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The facets and the symmetries of the approval-voting polytope

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Abstract

Characterizing the size-independent model for approval voting of Falmagne and Regenwetter (J. Math. Psychol. 40 (1996) 152) was shown by Doignon and Regenwetter (J. Math. Psychol. 41 (1997) 171) to be equivalent to determining all facets of the approval-voting polytope. Here, we prove that the facets of this polytope correspond in a natural way to certain antichains in a power set. Several results on the approval-voting polytope are then derived. For instance, all facet-defining inequalities are characterized, and the group of automorphisms is completely described. On the other hand, providing an explicit listing of all of the facets is shown to be at least as intricate as listing all connected graphs on a given, finite set.

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1. Introduction

Throughout the text, S is a finite set of cardinality n, with $n \geqslant 2$. We denote by $\mathcal{P}(S)$ the power set of S, and by $\mathcal{P}(S,k)$ the collection of all k-sets contained in S, for $0 \leqslant k \leqslant n$. In $\mathcal{P}(S)$ partially ordered by inclusion, a *chain* (resp. *antichain*) $\mathcal{B}$ is a collection $\mathcal{B}$ of subsets of S such that for every pair of distinct sets in $\mathcal{B}$, one (resp. none) is included in the other. A chain $\mathcal{C}$ is *complete* if it meets $\mathcal{P}(S,k)$ for all k with $0 \leqslant k \leqslant n$. A *ranking* of S is a bijection from $\{1,2,\dots,n\}$ to S. Each ranking ρ determines a complete chain defined by

$$\mathcal{C} = \{\emptyset, \{\rho(1)\}, \{\rho(1), \rho(2)\}, \dots, \{\rho(1), \rho(2), \dots, \rho(n)\}\}.$$

This yields a one-to-one correspondence between the rankings of S and the complete chains in $\mathcal{P}(S)$. Consequently, there are $n!$ complete chains in $\mathcal{P}(S)$.

Let $\mathcal{I}$ be any finite set. The real vector space $\mathbb{R}^{\mathcal{I}}$ has one coordinate per element of $\mathcal{I}$. We denote, the coordinate corresponding to $i \in \mathcal{I}$ by x_i. More generally, for each subset $\mathcal{J}$ of $\mathcal{I}$, we define $x(\mathcal{J})$ as the formal sum of the coordinates x_j in $\mathbb{R}^{\mathcal{I}}$ with $j \in \mathcal{J}$, that is, we let $x(\mathcal{J}) = \sum_{j \in \mathcal{J}} x_j$. The *characteristic vector* of a subset $\mathcal{J}$ of $\mathcal{I}$ is the vector $v_{\mathcal{J}}$ in $\mathbb{R}^{\mathcal{I}}$ whose i-coordinate equals 1 if $i \in \mathcal{J}$ and 0 if $i \notin \mathcal{J}$.

Assume $\mathcal{I} = \mathcal{P}(S)$ in this paragraph. To any collection $\mathcal{B}$ of subsets of S, we associate as above its characteristic vector $v_{\mathcal{B}}$ in $\mathbb{R}^{\mathcal{P}(S)}$. The *approval-voting polytope* P^n_{AV} is the convex hull of the characteristic vectors of all complete chains in $\mathcal{P}(S)$, that is,

$$P^n_{\mathrm{AV}} = \mathrm{conv}\{v_{\mathcal{C}} \in \mathbb{R}^{\mathcal{P}(S)} \mid \mathcal{C} \text{ is a complete chain in } \mathcal{P}(S)\}.$$

Obviously, this 0/1-polytope has $n!$ vertices. It was introduced by Doignon and Regenwetter [8], with vertices corresponding to rankings of the set S. The motivation is as follows: Falmagne and Regenwetter [10] define a size-independent model for approval voting and raise the question of characterizing all probability distributions generated by this model. Later, Doignon and Regenwetter [8] show that this question is equivalent to the problem of finding a linear description of the approval-voting polytope P^n_{AV}, that is, a description by linear equations and inequalities. In geometric terms, such a description with a minimum number of (in)equalities amounts to a description of the affine hull together with an enumeration of all the facets of this polytope. Doignon and Regenwetter [8] prove that the dimension of P^n_{AV} is $2^n - n - 1$, and describe several families of facets. For $n \leqslant 5$, they find using the `porta` software that their list contains all facets. Here, for any natural number n at least 2, we put the facets of P^n_{AV} in a one-to-one correspondence with certain antichains of $\mathcal{P}(S)$. Two classical results will be the main ingredients of our proof: first, a polyhedral characterization of perfect graphs, due to Fulkerson [11–13], Lovász [14] and Chvátal [3] and second, Dilworth's Theorem [6] on chain coverings of posets. As we will see, the combination of these two results directly yields a minimum, linear characterization of a similar polytope, whose vertices are all characteristic vectors of chains (complete or not) of $\mathcal{P}(S)$. Because P^n_{AV} is a face of this larger polytope, its facets can be identified through a careful analysis of some antichains of $\mathcal{P}(S)$. Several problems about P^n_{AV} are then easily solved. For instance, we describe here the group of automorphisms. Further results on facet-producing antichains are collected elsewhere [7].

As regards terminology, we generally follow Bollobás [1] for graphs, Trotter [18] for posets, and Ziegler [20] for convex polytopes.

2. A complete linear description

The following result, of central interest to us, can also be found in [4,17]:

Theorem 1 (Fulkerson [11–13], Lovász [14], Chvátal [3]). *For a finite graph G, the following two conditions are equivalent*:

(i) *G is perfect*;
(ii) *the polytope*

$$P(G) = \{x \in \mathbb{R}^V \mid 0 \leqslant x \text{ and } x(K) \leqslant 1 \text{ for all complete subgraphs } K \text{ of } G\}$$

has only integral vertices.

The next proposition is a well-known reformulation of Dilworth's chain covering theorem [6].

Theorem 2. *The incomparability graph of any finite poset is perfect.*

Let G be the incomparability graph of $\mathcal{P}(S)$, and let $P = P(G)$. Because the complete subgraphs in G coincide with the antichains in $\mathcal{P}(S)$, we have

$$P = \{x \in \mathbb{R}^{\mathcal{P}(S)} \mid 0 \leqslant x \text{ and } x(\mathcal{A}) \leqslant 1 \text{ for all antichains } \mathcal{A} \text{ in } \mathcal{P}(S)\}.$$

Theorems 1 and 2 imply that P is a 0/1-polytope. Its vertices are exactly the characteristic vectors of chains in $\mathcal{P}(S)$. As a consequence, we derive in this way a complete linear description of what should be called the *approval-voting polytope for weak orders* on S, where a *weak order* on S is any complete and transitive relation on the finite set S (with 'ties' allowed). Moreover, the facets of this polytope correspond to maximal antichains in $\mathcal{P}(S)$: this is a particular case of a general result of Padberg [16], which implies that the facets of the polytope $P(G)$ in Theorem 1 are defined by the inequalities $0 \leqslant x_u$ for each node u of the graph G, and $x(K) \leqslant 1$ for all maximal cliques K of G.

Now, consider the face of P which is defined by the valid inequality $x(\mathcal{P}(S)) \leqslant n+1$. By definition, this face is a 0/1-polytope whose vertices are exactly the characteristic vectors of complete chains in $\mathcal{P}(S)$, hence it is equal to P^n_{AV}. This is rephrased in the next theorem.

Proposition 1. *For all $n \geqslant 2$, the approval-voting polytope P^n_{AV} equals the set of points in $\mathbb{R}^{\mathcal{P}(S)}$ that satisfy*

$$x(\mathcal{P}(S,k)) = 1 \quad \text{for } 0 \leqslant k \leqslant n, \tag{1}$$

$$x(\mathcal{A}) \leqslant 1 \quad \text{for all antichains } \mathcal{A} \text{ in } \mathcal{P}(S). \tag{2}$$

We refer to Eq. (1) as *layer equations* and to inequalities (2) as *antichain inequalities*.

Remark 1. The nonnegativity inequality $0 \leqslant x_X$ is implied by the antichain inequality $x(\mathcal{A}) \leqslant 1$ for $\mathcal{A} = \mathcal{P}(S, |X|) \backslash \{X\}$ and by the layer equation $x(\mathcal{P}(S, |X|)) = 1$.

Remark 2. The layer equations are readily seen to be independent. Furthermore, they form a *complete* system of equations for P^n_{AV} in the sense that every linear equation that is satisfied by all points of P^n_{AV} is implied by the system (cf. [8]).

3. The facets

When $\mathcal{A}$ is an antichain in $\mathcal{P}(S)$, we denote by $F(\mathcal{A})$ the face of the approval-voting polytope P^n_{AV} defined by the antichain inequality $x(\mathcal{A}) \leqslant 1$. Proposition 1 implies that for each facet F of P^n_{AV}, there exists an antichain $\mathcal{A}$ in $\mathcal{P}(S)$ such that $F = F(\mathcal{A})$. In this section, we characterize the antichains whose corresponding face is a facet. Notice that the vertex $v_{\mathcal{C}}$ of P^n_{AV} lies in the face $F(\mathcal{A})$ if and only if the complete chain $\mathcal{C}$ meets the antichain $\mathcal{A}$.

For $0 \leqslant k \leqslant n$, we refer to the antichain $\mathcal{P}(S,k)$ as the *kth layer*, and if moreover $0 < k < n$, as a *nontrivial layer*. The *Johnson graph* $\mathrm{J}(S,k)$ has $\mathcal{P}(S,k)$ as its set of nodes, with adjacency of two distinct nodes A, B being defined by any of the following four equivalent conditions:

$$\begin{aligned} A \sim B \quad &\Leftrightarrow \quad \exists X \in \mathcal{P}(S, k+1) : A \subset X \text{ and } B \subset X \\ &\Leftrightarrow \quad \exists Y \in \mathcal{P}(S, k-1) : Y \subset A \text{ and } Y \subset B \\ &\Leftrightarrow \quad |A \cup B| = k+1 \\ &\Leftrightarrow \quad |A \cap B| = k-1. \end{aligned}$$

It is easy to prove that $\mathrm{J}(S,k)$ is a connected graph which remains connected after deletion of any vertex (Lemma 3 provides a stronger property of the graph).

The next lemma proves that the antichains $\mathcal{A}$ such that $F(\mathcal{A})$ is the whole approval-voting polytope P^n_{AV} are exactly the layers.

Lemma 1. *Let $\mathcal{A}$ be an antichain in $\mathcal{P}(S)$. Then $\mathcal{A}$ meets every complete chain in $\mathcal{P}(S)$ if and only if $\mathcal{A}$ is a layer.*

Proof. By definition, any layer meets every complete chain. To prove the converse, assume now that $\mathcal{A}$ is not a layer, and denote its complement in $\mathcal{P}(S)$ by $\mathcal{B}$. There must then be some nontrivial layer $\mathcal{P}(S,k)$ that meets both $\mathcal{A}$ and $\mathcal{B}$. By the connexity of the Johnson graph $\mathrm{J}(S,k)$, there are subsets A and B such that $A \in \mathcal{A}$, $B \in \mathcal{B}$ with moreover $A \sim B$. Take any complete chain $\mathcal{C}$ containing the three subsets $A \cap B$, A and $A \cup B$ of respective sizes $k-1$, k and $k+1$. This chain $\mathcal{C}$ meets the antichain $\mathcal{A}$ only at A. Then, the complete chain $(\mathcal{C} \backslash \{A\}) \cup \{B\}$ does not meet $\mathcal{A}$. $\square$

In the following proofs, Lemma 1 will often be applied to intervals $[\emptyset, X]$ or $[X, S]$ of $\mathcal{P}(S)$. This makes sense because any such interval is isomorphic to some power set.

Let $\aleph$ denote the set of all nonempty antichains in $\mathcal{P}(S)$ which are not layers. For $\mathcal{A}, \mathcal{A}' \in \aleph$, we say that $\mathcal{A}$ is *dominated* by $\mathcal{A}'$ if every complete chain intersecting $\mathcal{A}$ also intersects $\mathcal{A}'$. When $\mathcal{A}$ is dominated by $\mathcal{A}'$, we write $\mathcal{A} \sqsubseteq \mathcal{A}'$. As usual, $\mathcal{A} \sqsubset \mathcal{A}'$ means $\mathcal{A} \sqsubseteq \mathcal{A}'$ and not $\mathcal{A}' \sqsubseteq \mathcal{A}$. Notice that $\mathcal{A} \subseteq \mathcal{A}'$ implies $\mathcal{A} \sqsubseteq \mathcal{A}'$.

Lemma 2. *The dominance relation $\sqsubseteq$ is a partial ordering of $\aleph$.*

Proof. The reflexivity and transitivity of $\sqsubseteq$ are obvious. To prove antisymmetry, suppose $\mathcal{A} \sqsubseteq \mathcal{A}'$ and $\mathcal{A}' \sqsubseteq \mathcal{A}$ for distinct $\mathcal{A}$ and $\mathcal{A}'$ in $\aleph$. By exchanging if necessary the two antichains, we may assume that some set X belongs to $\mathcal{A} \backslash \mathcal{A}'$. Let $k = |X|$. We claim that every Y in $\mathcal{P}(S,k)$ which is adjacent to X belongs to $\mathcal{A} \backslash \mathcal{A}'$. Because the Johnson graph $\mathrm{J}(S,k)$ is connected, the claim implies that $\mathcal{A}$ is a layer, a contradiction.

To prove the claim, consider the two intervals $[\emptyset, X]$ and $[X, S]$ of $\mathcal{P}(S)$. By Lemma 1, the restriction of $\mathcal{A}'$ to one of these intervals is a nontrivial layer inside this interval because otherwise there would exist a complete chain in $\mathcal{P}(S)$ intersecting $\mathcal{A}$ and disjoint of $\mathcal{A}'$, a contradiction. In particular, for every Y adjacent to X in $\mathrm{J}(S,k)$, there exists a $Z \in \mathcal{A}'$ such that X and Y are comparable to Z in $\mathcal{P}(S)$. Hence $Y \notin \mathcal{A}'$. Considering the two intervals $[\emptyset, Z]$ and $[Z, S]$, we derive this time that the restriction of $\mathcal{A}$ to one of these two intervals is a layer in that interval. Because this layer contains X, it also contains Y, and thus Y belongs to $\mathcal{A}$. Consequently, the claim holds. $\square$

Let $\mathcal{F}$ denote the collection of all faces of the approval-voting polytope P^n_{AV} that are of the form $F(\mathcal{A})$ with $\mathcal{A} \in \aleph$. All facets belong to $\mathcal{F}$ because of Proposition 1. Clearly, $\mathcal{A} \sqsubseteq \mathcal{A}'$ if and only if $F(\mathcal{A}) \subseteq F(\mathcal{A}')$. Hence, it follows from Lemma 2 that $F(\mathcal{A}) = F(\mathcal{A}')$ implies $\mathcal{A} = \mathcal{A}'$. In conclusion, the mapping $\mathcal{A} \mapsto F(\mathcal{A})$ is an isomorphism from the poset $(\aleph, \sqsubseteq)$ onto the poset $(\mathcal{F}, \subseteq)$. This proves the next proposition.

Proposition 2. *The facets of the approval-voting polytope P^n_{AV} biunivocally correspond to the maximal elements of $(\aleph, \sqsubseteq)$, with $F(\mathcal{A})$ being the facet corresponding to the maximal element $\mathcal{A}$.*

The next proposition characterizes the maximal elements of $(\aleph, \sqsubseteq)$. Given any subset $\mathcal{A}$ of $\mathcal{P}(S)$, we define $\mathcal{A}_i = \mathcal{A} \cap \mathcal{P}(S, i)$ for $0 \leqslant i \leqslant n$.

Proposition 3. *An element $\mathcal{A}$ of $\aleph$ is maximal for $\sqsubseteq$ if and only if it is of one of the following two types.*

Unilayer antichain with parameter k. *There exists some k with $0 < k < n$ and some Z in $\mathcal{P}(S,k)$ such that $\mathcal{A} = \mathcal{P}(S,k) \backslash \{Z\}$.*

Bilayer antichain with parameter k. *There exists some k with $0 < k < n-1$ such that*

(C1) *the graph induced on $\mathcal{A}_k$ by $\mathrm{J}(S,k)$ is connected,*
(C2) *the graph induced on $\mathcal{A}_{k+1}$ by $\mathrm{J}(S,k+1)$ is connected,*
(C3) $\mathcal{A}_k = \{X \in \mathcal{P}(S,k) \mid \forall Y \in \mathcal{A}_{k+1} : X \not\subset Y\}$ *and*
(C4) $\mathcal{A}_{k+1} = \{Z \in \mathcal{P}(S,k+1) \mid \forall T \in \mathcal{A}_k : T \not\subset Z\}$.

When $\mathcal{A}$ is a uni- or bilayer antichains as in Proposition 3, we call $F(\mathcal{A})$ a *unilayer* or *bilayer facet* with *parameter k*. For a bilayer antichain, conditions (C3) and (C4) imply $\mathcal{A}_i \neq \emptyset$ if and only if $i \in \{k, k+1\}$, because $\mathcal{A}$ is an antichain. The four conditions (C1)–(C4) can be shown to be mutually independent.

Proof. Assume that $\mathcal{A}$ is a maximal element of $\aleph$ for $\sqsubseteq$. Consider the least k such that $\mathcal{A}$ meets the kth layer. If $\mathcal{A} \subseteq \mathcal{P}(S,k)$, it is easily seen that $\mathcal{A}$ is a unilayer antichain. Assume now $\mathcal{A} \nsubseteq \mathcal{P}(S,k)$. Then the collection $\mathcal{B}$ defined by

$$\mathcal{B} = (\mathcal{A} \backslash \mathcal{A}_k) \cup \{Y \in \mathcal{P}(S,k+1) \mid \exists X \in \mathcal{A}_k : X \subset Y\}$$

is again an antichain, and it satisfies $\mathcal{A} \sqsubseteq \mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$. Thus $\mathcal{B} \notin \aleph$, so $\mathcal{B}$ is a layer. Hence there is some k such that $\mathcal{A} \subset \mathcal{P}(S,k) \cup \mathcal{P}(S,k+1)$. The maximality of $\mathcal{A}$ for $\sqsubseteq$, and thus for $\subseteq$, implies (C3) and (C4). Finally, the connectedness conditions (C1) and (C2) can be established using arguments as above. Indeed, if $\mathcal{A}_k$ were not connected, then denote by $\mathcal{D}$ any connected component of $\mathcal{A}_k$. Because

$$\mathcal{B} = (\mathcal{A} \backslash \mathcal{D}) \cup \{Y \in \mathcal{P}(S,k+1) \mid \exists X \in \mathcal{D} : X \subset Y\}$$

is in $\aleph$ and satisfies $\mathcal{A} \sqsubset \mathcal{B}$, a contradiction is reached.

Conversely, we show that any uni- or bilayer antichain $\mathcal{A}$ with parameter k is maximal for $\sqsubseteq$. Assume that there exists some antichain $\mathcal{A}'$ in $\aleph$ with $\mathcal{A} \sqsubseteq \mathcal{A}'$. In view of Lemma 2, it suffices to show $\mathcal{A}' \sqsubseteq \mathcal{A}$.

First, note that $\mathcal{A}$ is maximal for $\subseteq$ in $\aleph$. In other words, any collection of subsets of S properly containing $\mathcal{A}$ either is not an antichain or is a layer. This is trivial in case $\mathcal{A}$ is a unilayer antichain and follows from conditions (C3) and (C4) in case $\mathcal{A}$ is a bilayer antichain.

For each $X \in \mathcal{A}$, we derive from Lemma 1 that $\mathcal{A}'$ contains a layer $\mathcal{L}_X$ of one of the two intervals $[\emptyset, X]$ and $[X, S]$. Let $h(X)$ denote the common cardinality of all elements of $\mathcal{L}_X$. As is easily seen, $h(X) = |X|$, $\mathcal{L}_X = \{X\}$ and $X \in \mathcal{A} \cap \mathcal{A}'$ are equivalent conditions on X. Furthermore, if $Y \in \mathcal{A}$ is such that $|Y| = |X|$ and $Y \sim X$, then we have $h(Y) = h(X)$.

Now, suppose that there exists a complete chain $\mathcal{C}$ meeting $\mathcal{A}'$ but disjoint of $\mathcal{A}$, let A' denote the unique subset in the intersection of $\mathcal{A}'$ and $\mathcal{C}$, and let ℓ' denote the cardinality of A'. We claim $h(X) = \ell'$ for all $X \in \mathcal{A}$. By maximality of $\mathcal{A}$ for $\sqsubseteq$ in $\aleph$, this claim implies that $\mathcal{A}'$ contains the layer $\mathcal{P}(S, \ell')$, a contradiction.

If $\mathcal{A}$ is a unilayer antichain, then the claim holds because the Johnson graph $\mathrm{J}(S,k)$ remains connected when any vertex is removed from this graph. Next, assume that $\mathcal{A}$ is a bilayer antichain. Let C and D denote the unique subsets of cardinality respectively k and $k+1$ in $\mathcal{C} \cap \mathcal{A}$. By conditions (C3) and (C4), we can find A in $\mathcal{A}_k$ and B in $\mathcal{A}_{k+1}$ such that $A \subset D$ and $C \subset B$. Then we have $h(A) = h(B) = \ell'$ and, by conditions (C1) and (C2), the claim holds. $\square$

Corollary 1. *The inequalities that define facets of the approval-voting polytope P^n_{AV} are exactly the inequalities of the form*

$$\lambda x(\mathcal{A}) + \sum_{i=0}^{n} \mu_i x(\mathcal{P}(S,i)) \quad \leqslant \quad \lambda + \sum_{i=0}^{n} \mu_i, \tag{3}$$

where $\lambda \in \mathbb{R}^+_*$, $\mathcal{A}$ is a maximal antichain in $(\aleph, \sqsubseteq)$ as in Proposition 3, *and $\mu_0, \mu_1, \ldots, \mu_n \in \mathbb{R}$. The facet defined by Eq.* (3) *is also defined by the antichain inequality $x(\mathcal{A}) \leqslant 1$.*

As will be seen in the next section, several other properties of P^n_{AV} can be easily derived from Propositions 2 and 3. Nevertheless, providing an explicit enumeration of all facets appears to be intricate. To see this, assume $n \geqslant 4$ and take any subset $\mathcal{A}$ of $\mathcal{P}(S,2) \cup \mathcal{P}(S,3)$ which meets both layers. We derive a graph $\Phi = (S, \mathcal{A}_2)$. Condition (C4) in Proposition 3 imposes that $\mathcal{A}_3$ equals the collection of all the stable sets of size 3 of Φ. If Φ has at least two isolated nodes, then the only additional requirement for $\mathcal{A}$ to be a maximal element of $(\aleph, \sqsubseteq)$ is that the graph Φ be *almost connected*, that is: besides its isolated nodes, Φ has only one connected component. From this subcase, we deduce that enumerating all facets of P^n_{AV} is at least as intricate as enumerating all labelled, connected graphs on at most $n-2$ nodes. Further results on the case $k = 2$ are given by Doignon and Fiorini [7].

4. The symmetries

In this section, we assume $n \geqslant 4$. Doignon and Regenwetter [8] exhibit a group G of $8 \cdot n!$ affine symmetries of the approval-voting polytope P^n_{AV}, and ask whether these affine symmetries induce the full group $\mathrm{Aut}(P^n_{\mathrm{AV}})$ of (combinatorial) automorphisms. The affirmative answer will be established here. As a consequence, any automorphism of P^n_{AV} is the restriction of some affine symmetry. Notice in passing that the graph of the polytope P^n_{AV} has a group of automorphisms which is much larger than G, as can be inferred from [9].

Each permutation α of S induces a permutation of the set of all rankings $\rho : \{1, 2, \ldots, n\} \to S$, that maps ranking ρ onto ranking $\alpha \circ \rho$. In terms of complete chains, we get the permutation $\mathcal{C} \mapsto \alpha(\mathcal{C})$. As in [8], there results the affine symmetry of P^n_{AV} mapping vertex $v_{\mathcal{C}}$ onto vertex $v_{\alpha(\mathcal{C})}$. The restriction to the vertex set of P^n_{AV} of such a symmetry is called a *relabelling automorphism* (induced by α). On the other hand, any permutation β of $\{1, 2, \ldots, n\}$ induces the permutation of the set of all rankings that maps ranking ρ onto ranking $\rho \circ \beta$, that is: the elements of S are shuffled according to the given permutation β of the ranks. The complete chain

$$\mathcal{C} = \{\emptyset, \{\rho(1)\}, \{\rho(1), \rho(2)\}, \ldots, \{\rho(1), \rho(2), \ldots, \rho(n)\}\}$$

is then mapped onto the complete chain

$$\mathcal{D} = \{\emptyset, \{\rho\circ\beta(1)\}, \{\rho\circ\beta(1), \rho\circ\beta(2)\}, \ldots, \{\rho\circ\beta(1), \rho\circ\beta(2), \ldots, \rho\circ\beta(n)\}\}.$$

Hence, we can associate to β the permutation $U_\beta : v_{\mathcal{C}} \mapsto v_{\mathcal{D}}$ of the set of all vertices of P^n_{AV}, that we call the *reshuffling permutation* induced by β. According to [9], this reshuffling permutation U_β is an automorphism of P^n_{AV}, which is the restriction of an affine symmetry, if and only if U_β belongs to the group generated by U_σ and U_τ, where

$$\sigma = (1, 2), \quad \tau = (1, n)\,(2, n-1) \cdots (\lfloor n/2 \rfloor, \lceil n/2 \rceil).$$

As in [9], U_σ is called the *initial switch automorphism* and U_τ the *reversing automorphism*. Together, U_σ and U_τ generate a group isomorphic to the dihedral group $\mathcal{D}_8$ of order 8 (remember that we assume $n \geqslant 4$).

Because every relabelling automorphism commutes with every reshuffling automorphism, all of these automorphisms generate a group G isomorphic to $\mathcal{D}_8 \times \mathrm{Sym}(n)$ (cf. [9]). To show $G = \mathrm{Aut}\ (P^n_{\mathrm{AV}})$, we first determine orbits of facets under the action of Aut (P^n_{AV}), by making intensive use of the numbers of vertices on the various facets. For a face F, we use F (resp. $\overline{\mathrm{vert}}\ F$) to denote the set of vertices on F (resp. not on F). Thus $\overline{\mathrm{vert}}\ F = (\mathrm{vert}\ P^n_{\mathrm{AV}}) \setminus \mathrm{vert}\ F$.

Consider a facet $F(\mathcal{A})$ with $\mathcal{A}$ as in Proposition 3. If $\mathcal{A}$ is a unilayer antichain with parameter k, the number of vertices on $F(\mathcal{A})$ equals

$$n! - k! \cdot (n-k)!. \tag{4}$$

This expression strictly increases when k varies from 1 to $\lfloor n/2 \rfloor$ and strictly decreases when k varies from $\lceil n/2 \rceil$ to $n-1$.

If $\mathcal{A}$ is a bilayer antichain with parameter k, we use $\alpha = |\mathcal{A}_k|$ and $\beta = |\mathcal{A}_{k+1}|$ to express the number of vertices on $F(\mathcal{A})$ as

$$k! \cdot (n-k-1)! \cdot (\alpha \cdot (n-k) + \beta \cdot (k+1)). \tag{5}$$

Together with the following lemma, Eqs. (4) and (5) produce some interesting consequences that we collect in Proposition 4. For the sake of simplicity, we set $\Delta_k = k(n-k)$. Note that Δ_k is the degree of any node in the Johnson graph $\mathrm{J}(S, k)$.

Lemma 3 (Watkins [19], Mader [15], Daven and Rodger [5]). *The Johnson graph* $\mathrm{J}(S, k)$ *is* Δ_k*-connected.*

For convenience, given $\mathcal{A} \subseteq \mathcal{P}(S)$ and $0 \leqslant i \leqslant n$, we define $\bar{\mathcal{A}}_i = \mathcal{P}(S, i) \setminus \mathcal{A}_i$.

Proposition 4. *Let $\mathcal{A}$ be a bilayer antichain with parameter k, thus $0<k<n-1$. Moreover, let $\mathcal{D}$ and $\mathcal{E}$ be unilayer antichains with parameter equal to, respectively, k and $k+1$. We have for the corresponding facets of P^n_{AV}:*

$$|\mathrm{vert}\ F(\mathcal{A})| \leqslant \min\{|\mathrm{vert}\ F(\mathcal{D})|, |\mathrm{vert}\ F(\mathcal{E})|\}.$$

Moreover, this inequality becomes an equality in exactly two cases:

(a) $k=1$, $|\mathcal{A}_1|=1$ *and then* $|\mathrm{vert}\ F(\mathcal{D})| = |\mathrm{vert}\ F(\mathcal{A})| < |\mathrm{vert}\ F(\mathcal{E})|$;
(b) $k=n-2$, $|\mathcal{A}_{k+1}|=1$ *and then* $|\mathrm{vert}\ F(\mathcal{D})| > |\mathrm{vert}\ F(\mathcal{A})| = |\mathrm{vert}\ F(\mathcal{E})|$.

Proof. We may assume $k \leqslant (n-1)/2$ because the case $k>(n-1)/2$ is transformed into this one by applying the reversing automorphism. Then, $|\mathrm{vert}\ F(\mathcal{D})| \leqslant |\mathrm{vert}\ F(\mathcal{E})|$.

Because $\mathcal{A}_k \neq \emptyset$, we derive $|\overline{\mathcal{A}}_{k+1}| \geqslant n-k$. Take $C \in \mathcal{A}_{k+1}$. Because $n-k \leqslant (k+1)(n-k-1) = \Delta_{k+1}$ follows from $1 \leqslant k \leqslant (n-1)/2$, Lemma 3 implies that the Johnson graph $\mathrm{J}(S,k+1)$ is $(n-k)$-connected. Thus, there exist in this graph $n-k$ paths starting at C and ending in $\overline{\mathcal{A}}_{k+1}$, with the further condition that any two of these paths have exactly node C in common (see Exercise 13 on p. 93 of [1]). On each of these paths, we find a pair (A,B) of adjacent nodes, with $A \in \mathcal{A}_{k+1}$ and $B \in \overline{\mathcal{A}}_{k+1}$. Then $A \cap B$ is in $\overline{\mathcal{A}}_k$, and the $k! \cdot (n-k-1)!$ complete chains through $A \cap B$ and B correspond to vertices not on the facet $F(\mathcal{A})$. The $n-k$ sets B being distinct, we find that the number of vertices not on $F(\mathcal{A})$ is at least $k! \cdot (n-k)!$, which is the number of vertices not on $F(\mathcal{D})$. There follows the desired inequality

$$|\mathrm{vert}\ F(\mathcal{A})| \leqslant |\mathrm{vert}\ F(\mathcal{D})|. \tag{6}$$

Moreover, by similar arguments, Eq. (6) gives a strict inequality when $|\mathcal{A}_k| \geqslant 2$. Indeed, we then have $\overline{\mathcal{A}}_k \geqslant n-k+1$. On the other hand, $n-k+1 \leqslant (k+1)(n-k-1)$ follows from $n \geqslant 4$ and k a natural number with $k \leqslant (n-1)/2$. Hence, we can use the $(n-k+1)$-connectivity of the Johnson graph $\mathrm{J}(S,k+1)$ to derive the strict inequality in Eq. (6).

There remains to show that the inequality is also strict when $|\mathcal{A}_k|=1$ together with $k>1$. This is left to the reader. □

The next three lemmas will be used in the proof of Proposition 5, the main result in this section.

Lemma 4 (Brouwer et al. [2]). *The automorphism group of the Johnson graph* $\mathrm{J}(S,k)$, *with* $0<k<n$, *is isomorphic to*

$$\begin{aligned} &\mathbb{Z}_2 \times \mathrm{Sym}\ (n) && \textit{if } n=2k \geqslant 4,\\ &\mathrm{Sym}\ (n) && \textit{otherwise.} \end{aligned}$$

Let Γ_n be the restriction of the comparability graph of $\mathcal{P}(S)$ to the union of the layers $\mathcal{P}(S,k)$ for $n/2-1 \leqslant k \leqslant n/2+1$. In other words, Γ_n is the graph whose vertex set is the union of the two central layers of $\mathcal{P}(S)$ if n is odd and of the three central layers of $\mathcal{P}(S)$ if n is even, and in which two distinct sets X and Y are adjacent if they are comparable for inclusion.

Lemma 5. *For all* $n \geqslant 2$, *the automorphism group of the graph* Γ_n *is isomorphic to* $\mathbb{Z}_2 \times \mathrm{Sym}(n)$.

Proof. We use Lemma 4. If n is odd, the structure of the Johnson graphs $\mathrm{J}(S,(n-1)/2)$ and $\mathrm{J}(S,(n+1)/2)$ can be derived from that of Γ_n, although their two sets of vertices are indistinguishable.

If n is even, all nodes of Γ_n in $\mathcal{P}(S,n/2)$ have the same degree, which differs from the degree of any other node in Γ_n. Moreover, the structure of $\mathrm{J}(S,(n/2))$ is recoverable from that of Γ_n. $\square$

When $Z \in \mathcal{P}(S,k)$, we write F_Z for the unilayer facet $F(\mathcal{P}(S,|Z|)\backslash\{Z\})$.

Lemma 6. *Let* $\mathcal{A}$ *be a bilayer antichain with parameter* k. *Then*

$$\mathrm{vert}\ F(\mathcal{A}) = \bigcap\{\mathrm{vert}\ F_A \cup \mathrm{vert}\ F_B \mid A \in \overline{\mathcal{A}}_k, B \in \overline{\mathcal{A}}_{k+1}, A \subset B\}.$$

The proof is easy and therefore skipped. For $n=2$ and $n=3$, the group $\mathrm{Aut}(P^n_{\mathrm{AV}})$ does not fit into the general scheme provided in the next proposition: it is then of order 2, resp. 72 (see [9]).

Proposition 5. *For* $n \geqslant 4$, *the automorphism group of the approval-voting polytope* P^n_{AV} *is isomorphic to* $\mathcal{D}_8 \times \mathrm{Sym}(n)$.

Proof. The case $n=4$ is established in [9], so we take here $n \geqslant 5$. It suffices to show that the total number of automorphisms does not exceed $8 \cdot n!$. This is due to the fact that Aut (P^n_{AV}) has a subgroup isomorphic to $\mathcal{D}_8 \times \mathrm{Sym}(n)$.

Let $M = \{k \in \mathbb{Z} \mid n/2 - 1 \leqslant k \leqslant n/2 + 1\}$. From Proposition 4, we infer that the unilayer facets with parameter in the set M are the facets of P^n_{AV} with the highest possible number of vertices when n is odd, and with the two highest possible numbers of vertices when n is even. Consequently, every automorphism of P^n_{AV} permutes the unilayer facets with parameter in M among themselves. By the canonical identification of the facet F_X with the set X, we obtain for each automorphism α of P^n_{AV} a permutation α' of the vertex set of Γ_n. Because two distinct sets X and Y are adjacent in Γ_n if and only if $\overline{\mathrm{vert}}\ F_X$ and $\overline{\mathrm{vert}}\ F_Y$ share a common vertex, the permutation α' is an automorphism of Γ_n for each automorphism α of P^n_{AV}. We thus derive a group homomorphism $f : \mathrm{Aut}(P^n_{\mathrm{AV}}) \to \mathrm{Aut}(\Gamma_n) : \alpha \mapsto \alpha'$. In view of Lemma 5, it suffices now to prove that the kernel of f has at most four elements.

From now on, assume that α belongs to the kernel of f, in other words, that α stabilizes every unilayer facet with parameter in the set M. First, we prove by induction on ℓ that α stabilizes any unilayer facet with parameter in

$$I_\ell = \{k \in \mathbb{Z} \mid n/2 - \ell \leqslant k \leqslant n/2 + \ell\}$$

for $\ell = 1, 2, \ldots, \lfloor (n+1)/2 \rfloor - 2$. Note that $I_1 = M$. Since α belongs to the kernel of f, the induction thesis holds for $\ell = 1$. Now assume that it holds for some ℓ with $1 \leqslant \ell \leqslant \lfloor (n+1)/2 \rfloor - 3$ and let us prove that it also holds when ℓ is replaced with $\ell + 1$. By Lemma 6, the automorphism α stabilizes every bilayer facet $F(\mathcal{A})$ such that $\mathcal{A}$ meets two layers $\mathcal{P}(S,j)$ with j in I_ℓ. Let D_ℓ denote the set $I_{\ell+1} \backslash I_\ell$. Thus $D_\ell = \{\frac{n}{2} - l - 1, \frac{n}{2} + l + 1\}$ if n is even and $D_\ell = \{\frac{n+1}{2} - l - 1, \frac{n-1}{2} + l + 1\}$ if n is odd. Clearly, each unilayer facet with parameter in D_ℓ is either stabilized by α or mapped by α to a distinct facet of same cardinality which is not stabilized by α. Then Proposition 4 implies that α permutes the unilayer facets with parameter in D_ℓ among themselves. Let X be a subset of S with $|X| \in D_\ell$. Without loss of generality, assume that $|X| < n/2$. Let $\mathcal{L}$ denote the collection of all subsets Y of S such that $X \subset Y$ and $|Y| = |X| + 1$. By hypothesis, we know that all unilayer facets F_Y with $Y \in \mathcal{L}$ are stabilized by α. Moreover, we know that α maps F_X to some unilayer facet $F_{X'}$ with $|X'| \in D_\ell$. As X is comparable to every $Y \in \mathcal{L}$, we conclude that X' is comparable to every $Y \in \mathcal{L}$, and therefore $X = X'$. In conclusion, the induction thesis holds for $\ell + 1$.

When X is any subset of S of cardinality 1 or $n-1$, we denote by H_X the unique bilayer facet $F(\mathcal{A})$ such that the intersection of $\mathcal{A}$ and $\mathcal{P}(S,1) \cup \mathcal{P}(S, n-1)$ is exactly $\{X\}$. By what precedes and by Proposition 4, the automorphism α maps every unilayer facet F_X with $|X| \in \{1, n-1\}$ either to a unilayer facet $F_{X'}$ with $|X'| \in \{1, n-1\}$ or to a bilayer facet $H_{X''}$ with $|X''| \in \{1, n-1\}$. If $\alpha(F_X) = F_{X'}$, then we have $X = X'$ by an argument of the last paragraph. By adapting the latter argument, it is easy to show that $\alpha(F_X) = H_{X''}$ implies $X = X''$. Moreover, if some unilayer facet with parameter 1 is stabilized, then each unilayer facet with parameter 1 is stabilized. This follows from the following three observations:

- vert P^n_{AV} is partitioned by the subsets $\overline{\mathrm{vert}}\ F_X$, for $X \in \mathcal{P}(S, 1)$,
- vert P^n_{AV} is partitioned by the subsets $\overline{\mathrm{vert}}\ H_Y$, for $Y \in \mathcal{P}(S, 1)$,
- for $X, Y \in \mathcal{P}(S, 1)$, the subsets $\overline{\mathrm{vert}}\ F_X$ and $\overline{\mathrm{vert}}\ H_Y$ have an empty intersection if and only if $X = Y$.

Similar arguments can be used for the unilayer facets with parameter $n - 1$. It follows that the kernel of f consists of at most four automorphisms. $\square$

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