}}{g(x^*, y) - (x^*, B\bar{x})^{1/2}} > \bar{k}.$$
The above inequality contradicts
\[ \sup_{y \in Y} f(x^*, y) + \langle x^*, Ax^* \rangle^{1/2} = k = \tilde{k}. \]

Therefore, we conclude that \( x^* = \tilde{z} \). Hence, the proof is completed. \( \square \)

**Remark 3.** If we take \( \phi_0, \phi_1 \) as the identity maps, and \( b_0 = 1 = b_1 \) in the above Theorems 4.1–4.3 we get Theorems 4.1–4.3 in [9].

### 5. Second duality model

In this section, we formulate the Wolfe-type dual model to problem (P) as follows:

\[
\begin{align*}
\text{(DII)} \quad \max_{(s, t, y) \in K(z)} & \quad \sup_{(z, u, v, s, t, \tilde{y}) \in H_2(s, t, \tilde{y})} F(z) \\
\text{s.t.} & \quad \sum_{i=1}^s t_i \{ (g(z, \tilde{y}_i) - \langle z, Bz \rangle^{1/2} (\nabla f(z, \tilde{y}_i) + Au) - (f(z, \tilde{y}_i) + \langle z, Az \rangle^{1/2} (\nabla g(z, \tilde{y}_i) - Bv) \} + \langle \mu, h(z) \rangle = 0, \\
& \quad \langle \mu, h(z) \rangle \geq 0, \\
& \quad \langle z, Az \rangle \leq 1, \quad \langle z, Bz \rangle \leq 1, \\
& \quad \langle z, Bz \rangle^{1/2} = \langle z, Au \rangle, \quad \langle z, Bz \rangle^{1/2} = \langle z, Bv \rangle.
\end{align*}
\]

where
\[
F(z) = \sup_{y \in Y} \frac{f(z, y) + \langle z, Az \rangle^{1/2}}{g(z, y) - \langle z, Bz \rangle^{1/2}}.
\]

\( y_i \in Y(z) \) and \( H_2(s, t, \tilde{y}) \) denotes the set of \( (z, \mu, u, v) \in \mathbb{R}^n \times \mathbb{R}^p_+ \times \mathbb{R}^n \times \mathbb{R}^n \) satisfying (34)–(36). If the set \( H_2(s, t, \tilde{y}) \) is empty, then we define the supremum over it to be \( -\infty \). In this section, we denote

\[
\psi_1(.) = \sum_{i=1}^s t_i \{ (g(z, \tilde{y}_i) - \langle z, Bv \rangle)(f(., \tilde{y}_i) + \langle ., Au \rangle) - (f(z, \tilde{y}_i) + \langle z, Au \rangle)(g(., \tilde{y}_i) - \langle ., Bv \rangle) \}.
\]

Now we establish the following duality theorems between (P) and (DII).

**Theorem 5.1.** (Weak duality). Let \( x \in \mathfrak{X}_p \) be a feasible solution for (P) and let \( (z, \mu, u, v, s, t, \tilde{y}) \) be a feasible solution for (DII). Assume that one of the following conditions holds:

(a) \( \psi_1(.) \) and \( \langle \mu, h(.) \rangle \) are \( \alpha \)-univex with respect to \( b_0, b_1, \phi_0, \phi_1, z_0 \) and \( \eta \) with \( \phi_0(V) \geq 0 \Rightarrow V \geq 0 \) and \( \phi_1(V) \geq V \);

(b) \( \psi_1(.) \) is pseudo \( \alpha \)-univex with respect to \( b_0, \phi_0, z_0 \) and \( \eta \) with \( V < 0 \Rightarrow \phi_0(V) < 0 \) and \( \langle \mu, h(.) \rangle \) is quasi \( \alpha \)-univex with respect to \( b_1, \phi_1, z_1 \) and \( \eta \) with \( V \leq 0 \Rightarrow \phi_1(V) \leq 0 \);

(c) \( \psi_1(.) \) is quasi \( \alpha \)-univex with respect to \( b_0, \phi_0, z_0 \) and \( \eta \) with \( V < 0 \Rightarrow \phi_0(V) < 0 \) and \( \langle \mu, h(.) \rangle \) is strictly pseudo \( \alpha \)-univex with respect to \( b_1, \phi_1, z_1 \) and \( \eta \) with \( V \leq 0 \Rightarrow \phi_1(V) \leq 0 \). Then

\[
\sup_{y \in Y} \frac{f(x, y) + \langle x, Ax \rangle^{1/2}}{g(x, y) - \langle x, Bx \rangle^{1/2}} \geq F(z).
\]

**Proof.** Suppose contrary to the result that for each \( x \in \mathfrak{X}_p \),

\[
\sup_{y \in Y} \frac{f(x, y) + \langle x, Ax \rangle^{1/2}}{g(x, y) - \langle x, Bx \rangle^{1/2}} < F(z).
\]
Since \( \bar{y}_i \in \bar{Y}(z) \), \( i = 1, 2, \ldots, s \), we have

\[
F(z) = \frac{f(z, \bar{y}_i) + \langle z, A\bar{y}_i \rangle^{1/2}}{g(z, \bar{y}_i) - \langle z, B\bar{y}_i \rangle^{1/2}}, \quad i = 1, 2, \ldots, s.
\]  

(24)

Following as in [6], we get

\[
\psi_1(x) < \psi_1(z).
\]  

(25)

Now if condition (a) holds, then

\[
b_0(x, z)\phi_0[\psi_1(x) - \psi_1(z)]
\geq \langle x_0(x, z)\nabla \psi_1(z), \eta(x, z) \rangle
\geq -b_1(x, z)\phi_1[\langle \mu, h(x) \rangle - \langle \mu, h(z) \rangle] \quad \text{(by the } \alpha\text{-univexity of } \langle \mu, h(.) \rangle)
\geq [\langle \mu, h(z) \rangle - \langle \mu, h(x) \rangle] \quad \text{(by the positivity of } b_1 \text{ and } \phi_1(V) \geq V)
\geq 0 \quad \text{(by the feasibility and (21)).}
\]

Since \( \phi_0(V) \geq 0 \Rightarrow V \geq 0 \) and \( b_0 \geq 0 \), we get

\[
\psi_1(x) \geq \psi_1(z),
\]

which contradicts (25).

If condition (b) holds, by the positivity of \( b_0 \), \( V < 0 \Rightarrow \phi_0(V) < 0 \) and from the inequality (25), we get

\[
b_0(x, z)\phi_0[\psi_1(x) - \psi_1(z)] < 0.
\]

By the pseudo \( \alpha \)-univexity of \( \psi_1 \), the above inequality gives

\[
\langle x_0(x, z)\nabla \psi_1(z), \eta(x, z) \rangle < 0.
\]  

(26)

By (26) and (20), we get

\[
\langle x_0(x, z)\{\nabla \langle \mu, h(z) \rangle}, \eta(x, z) \rangle < 0,
\]

by the positivity of \( x_0 \), we get

\[
-\langle \nabla \langle \mu, h(z) \rangle, \eta(x, z) \rangle < 0.
\]

i.e.

\[
\langle \nabla \langle \mu, h(z) \rangle, \eta(x, z) \rangle > 0.
\]  

(27)

Since \( x \in \mathcal{I}_P, \mu \in R^p_+ \), from (21), we get

\[
[\langle \mu, h(x) \rangle - \langle \mu, h(z) \rangle] \leq 0.
\]  

(28)

By the condition \( V \leq 0 \Rightarrow \phi_1(V) \leq 0 \) and the positivity of \( b_1 \), (28) gives

\[
b_1(x, z)\phi_1[\langle \mu, h(x) \rangle - \langle \mu, h(z) \rangle] \leq 0.
\]

By the quasi \( \alpha \)-univexity of \( \langle \mu, h(.) \rangle \) and from the above inequality, we get

\[
\langle x_1(x, z)\nabla \langle \mu, h(z) \rangle, \eta(x, z) \rangle \leq 0.
\]

By the positivity of \( x_1 \), we get

\[
\langle \nabla \langle \mu, h(z) \rangle, \eta(x, z) \rangle \leq 0,
\]

which contradicts (27).
Theorem 5.2. (Strong duality). Assume that \(x^*\) is an optimal solution for \((P)\) and \(x^*\) satisfies a constraints qualification for \((P)\). Then there exist \((s^*, t^*, \tilde{y}^*) \in K(x^*)\) and \((x^*, \mu^*, k^*, u^*, v^*) \in H_2(s^*, t^*, \tilde{y}^*)\) such that \((x^*, \mu^*, k^*, u^*, v^*, s^*, t^*, \tilde{y}^*)\) is feasible for \((DII)\). If any of the conditions of Theorem 5.1 holds, then \((x^*, \mu^*, k^*, u^*, v^*, s^*, t^*, \tilde{y}^*)\) is an optimal solution for \((DII)\), and problem \((P)\) and \((DII)\) have the same optimal value.

Proof. By Lemma 1, there exist \((s^*, t^*, \tilde{y}^*) \in K(x^*)\) and \((x^*, \mu^*, k^*, u^*, v^*) \in H_2(s^*, t^*, \tilde{y}^*)\) such that \((x^*, \mu^*, k^*, u^*, v^*, s^*, t^*, \tilde{y}^*)\) is feasible for \((DII)\), and

\[
k_0 = \frac{f(x^*, \tilde{y}^*) + (x^*, Ax^*)^{1/2}}{g(x^*, \tilde{y}^*) - (x^*, Bx^*)^{1/2}}.
\]

The optimality of this feasible solution for \((DII)\) follows from Theorem 5.1. □

Theorem 5.3. (Strict converse duality). Let \(x^*\) and \((z, \mu, u, v, s, t, \tilde{y})\) be optimal for \((P)\) and \((DII)\), respectively. Assume that the hypothesis of Theorem 5.2 is fulfilled. Assume that any one of the following conditions holds:

(a) \(\psi_1(.)\) is strictly \(z\)-univex with respect to \(b_0, \phi_0, \varphi_0\) and \(\eta\) and \(\langle \mu, h(.) \rangle\) is \(z\)-univex with respect to \(b_1, \phi_1, \varphi_1\) and \(\eta\) with \(V \leq 0 \Rightarrow \phi_0(V) \leq 0\) and \(V \leq 0 \Rightarrow \phi_1(V) \leq 0\);

(b) \(\psi_1(.)\) is strictly pseudo \(z\)-univex with respect to \(b_0, \phi_0, \varphi_0\) and \(\eta\) with \(V \leq 0 \Rightarrow \phi_0(V) \leq 0\) and \(\langle \mu, h(.) \rangle\) is quasi \(z\)-univex with respect to \(b_1, \phi_1, \varphi_1\) and \(\eta\) with \(V \leq 0 \Rightarrow \phi_1(V) \leq 0\).

Then \(x^* = z\); that is, \(z\) is an optimal solution for \((P)\).

Proof. Suppose on the contrary that \(x^* \neq z\). As in Theorem 5.1, we get

\[
\sup_{y \in Y} \frac{f(x^*, y) + (x^*, Ax^*)^{1/2}}{g(x^*, y) - (x^*, Bx^*)^{1/2}} \leq F(z).
\] (29)

Following as in [6], we get

\[
\psi_1(x^*) \leq \psi_1(z).
\] (30)

If condition (a) holds, from (30), we get

\[
b_0(x^*, z) \phi_0[\psi_1(x^*) - \psi_1(z)] \leq 0.
\]

By the strict \(z\)-univexity of \(\psi_1(.)\) and from the above inequality, we get

\[
\langle \varphi_0(x^*, z) \nabla \psi_1(z), \eta(x^*, z) \rangle < 0.
\] (31)

Now from (31) and (20), we get

\[
\langle \varphi_0(x^*, z) [-\nabla \langle \mu, h(z) \rangle], \eta(x^*, z) \rangle < 0.
\]

By the positivity of \(\varphi_0\), we get

\[
-\nabla \langle \mu, h(z) \rangle, \eta(x^*, z) \rangle < 0.
\]

i.e.,

\[
\langle \nabla \langle \mu, h(z) \rangle, \eta(x^*, z) \rangle > 0.
\] (32)

Since \(x^* \in \mathcal{F}_P, \mu \in \mathcal{R}_P^0\), from (21), we get

\[
[\langle \mu, h(x^*) \rangle - \langle \mu, h(z) \rangle] \leq 0.
\] (33)
By the condition $V \leq 0 \Rightarrow \phi_1(V) \leq 0$ and the positivity of $b_1$, (33) gives
\[ b_1(x^*, z)\phi_1(\mu, h(x^*)) - \langle \mu, h(z) \rangle \leq 0. \]

By the $z$-univexity of $\langle \mu, h(.) \rangle$, from the above inequality, we get
\[ \langle z_1(x^*, z)\nabla \langle \mu, h(z) \rangle, \eta(x^*, z) \rangle \leq 0. \]

By the positivity of $z_1$, we get
\[ \langle \nabla \langle \mu, h(z) \rangle, \eta(x^*, z) \rangle \leq 0, \]
which is a contradiction to (32). Hence (29) is false, so we have
\[ \sup_{y \in Y} f(x^*, y) + \langle x^*, Ax^* \rangle^{1/2} > F(z). \quad (34) \]

Since $x^*$ is an optimal solution for (P), from Theorem 5.2 there exist $(s^*, t^*, \bar{y}^*) \in K(x^*)$ and $(x^*, \mu^*, u^*, v^*) \in H_2(s^*, t^*, \bar{y}^*)$ such that $(x^*, \mu^*, u^*, v^*, s^*, t^*, \bar{y}^*)$ is an optimal solution for (DII) with the optimal value
\[ \sup_{y \in Y} f(x^*, y) + \langle x^*, Ax^* \rangle^{1/2} = F(x^*) = F(z), \]
which contradicts (34). Hence $x^* = z$; that is, $z$ is an optimal solution for (P).

Remark 4. If we take $\phi_0, \phi_1$ as the identity maps, and $b_0 = 1 = b_1$ in the above Theorem 5.1–5.3 we get Theorem 5.1–5.3 in [9].

6. Third duality model

In this section we take the following form of Lemma 1:

Lemma 2. Let $x^*$ be an optimal solution for (P). Assume that $\nabla g_j(x^*), j \in J(x^*)$ are linearly independent. Then there exist $(s, t^*, \bar{y}^*) \in K$ and $\mu^* \in R_+^p$ such that
\[ \nabla \left( \sum_{i=1}^{s^*} t_i^*(f(x^*, \bar{y}_i) + \langle x^*, Au_i \rangle + \langle \mu^*, h(x^*) \rangle) \right) = 0, \]
\[ \langle \mu^*, h(x^*) \rangle = 0, \]
\[ \langle u, Au \rangle \leq 1, \quad \langle v, Bv \rangle \leq 1, \]
\[ \langle x^*, Ax^* \rangle^{1/2} = \langle x^*, Au \rangle, \quad \langle x^*, Bx^* \rangle^{1/2} = \langle x^*, Bv \rangle, \]
\[ \mu^* \in R_+^p, \quad t_i^* \geq 0, \quad \sum_{i=1}^{s} t_i^* = 1, \quad y_i \in Y(x^*), \quad i = 1, 2, \ldots, s^*. \]
In this section, we consider the following parameter-free dual problem for (P):

\[
\text{max}_{(s,t,\bar{y}) \in K(t)} \quad \sup_{(z,\mu,u,v) \in H_3(s,t,\bar{y})} \left( \frac{\sum_{i=1}^{s} t_i^*(f(z, \bar{y}_i) + \langle z, Au \rangle + \langle \mu, h(z) \rangle)}{\sum_{i=1}^{s} t_i^*(g(z, \bar{y}_i) - \langle z, Bv \rangle)} \right)
\]

s.t. \( \nabla \left( \frac{\sum_{i=1}^{s} t_i^*(f(z, \bar{y}_i) + \langle z, Au \rangle + \langle \mu, h(z) \rangle)}{\sum_{i=1}^{s} t_i^*(g(z, \bar{y}_i) - \langle z, Bv \rangle)} \right) = 0, \) \hspace{1cm} (39)

\( \langle \mu, Au \rangle \leq 1, \quad \langle v, Bv \rangle \leq 1, \)

\( \langle z, Az \rangle^{1/2} = \langle z, Au \rangle, \quad \langle z, Bz \rangle^{1/2} = \langle z, Bv \rangle. \)

where \( H_3(s,t,\bar{y}) \) denotes the set of \((z,\mu,u,v) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^p \) satisfying (39). if the set \( H_3(s,t,\bar{y}) \) is empty, then we define the supremum over it to be \((-\infty)\). Throughout this section for the sake of simplicity, we denote by \( \psi_2(\cdot) \)

\[
[t_i^*(g(z, \bar{y}_i) - \langle z, Bv \rangle)] \left[ \sum_{i=1}^{s} t_i f(\cdot, y_i) + \sum_{j=1}^{p} \mu_j g_j(\cdot) \right] - \left[ \sum_{i=1}^{s} t_i^*(f(z, \bar{y}_i) + \langle z, Au \rangle + \langle \mu, h(z) \rangle \right] \left[ t_i^*(g(\cdot, \bar{y}_i) - \langle \cdot, Bv \rangle) \right].
\]

Now we shall state weak, strong and converse duality theorems without proof as they can be proved in light of Theorems 5.1 and 5.2, proved in the previous section.

**Theorem 6.1. (Weak duality).** Let \( x \in \mathcal{X}_P \) be a feasible solution for (P) and let \((z, \mu, u, v, s, t, \bar{y}) \) be a feasible solution for (DIII). If \( \psi_2(\cdot) \) is pseudo \( \eta \)-univex with respect to \( b_0, \phi_0, \alpha_0 \) and \( \eta \) with \( V \leq 0 \Rightarrow \phi_0(V) \leq 0 \), then

\[
\sup_{y \in Y} \frac{f(x,\cdot) + \langle x, Ax \rangle^{1/2}}{g(x,\cdot) - \langle x, Bx \rangle^{1/2}} \geq \left( \frac{\sum_{i=1}^{s} t_i (f(z, \bar{y}_i) + \langle z, Au \rangle + \langle \mu, h(z) \rangle)}{\sum_{i=1}^{s} t_i (g(z, \bar{y}_i) - \langle z, Bv \rangle)} \right).
\]

**Theorem 6.2. (Strong duality).** Assume that \( x^* \) is an optimal solution for (P) satisfying the hypothesis of Theorem 6.1. Then there exist \((s^*, t^*, \bar{y}^*) \in K(x^*) \) and \((x^*, \mu^*, u^*, v^*) \in H_3(s^*, t^*, \bar{y}^*) \) such that \((x^*, \mu^*, u^*, v^*, s^*, t^*, \bar{y}^*) \) is feasible for (DIII). If any of the conditions of Theorem 6.1 holds, then \((x^*, \mu^*, u^*, v^*, s^*, t^*, \bar{y}^*) \) is an optimal solution for (DIII) and problem (P) and (DIII) have the same optimal value.

**Theorem 6.3. (Strict converse duality).** Let \( x^* \) be an optimal solution for (P) and \((z, \mu, u, v, s, t, \bar{y}) \) be an optimal solution for (DIII). Assume that the hypothesis of Theorem 6.2 is fulfilled and \( \psi_2(\cdot) \) is strictly pseudo \( \eta \)-univex with respect to \( b_0, \phi_0, \alpha_0 \) and \( \eta \) with \( V \leq 0 \Rightarrow \phi_0(V) \leq 0 \). Then \( z = x^* \) is an optimal solution of (P).

**Remark 5.** If we take \( \phi_0, \phi_1 \) as the identity maps, and \( b_0 = 1 = b_1 \) in the above Theorems 6.1–6.3 we get Theorems 6.1–6.3 in [9].

**References**


Further Reading