Algebraic and Topological Classification of the Homogeneous Cubic Vector Fields in the Plane

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Introduction

Tsutomu Date and Masao Iri in [DI] gave an algebraic classification of systems \( \dot{x} = P(x, y), \dot{y} = Q(x, y) \), where \( P \) and \( Q \) are homogeneous polynomials of degree 2. For this, they used the classification of the binary cubic forms and also the simultaneous classification of a linear binary form and a cubic binary form given by the algebraic invariant theory.

We begin by doing a similar study for systems \( \dot{x} = P(x, y), \dot{y} = Q(x, y) \), where \( P \) and \( Q \) are homogeneous polynomials of degree three (i.e., cubic systems). The classification's theorem of such systems is based on the classification of fourth-order binary forms. Gurevich in [Gu] did the classification of fourth-order binary forms on the field of complex numbers. Since we did not find the classification on the real domain, we adapt Gurevich's proof to obtain it. In Section 1 we give some definitions and preliminary results, while in Section 2 we give the theorem of classification of fourth-order binary forms on the real domain. The method used in the proof (Caley's method) let us obtain canonical forms of the fourth-order binary forms and the algebraic characteristics. Section 3 is devoted to obtaining the algebraic classification of systems \( \dot{x} = P(x, y), \dot{y} = Q(x, y) \), where \( P \) and \( Q \) are homogeneous polynomials of degree 3. Given an arbitrary system \( X = (P, Q) \) with \( P \) and \( Q \) homogeneous polynomials of degree 3, we can know the equivalence-class at which it belongs through the algebraic characteristics.

In Section 4 we study the phase-portraits of systems \( \dot{x} = P(x, y), \dot{y} = Q(x, y) \), where \( P \) and \( Q \) are homogeneous polynomials of degree \( n \) and \( P \) and \( Q \) have no common factor. Such systems have been studied by J. Argemi in [A]. Here we give a shorter new proof of his results by using...
standard tools of plane vector fields. Ending Section 4, we prove that these results can be applied to determine generically the behaviour at infinity of systems $X = (P, Q)$ not necessarily homogeneous.

In Section 5 we return to the case $n = 3$ obtaining all the possible phase-portraits of homogeneous cubic systems $X = (P, Q)$ with degree($P$) = degree($Q$) = 3. In the case that $P$ and $Q$ have no common factor, our method allows us to know the phase-portrait of cubic systems using only the algebraic classification.

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1. Definitions and Some Preliminary Results

Let $\Omega$ be a field of characteristic zero ($\Omega$ eventually shall be $\mathbb{R}$ or $\mathbb{C}$). Consider the ring of polynomials in $n$ variables $\Omega[x_1, x_2, \ldots, x_n]$. A homogeneous polynomial in $\Omega[x_1, x_2, \ldots, x_n]$ of degree $r$ is called an $n$-ary form of order $r$. The isomorphism between the space of homogeneous polynomials in $n$ variables of degree $r$ and the space $S_r(E)$ of symmetric covariant tensors of order $r$ in an $n$-dimensional vector space $E$ is well known. Thus, given any $n$-ary form of order $r$,

$$f(x) = \sum \beta_{r_1\ldots r_n} x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n},$$

with $r_1 + r_2 + \cdots + r_n = r$, there is associated a unique symmetric covariant tensor of order $r$ expressed as

$$\varphi = \sum \beta_{r_1\ldots r_n} u_{i_1}^{r_1} \otimes u_{i_2}^{r_2} \otimes \cdots \otimes u_{i_n}^{r_n},$$

where $u_1, \ldots, u_n$ is the dual basis of the basis $e_1, \ldots, e_n$ of $E$ at which $x = \sum_{i=1}^n x_i e_i$.

Let $\Gamma$ be a subgroup of the linear group $GL(n; \Omega)$. Then, every $\sigma \in \Gamma$ induces a linear morphism $\sigma: S_r(E) \to S_r(E)$ defined by $\sigma(f)(x) = f(\sigma(x))$. We shall say that $f, g \in S_r(E)$ are equivalents through $\Gamma$ if there exists $\sigma \in \Gamma$ such that $\sigma(f) = g$.

Our goal is to classify the fourth-order binary forms under the above equivalence relation for $\Omega = \mathbb{R}$ and $\Gamma = GL(2; \mathbb{R})$, by giving canonical forms which represent each class of equivalence.
Let $e_1, e_2, \ldots, e_n$ be a fixed basis of $E$. Then $\Gamma \subset GL(n; \Omega)$ acts on $E$ in a natural way. That is, we have the map $\Gamma \times E \to E$ defined by $(\sigma, x) \mapsto \sigma \cdot x = \sigma(x)$. This map has the properties

1. $(\sigma \circ \tau) \cdot x = \sigma \cdot (\tau \cdot x)$ for all $x \in E$ and for all $\sigma, \tau \in \Gamma$ and,
2. $\varepsilon \cdot x = x$ for all $x \in E$, where $\varepsilon$ is the identity of $\Gamma$.

The group $\Gamma$ also acts on $S_r(E)$ in the following way. Consider the map $\Gamma \times S_r(E) \to S_r(E)$, defined by $(\sigma, f) \mapsto \sigma \circ f - \sigma_r(f)$, where $(\sigma)$ is the morphism induced by $\sigma$.

Every $\sigma \in \Gamma$ can be represented by a square matrix $A$ with $\det A = |\sigma| \neq 0$. We shall say that $u : S_r(E) \to \Omega$ is a relative invariant of $\Gamma$ of weight $q$, if $u(\sigma_r(f)) = |\sigma|^q u(f)$ for all $\sigma \in \Gamma$. A relative invariant of weight $q = 0$ is called an absolute invariant of $\Gamma$.

**Examples of Invariants.** Let $f \in S_4(\Omega^2)$ be defined by

$$f(x, y) = a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 x y^3 + a_4 y^4.$$ 

1. The map $S_4(\Omega^2) \to \Omega$ given by $f \to i_f$, where

$$i_f = a_0 a_4 - 4a_1 a_3 + 3a_2^2$$

is a relative invariant of weight $q = 4$ for $\Gamma = GL(2; \Omega)$ (see [Gu, p. 155]).

2. The map $S_4(\Omega^2) \to \Omega$ given by $f \to j_f$, where

$$j_f = a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_1^2 a_4 - a_0 a_3^2 - a_2^3$$

is a relative invariant of weight $q = 6$ for $\Gamma = GL(2; \Omega)$ (see [Gu, p. 206 (Theorem 1.8) and p. 208 (Exercise 15)]).

3. The map $S_4(\Omega^2) \to \Omega$ given by $f \to D_f$, where

$$D_f = i_f^3 - 27 j_f^2$$

is a relative invariant of weight $q = 12$ for $\Gamma = GL(2; \Omega)$ because

$$D_{\sigma(f)} = i_{\sigma(f)}^3 - 27 j_{\sigma(f)}^2 = (|\sigma|^3 i_f)^3 - 27 (|\sigma|^{6} j_f)^2 = |\sigma|^{12} D_f.$$ 

The invariant $D_f$ is called the discriminant of $f$, and it coincides with the classical definition as a product of the squares of the differences of the roots of $f$ (through a constant). Hence $f = 0$ has a multiple root if and only if $D_f = 0$.

Let $F$ and $G$ be defined by $F = S_r(E)$ and $G = S_r(E)$. Then $\Gamma$ acts on $G^F$, the set of all mappings $F \to G$, in the following way. Consider $\Gamma \times G^F \to G^F$ defined by $(\sigma, \varphi) \to \sigma \circ \varphi$ with $(\sigma \circ \varphi)(f) = \sigma_r(\varphi(\sigma_r^{-1}(f)))$, where $\sigma$, and $\sigma_r$ are the morphisms induced by $\sigma$ on $F$ and $G$, respectively.
We shall say that $u: F \to G$ is a concomitant of $\Gamma$ if $u$ is invariant under the action of $\Gamma$ on $G^F$; i.e., if for all $\sigma \in \Gamma$, $\sigma \circ u = u$; or equivalently if for all $\sigma \in \Gamma$, $\sigma_x \circ u = u \circ \sigma_x$.

**Examples of Concomitants.**

1. Let $F = G = S_4(\Omega^2)$ and let $H: F \to G$ be defined by $f \to H_f$, where

$$H_f = \frac{1}{3^24^2} \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix},$$

where $f_{xx}, f_{xy},$ and $f_{yy}$ are the partial derivatives of $f$ of second order. For each $f \in S_4(\Omega^2)$, $H_f$ is called the hessian of $f$ and it is a concomitant of $\Gamma = GL(2; \Omega)$ (see [Gu, p. 153 (Theorem 14.7)]).

2. Let $F = S_4(\Omega^2)$, $G = S_6(\Omega^2)$ and let $Q: F \to G$ be defined by $f \to Q_f$, where

$$Q_f = \frac{1}{4^2} \begin{vmatrix} f_x & 2(H_f)_x \\ f_y & 2(H_f)_y \end{vmatrix},$$

where $f_x, f_y$ and $(H_f)_x, (H_f)_y$ are the partial derivatives of $f$ and $H_f$ of first order, respectively. For each $f \in S_4(\Omega^2)$, $Q_f$ is called the jacobian of $f$ and it is a concomitant of $\Gamma = GL(2; \Omega)$ (see [Gu, p. 286]).

3. Every linear combination of concomitants is also a concomitant. So, $2i_f H_f - 3j_f f, i_f H_f + 3j_f f$, and $12H_f^2 - i_f f^2$ are also concomitants.

**Proposition 1.1.** Let $f \in S_4(\Omega^2)$ be a fourth-order binary form and let $i_f, j_f, H_f,$ and $Q_f$ be the invariants and concomitants defined above. Then:

(a) $i_f, j_f, H_f,$ and $Q_f$ are related by the syzygy

$$Q_f^2 + (j_f f - i_f H_f) f^2 + 4H_f^3 = 0.$$

(b) The hessian $H_{H_f}$ of the hessian can be expressed by the formula

$$H_{H_f} = \frac{1}{4} i_f f - \frac{1}{12} i_f H_f.$$

For a proof see [Gu, p. 287 for (a) and p. 284 for (b)].

A $k$th transvectant of the two binary forms $f$ and $\varphi$ of orders $r$ and $p$, respectively, is defined by

$$(f, \varphi)^{(k)} = \frac{(r-k)! (p-k)!}{r! p!} \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{r-h} \partial y^h} \frac{\partial^h \varphi}{\partial x^h \partial y^{r-h}}, \quad (1.1)$$

for $0 \leq k \leq \min\{r, p\}$. Otherwise, $(f, \varphi)^{(k)} = 0.
Transvectants have the two following properties

\[(\varphi, f)^{(k)} = (-1)^k (f, \varphi)^{(k)}, \quad (1.2)\]

\[(f, \lambda \varphi + \mu \psi)^{(k)} = \lambda (f, \varphi)^{(k)} + \mu (f, \psi)^{(k)}. \quad (1.3)\]

It follows from (1.2) that for \(k\) odd

\[(f, f)^{(k)} = 0. \quad (1.4)\]

Some simple computations show the following result.

**Proposition 1.2.** Let \(f \in S_d(\Omega^2)\) and \(i_f, j_f, Q_f, \) and \(H_f\) be the invariants and concomitants defined above.

(a) \((f, f)^{(1)} = 2H_f, (f, f)^{(2}) = 2i_f, \) and \((f, f)^{(k)} = 0\) for \(k = 1, 3.\)

(b) \((H_f, f)^{(1)} = -\frac{1}{2}Q_f, (H_f, f)^{(2)} = \frac{1}{6}i_f, (H_f, f)^{(3)} = 0, \) and \((H_f, f)^{(4)} = 3j_f.\)

In order to classify the real fourth-order binary forms we shall use the classification of the real binary forms of the second- and third-order.

For each \(f \in S_2(\Omega^2),\) we define the discriminant of \(f, D_f,\) as \(D_f = ac - b^2\) being \(f = ax^2 + 2bxy + cy^2.\)

**Theorem 1.3.** For each real quadratic binary form \(f,\) there exists some \(\sigma \in GL(2; \mathbb{R})\) which transforms \(f\) in one and only one of the following canonical forms:

I. \(f = \alpha(x^2 + y^2), \quad \alpha = \pm 1\) with \(D_f > 0, \alpha f > 0.\)

II. \(f = x^2 - y^2,\) with \(D_f < 0.\)

III. \(f = \alpha x^2, \quad \alpha = \pm 1,\) with \(D_f = 0, \alpha f > 0.\)

IV. \(f = 0.\)

For a proof see [Gu, p. 252].

For each \(f \in S_3(\Omega^2),\) we define the discriminant of \(f, D_f,\) and the hessian of \(f, H_f,\) as

\[D_f = 3a_1^2a_2^2 + 6a_0a_1a_2a_3 - 4a_0a_2^3 - 4a_1^3a_3 - a_2^2a_3^2,\]

\[H_f = (a_0a_2 - a_1^2)x^2 + (a_0a_3 - a_1a_2)xy + (a_1a_3 - a_2^2)y^2,\]

where

\[f = a_0x^4 + 3a_1x^2y + 3a_2xy^2 + a_3y^3.\]
THEOREM 1.4. For each real cubic binary form $f$, there exists some $\sigma \in \text{GL}(2; \mathbb{R})$ which transforms $f$ in one and only one of the following canonical forms:

I. $f = x^3 + y^3$ with $D_f < 0$.

II. $f = x(x^2 - 3y^2)$ with $D_f > 0$.

III. $f = 3x^2y$ with $D_f = 0$ and $H_f \neq 0$.

IV. $f = x^3$ with $D_f = 0$ and $H_f = 0$ and $f \neq 0$.

V. $f = 0$.

For a proof see [Gu, pp. 263, 265].

2. Real Classification of the Fourth-Order Binary Forms

In order to do the real classification of the fourth-order binary forms $f \in S_4(\Omega^2)$, we shall use the Caley method. Consider the pencil of forms

$$f_\lambda = H_f - \lambda f.$$ (2.1)

LEMMA 2.1. (a) Let $f = \alpha \varphi^2$, where $\varphi$ is a quadratic form and $\alpha = \pm 1$. Then, $H_f = (\frac{1}{3}) D_\varphi \varphi^2$, where $D_\varphi$ is the discriminant of the form $\varphi$.

(b) $H_f = \lambda f$ implies $f = \alpha \varphi^2$, where $\varphi$ is a quadratic form and $\alpha = \pm 1$.

Proof. Assume $f = \alpha \varphi^2$ with $\varphi = ax^2 + 2bxy + cy^2$. Then,

$$H_f = \frac{4\alpha^2}{4a^2} \begin{vmatrix} \varphi_x^2 + \varphi \varphi_{xx} & \varphi_y \varphi_x + \varphi \varphi_{xy} \\ \varphi_x \varphi_y + \varphi \varphi_{xy} & \varphi_y^2 + \varphi \varphi_{yy} \end{vmatrix}$$

$$= \frac{1}{36} [\varphi^2(\varphi_{xx} \varphi_{yy} - \varphi_{xy}^2) + \varphi(\varphi_x^2 \varphi_{yy} - 2\varphi_x \varphi_y \varphi_{xy})]$$

$$= \frac{1}{36} [4D_\varphi \varphi^2 + \varphi(8c(ax + by)^2 + 8a(bx + cy)^2 - 16b(ax + by)(bx + cy))]$$

$$= \frac{1}{36} [4D_\varphi \varphi^2 + 8D_\varphi \varphi^2]$$

$$= \frac{1}{3} [D_\varphi \varphi^2].$$

and (a) is proved.

Now, assume $H_f = \lambda f$. This means that $f = \psi^2$, where $\psi$ is a quadratic form with complex coefficients (see Theorem 25.2, p. 288 of [Gu]). Let
\[ \psi = ax^2 + bxy + cy^2. \]
Then \( a_0 = a^2, \ a_1 = 2ab, \ a_2 = b^2 + 2ac, \ a_3 = 2bc, \) and
\( a_4 = c^2, \) where \( f(x, y) = a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 xy^3 + a_4 y^4. \)
From these equalities, it is easy to check that either \( a, b, c \in \mathbb{R} \) or \( a = ai, \ b = bi, \ c = ci \) with \( a, b, c \in \mathbb{R}. \) So, either \( \psi^2 = \varphi^2 \) with \( \varphi = ax^2 + bxy + cy^2 \) or \( \psi^2 = -\varphi^2 \) with \( \varphi = ax^2 + bxy + cy^2. \) In every case, \( f = \alpha \varphi^2 \) with \( \alpha = \pm 1 \) and \( \varphi \) a real quadratic binary form. \( \Box \)

**Lemma 2.2.** \( H_f = 0 \) if and only if \( f = \alpha l^4, \) where \( l \) is a linear form and \( \alpha = \pm 1. \)

**Proof.** Assume \( f = \alpha l^4. \) Then \( f = \alpha \varphi^2 \) with \( \varphi = l^2 \) and \( D_\varphi = 0. \) By Lemma 2.1(a), \( H_f = \frac{1}{2} D_\varphi \varphi^2 = 0. \) Now, assume \( H_f = 0. \) Then \( f = \psi^2, \) where \( \varphi \) is a linear form with complex coefficients (see [Gu, Theorem 5.3, p. 288]). Let \( \psi = ax + by. \) Then, \( a_0 = a^4, \ a_1 = a^3 b, \ a_2 = a^2 b^2, \ a_3 = a b^3, \) and \( a_4 = b^4, \) where \( f = a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 xy^3 + a_4 y^4. \) Writing \( a = m + ni \) and \( b = p + qi, \) some simple computations show that either \( a = ni \) and \( b = qi, \) or \( a = m \) and \( b = p, \) or \( a = m(1 + i) \) and \( b = p(1 + i), \) or \( a = m(1 - i) \) and \( b = p(1 - i). \) From \( f = \psi^4 \) it follows that either \( f = (nx + qy)^4, \) or \( f = (mx + py)^4, \) or \( f = -4(mx + ny)^4. \) In every case, \( f = \alpha l^4 \) with \( \alpha = \pm 1 \) and \( l \) a real linear binary form. \( \Box \)

For each \( f \in S_4(\mathbb{R}^2), \) we define the cubic resolvent \( \omega(\lambda) \) as \( \omega(\lambda) = 4\lambda^3 - if\lambda + j_f. \) Notice that the discriminant of \( \omega(\lambda), \ D_\omega = \frac{15}{27}(i_f^3 - 27j_f^2) \) differs only in a positive constant factor from the discriminant of \( f, D_f. \)

**Lemma 2.3.** \( \lambda \) is a root of the cubic resolvent if and only if there exists a quadratic form \( \varphi \) for which \( f, = H_f - \lambda f = \alpha \varphi^2, \) where \( \alpha = \pm 1. \)

**Proof.** Denoting by \( H_\lambda \) the hessian of the form \( f, = H_f - \lambda f, \) we obtain from (1.2), (1.3), and Proposition 1.2(a) that
\[
H_\lambda = \frac{1}{2} (f, f, f)^{(2)} = \frac{1}{2} (H_f - \lambda f, H_f - \lambda f)^{(2)}
= \frac{1}{2} (H_f, H_f)^{(2)} - \lambda (H_f, f)^{(2)} + \frac{1}{2} \lambda^2 (f, f)^{(2)}.
\]
From Proposition 1.2, it follows that \( H_\lambda = H_{if} - \frac{1}{6} i_f f_{ix} + H_f \lambda^2. \) Then from Proposition 1.1(b), we have that
\[
H_\lambda = (\lambda^2 - \frac{1}{12} i_f) H_f + \left( \frac{1}{4} i_f - \frac{1}{6} \lambda i_f \right) f.
\]
(2.2)
Now, assume \( \omega(\lambda) = 0. \) Then,
\[
H_\lambda = (\lambda^2 - \frac{1}{12} i_f) H_f + \left[ - (\lambda^2 - \frac{1}{12} i_f) \lambda + (\lambda^2 - \frac{1}{12} i_f) \frac{4}{3} i_f \right] f
= (\lambda^2 - \frac{1}{12} i_f)(H_f - \lambda f) + (\lambda^3 - \frac{4}{3} i_f \lambda + \frac{1}{2} i_f f, f).
\]
(2.3)
By Lemma 2.1(b), the "if" part follows.
Now, assume \( f_\lambda = \alpha \phi^2 \). By Lemma 2.1(a), we obtain that \( H_\lambda = \frac{1}{3} D_\phi \phi^2 = \frac{1}{3} \alpha D_\phi (H_f - \lambda f) \). From (2.2), we obtain

\[
\lambda^2 - \frac{1}{4} f = \frac{1}{3} \alpha D_\phi, \quad \frac{1}{4} f - \frac{1}{6} \lambda = - \frac{1}{3} \alpha D_\phi \lambda.
\]

From both it follows that \( \omega(\lambda) = 0 \).

**Lemma 2.4.** Let \( \lambda_1, \lambda_2, \lambda_3 \) be the roots of the cubic resolvent \( \omega(\lambda) = 4\lambda^3 - i_f \lambda + j_f \). Then

\[
(H_f - \lambda_1 f)(H_f - \lambda_2 f)(H_f - \lambda_3 f) = -\frac{1}{4} Q_f^2,
\]

(2.4)

\( H_f - \lambda_1 f, H_f - \lambda_2 f, H_f - \lambda_3 f \) are the roots of the equation

\[
\Phi(\theta) = \theta^3 - 3H_f \theta^2 + \frac{1}{4}(12H_f^2 - i_f f^2) \theta + \frac{1}{4} Q_f^2 = 0,
\]

(2.5)

and

\[
\omega'(\lambda_i) = 4\alpha_i D_\phi, \quad \omega(\lambda_i) = 0 \quad \text{if} \quad H_f - \lambda_i f = \alpha_i \phi_i^2 \quad \text{(see Lemma 2.3)}.
\]

**Proof.** Since \( \lambda_1, \lambda_2, \lambda_3 \) are the roots of \( \omega(\lambda) \), \( \lambda_1 + \lambda_2 + \lambda_3 = 0 \), \( \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = -\frac{1}{4} f \), and \( \lambda_1 \lambda_2 \lambda_3 = -\frac{1}{4} j_f \). So, from the syzygy given in Proposition 1.1(a), we have

\[
(H_f - \lambda_1 f)(H_f - \lambda_2 f)(H_f - \lambda_3 f) = -\frac{1}{4} Q_f^2.
\]

(2.5)

By using the above syzygy, a simple computation shows that \( \omega(\lambda) = 4\lambda^3 - i_f \lambda + j_f = 0 \) implies that \( H_f - \lambda f \) is a root of \( \Phi(\theta) = 0 \).

By Lemma 2.1(a), if \( H_f - \lambda_i f = \alpha_i \phi_i^2 \) then \( H_{\lambda_i} = \frac{1}{3} D_\phi \phi_i^2 = \frac{1}{3} \alpha_i D_\phi \). (2.3) we have \( H_{\lambda_i} = (\lambda_i^2 - \frac{1}{4} f)(H_f - \lambda_i f) = \frac{1}{12} \omega'(\lambda_i)(H_f - \lambda_i f) \). So, \( \omega'(\lambda_i) = 4\alpha_i D_\phi \).

**Lemma 2.5.** Let \( D_\phi \) be the discriminant of \( f \). Then, \( D_\phi \neq 0 \) implies that the form \( H_f - \lambda_i f \) splits into real linear factors at least for one of the roots of the resolvent. In fact, \( H_f - \lambda_i f = \alpha \phi^2 \) with \( \alpha = \pm 1 \), \( \phi \) a quadratic form, and \( D_\phi < 0 \).

**Proof.** First assume \( D_\phi < 0 \). This implies that \( D_\omega < 0 \) and by Theorem 1.4, \( \omega \) has a unique real root \( \lambda_1 \). Then

\[
\omega(\lambda) = (\lambda - \lambda_1)(4\lambda^2 + 4\lambda_1 \lambda + 4\lambda_1^2 - i_f) = (\lambda - \lambda_1) g(\lambda),
\]
with
\[
D_\theta = \begin{vmatrix} 4 & 2i_i \\ 2i, & 4i_i^2 - i_i \end{vmatrix} = 4(3i_i^2 - i_i) > 0.
\]

This implies that \( \omega'(i_1) = 12i_1^2 - i_i > 0 \). By Lemmas 2.3 and 2.4, \( \omega'(i_1) = 4x_1D_{\varphi_1} \) and so
\[
\alpha_1D_{\varphi_1} > 0. \quad (2.7)
\]

On the other hand, write \( \lambda_2 = a + bi, \lambda_3 = a - bi \). Then, \((Hf - \lambda_2 f)(Hf - \lambda_3 f) = [(Hf - af) - bif][(Hf - af) + bif] = (Hf - af)^2 + b^2f^2 > 0\). Since \( \prod_{i=1}^{3} (Hf - \lambda_if) = -\frac{3}{4}Q_i^2 < 0 \) (see Lemma 2.4), we obtain that \( Hf - \lambda_if < 0 \) and so \( Hf - \lambda_if = \alpha_i\varphi_i^2 \) with \( \alpha_1 = -1 \). From (2.7), \( D_{\varphi_1} < 0 \).

By Theorem 1.3, \( D_{\varphi_1} < 0 \) implies that \( \varphi_1 \) splits into real linear factors and, so, the same is true for \( Hf - \lambda_if \).

Now suppose \( D_{\varphi_1} > 0 \). This implies that \( D_{\omega} > 0 \) and by Theorem 1.4, \( \omega \) has three real roots \( \lambda_1, \lambda_2, \lambda_3 \). Assume \( \lambda_1 < \lambda_2 < \lambda_3 \) and let \( \varphi_i \) be such that \( Hf - \lambda_if = \alpha_i\varphi_i^2 \) for \( i = 1, 2, 3 \) (see Lemma 2.3). Suppose that \( D_{\varphi_i} > 0 \) for \( i = 1, 2, 3 \). Since \( \omega(\lambda) = 0 \) has three real roots, the signs of \( \omega'(\lambda_i) \) for \( i = 1, 2, 3 \) are alternate. Since the sign of \( \omega'(\lambda_i) \) is equal to the sign of \( \alpha_i \) (see Lemma 2.4), we have that
\[
\omega'(\lambda_1) > 0, \quad \omega'(\lambda_2) < 0, \quad \omega'(\lambda_3) > 0. \quad (2.8)
\]

If not, \( \prod_{i=1}^{3} (Hf - \lambda_if) = \prod_{i=1}^{3} \varphi_i^2 > 0 \) in contradiction with (2.4). On the other hand, from Lemma 2.4, we have \( \Phi'(\theta) = 3\theta^2 - 6Hf\theta + 3H^2f^2 - \frac{3}{4}i_if^2 \). Taking \( \theta_i = Hf - \lambda_if \) we obtain \( \Phi'(\theta_i) = (3\lambda_i^2 - \frac{1}{4}i_i)f^2 = \frac{3}{4}\omega'(\lambda_i)f^2 \). Therefore, from (2.8) we deduce
\[
\Phi'(Hf - \lambda_1 f) > 0, \quad \Phi'(Hf - \lambda_2 f) < 0, \quad \Phi'(Hf - \lambda_3 f) > 0. \quad (2.9)
\]

Since we have that \( Hf - \lambda_1 f = \varphi_1^2 > 0, Hf - \lambda_2 f = \varphi_2^2 > 0, \) and \( Hf - \lambda_3 f = -\varphi_3^2 < 0 \), from Lemma 2.4 we have a contradiction with (2.9). So, there exists some \( i \in \{1, 2, 3\} \) with \( D_{\varphi_i} < 0 \) (note that since \( \omega'(\lambda_i) = 4\alpha_iD_{\varphi_i} \), from (2.8) it follows that \( D_{\varphi_i} \neq 0 \)). This implies that \( \varphi_i \) splits into real factors and, therefore, the same is true for \( Hf - \lambda_if \).

**Theorem 2.6.** For each real fourth-order binary form \( f \), there exists some \( \sigma \in \text{GL}(2; \mathbb{R}) \) which transforms \( f \) in one and only one of the following canonical forms:
I. $f = x^4 + 6\mu x^2 y^2 + y^2, \quad \mu < -\frac{1}{3}$  
with $D_f > 0, H_f < 0, 12H_f^2 i f^2 > 0$.

II. $f = \alpha(x^4 + 6\mu x^2 y^2 + y^4), \quad \alpha = \pm 1, \quad \mu > -\frac{1}{3}, \quad \mu \neq \frac{1}{3}$  
with $D_f > 0, \alpha f > 0$ and $H_f > 0$ or $12H_f^2 - i f^2 < 0$.

III. $f = x^4 + 6\mu x^2 y^2 - y^4, \quad \alpha = \pm 1$  
with $D_f < 0$.

IV. $f = \alpha y^2(6x^2 + y^2), \quad \alpha = \pm 1$  
with $D_f = 0, \alpha j f < 0, 2i f H_f - 3j f > 0$.

V. $f = \alpha y^2(6x^2 - y^2), \quad \alpha = \pm 1$  
with $D_f = 0, \alpha j f < 0, 2i f H_f - 3j f < 0$.

VI. $f = \alpha(x^2 + y^2)^2, \quad \alpha = \pm 1$  
with $D_f = 0, \alpha j f > 0, 2i f H_f - 3j f f = 0, H_f > 0$.

VII. $f = 6\alpha x^2 y^2, \quad \alpha = \pm 1$  
with $D_f = 0, \alpha j f < 0, 2i f H_f - 3j f f = 0, H_f < 0$.

VIII. $f = 4x^3 y, \quad \alpha = \pm 1$  
with $D_f = 0, j f = 0, i f = 0, H_f \neq 0$.

IX. $f = \alpha x^4, \quad \alpha = \pm 1$  
with $D_f = 0, j f = 0, i f = 0, H_f = 0, \alpha f > 0$.

X. $f = 0$.

Proof: Let $f$ be a fourth-order binary form such that

$$f(x, y) = a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 x y^3 + a_4 y^4.$$  

Consider the pencil of forms $f_\lambda = H_f - \lambda f$ and the cubic resolvent $\omega(\lambda) = 4\lambda^3 - i f \lambda + j f$. In what follows, we shall distinguish several cases.

Case 1. The resolvent has no multiple roots: i.e., $D_\omega \neq 0$ (see Theorem 1.4). Then, $D_f \neq 0$ and by Lemma 2.5 we know that $H_f - \lambda f$ splits into real linear factors for at least one root of $\omega(\lambda)$; i.e., $H_f - \lambda f = \alpha \varphi^2$ with $\alpha = \pm 1$, $\varphi$ a quadratic binary form, and $D_\varphi < 0$. From Theorem 1.3 there exists a linear transformation of the variables which reduces $\varphi$ to the form $\varphi = \sqrt{6} xy$. Consequently, $f_\lambda = \alpha \varphi^2 = 6\alpha x^2 y^2$. By Proposition 1.2 and (1.3) we have

$$(H_f - \lambda f, f)^{(3)} = (H_f, f)^{(3)} - \lambda(f, f)^{(3)} = 0,$$
or equivalently

\[(H_f, \Delta f, f)^{(3)} = 6x^2y^2, f)^{(3)} = 3x(a_1x^2 a_3y^2) = 0.\]

Thus, in the new variables

\[f = a_0x^4 + 6a_2x^2y^2 + a_4y^4,\]

where

\[D_f = i^3 - 27j^2 = a_0a_4(a_0a_4 - 9a_2^2) \neq 0.\]

**Case 1.1.** $D_f > 0$. Then $a_0a_4 > 0$ and we do the change of variables $X = (a_0)^{1/4}x$, $Y = (a_4)^{1/4}y$, where $\alpha = \pm 1$ depending on the sign of $a_0$ and $a_4$. Denoting by $\mu = n\alpha_2/\sqrt{a_0a_4}$, we obtain

\[f = \alpha(x^4 + 6\mu x^2y^2 + y^4),\]

with

\[H_f = \mu x^4 + (1 - 3\mu^2)x^2y^2 + \mu y^4,\]

and

\[\omega(\lambda) = 4\lambda^3 - (1 + 3\mu^2)\lambda + \alpha(\mu - \mu^3),\]

where we substitute $X$ and $Y$ by $x$ and $y$, respectively. The cubic resolvent has the roots $\alpha \mu$, $(1 - \alpha \mu)/2$, and $-(1 + \alpha \mu)/2$ and the following hold:

\[H_f - \omega f = (1 - 9\mu^2)x^2y^2,\]

\[H_f - \left[(1 - \alpha \mu)/2\right] f = \left[(3\mu - \alpha)/2\right](x^2 - \alpha y^2)^2,\]

\[H_f + \left[(1 + \alpha \mu)/2\right] f = \left[(3\mu + \alpha)/2\right](x^2 + \alpha y^2)^2.\]  

Notice that $\mu \neq \pm 1/2$ since for $\mu = \pm 1/2$, $\alpha_2/\sqrt{a_0a_4} = \pm 1/2$; i.e., $3\alpha_2 = \pm \sqrt{a_0a_4}$ which implies that $D_f = 0$.

Assume $\mu < -1/2$. Then $f = \alpha(x^2 - Ay^2)(x^2 - By^2)$ with $A = -3\mu + \sqrt{9\mu^2 - 1} > 0$, $B = -3\mu - \sqrt{9\mu^2 - 1} > 0$. Hence, $f$ determines four real points. On the other hand, $\mu < -1/2$ implies that $1 - 9\mu^2 < 0$, $3\mu - \alpha < -1 - \alpha \leq 0$, and $3\mu + \alpha < -1 + \alpha \leq 0$. So, from (2.10), $\alpha_1 = \alpha_2 = \alpha_3 = -1$ and the three roots of $\Phi(\theta)$ are negatives. By applying Descartes' theorem we obtain that $H_f < 0$ and $12H_f^2 - i_f f^2 > 0$. By the properties of the invariants and concomitants, it is clear that the inequalities $D_f > 0$, $H_f < 0$, and $12H_f^2 - i_f f^2 > 0$ characterize the fourth-order binary forms with four real roots.
To complete Case I of the theorem, assume \( f = -x^4 - 6\mu x^2 y^2 - y^4 \) with \( \mu < -\frac{1}{3} \). The change \( x = X + Y \) and \( y = Y - X \) transforms \( f \) in \( -(6\mu + 2) X^4 - 12(1 - \mu) X^2 Y^2 - (6\mu + 2) Y^4 \). Now, calling \( x_1 = [( - (6\mu + 2)]^{1/4} X \) and \( y_1 = [( - (6\mu + 2)]^{1/4} Y \) we obtain \( x_1^4 + 6\mu x_1^2 y_1^2 + y_1^4 \) with \( \hat{\mu} = (1 - \mu)/(1 + 3\mu) < -\frac{1}{3} \).

Assume \( \mu \in (-\frac{1}{3}, \frac{1}{3}) \). Then \( f = \psi(x^2 + 2axy + y^2)(x^2 - 2axy + y^2) = \alpha \psi^2 \), where \( a^2 = \frac{1}{2} - \frac{3}{2} \mu > 0 \), \( D_\psi = 1 - a^2 = \frac{1}{2} \mu + \frac{1}{2} > 0 \), and \( D_\psi = 1 - a^2 > 0 \). Hence, \( f \) determines four complex (not real) points (see Theorem 1.3). On the other hand, \( \mu \in \left( -\frac{1}{3}, \frac{1}{3} \right) \) implies that \( 1 - 9\mu^2 > 0 \). Since \( \prod_{i=1}^{3} (H_f - \lambda_i f) = \prod_{i=1}^{3} \alpha_i \phi_i^2 < 0 \) (see Lemma 2.4), by (2.10) we obtain that \( \Phi(\theta) \) has two positive roots and only one is negative. So, we must compute two changes of sign in \( \Phi(\theta) \); i.e., \( H_f > 0 \) or \( 12H_f^2 - i_f f^2 < 0 \). In this case \( f \) is of Type II.

Assume \( \mu \in (\frac{1}{3}, + \infty) \). Then \( f = \psi(x^2 - Ay^2)(x^2 - By^2) \) with \( A \) and \( B \) defined as in the case \( \mu < -\frac{1}{3} \) but now \( A < 0 \) and \( B < 0 \). Hence, \( f \) determines four complex (not real) points. Since \( 3\mu + \alpha > 1 + \alpha \geq 0 \), by (2.10) \( \Phi(\theta) \) has at least a positive root. The same argument as above gives \( H_f > 0 \) or \( 12H_f^2 - i_f f^2 < 0 \) and \( f \) is of Type II.

**Case 1.2.** \( D_f < 0 \). Then \( a_0 a_4 < 0 \). If \( a_0 > 0 \) and \( a_4 < 0 \), by the linear change \( X = a_0^{1/4} x, Y = (-a_4)^{1/4} y \), \( f \) adopts the form \( f = x^4 + 6\mu x^2 y^2 - y^4 \), where \( \mu = a_2/\sqrt{-a_0 a_4} \). If \( a_0 < 0 \) and \( a_4 > 0 \), by the linear change \( X = a_0^{1/4} x, Y = (-a_4)^{1/4} x \), \( f \) adopts the same form. In whichever case, \( f = x^4 + 6\mu x^2 y^2 - y^4 \) with \( \mu \in \mathbb{R} \). Then, \( f = (x^2 - Ay^2)(x^2 - By^2) \), where \( A = \sqrt{9\mu^2 + 1 - 3\mu} > 0 \), \( B = -\sqrt{9\mu^2 + 1 + 3\mu} < 0 \). Hence, \( f \) has two real and two complex roots. So, \( f \) is of Type III.

**Case 2.** The resolvent has a double root and a simple root. Then \( \omega(\lambda) = 27j_i^2 = 0, i_f \neq 0, j_f \neq 0 \) (see Theorem 1.4). Furthermore, if \( \omega(\lambda) = \omega'(\lambda) = 0 \) then \( \lambda_1 = 3j_f/2i_f \) and the other root is \( \lambda_2 = -3j_f/2i_f \). Since \( \omega'(\lambda) = 0 \), by (2.6), \( D_\psi = 0 \) where \( f_{\lambda_1} = \alpha_1 \phi_1^2 \). Now, by Lemma 2.1(a), \( H_{\lambda_1} = \frac{1}{3} D_\psi \phi_1^2 = 0 \). From Lemma 2.2, \( f_{\lambda_1} = H_f - \lambda_1 f = \alpha t^4 \), where \( \alpha = \pm 1 \) and \( l \) is a linear form. Since \( \lambda_1 = 3j_f/2i_f \), we obtain

\[
f_{\lambda_1} = H_f - \frac{3j_f}{2i_f} f = \frac{1}{2i_f} (2i_f H_f - 3j_f f).
\]

**Case 2.1.** \( 2i_f H_f - 3j_f f > 0 \). Since \( i_f^3 = 27j_f^2 \) we obtain \( i_f > 0 \) and so \( H_f - \lambda_1 f = [(1/(2i_f))](2i_f H_f - 3j_f f) = t^4 \). Let \( \beta = \pm 1 \) be such that \( \beta j_f < 0 \). Then,

\[
\frac{2i_f H_f - 3j_f f}{-9\beta j_f} = \frac{2i_f}{-9\beta j_f} t^4 = \left[ \frac{2i_f}{-9\beta j_f} \right]^4 = \left[ \frac{2i_f}{-9\beta j_f} \right]^4.
\]
and we can do a change of variables to obtain
\[ 2i_f H_f - 3j_f f = -9\beta_f y^4. \] (2.11)

Now, \((H_f - \lambda f, f)^{(3)} = 0\) (see Proposition 1.2) implies \((y^4, f)^{(3)} = 0\) and so \(a_0 = a_1 = 0\) (see (1.1)). Therefore \(f\) (and consequently \(H_f\)) is divisible by \(y^2\). Since the root \(\lambda_2\) is simple, we have \(\omega'(\lambda_2) = \alpha_2 \alpha_0^2 \neq 0\); so, by (2.6), \(D_{\phi_2} \neq 0\) and by Lemma 2.1(a), \(H_{\phi_2} \neq 0\). So, \(H_f - \lambda_2 f = \alpha_2 \alpha_0^2 \neq 0\) but it is not \(\alpha f^4\) with \(l\) linear. On the other hand, since \(\prod_{i=1}^{3} (H_f - \lambda_i f) = -\frac{1}{4}Q_f < 0\), we obtain \(H_f - \lambda_2 f < 0\) and, so, \(H_f - \lambda_2 f = (i_f H_f + 3j_f f)/i_f = -y^2l^2\), where \(l\) is linear and \(l(x, y) \neq 0\) for \(y = 0\). If \(\beta f < 0\) as before, we have that \((i_f H_f + 3j_f f)/27\beta f y^2 = (-i_f/27\beta f)l^2\). Now we can select a new variable \(x\) such that
\[ i_f H_f + 3j_f f = 27\beta f x^2 y^2. \] (2.12)

It follows from (2.11) and (2.12) that \(f = \beta(6x^2 y^2 + y^4)\) with \(\beta = \pm 1\), \(\beta f < 0\), \(D_f = 0\), and \(2i_f H_f - 3j_f f > 0\). So, \(f\) is of Type IV.

**Case 2.2.** \(2i_f H_f - 3j_f f < 0\). Taking into account the signs, we have, in a similar way
\[ 2i_f H_f - 3j_f f = 9\beta f y^4, \] (2.13)
and
\[ i_f H_f + 3j_f f = 27\beta f x^2 y^2, \] (2.14)
which together tell us that \(f = \beta(6x^2 y^2 - y^4)\) with \(\beta = \pm 1\), \(\beta f < 0\), \(D_f = 0\), and \(2i_f H_f - 3j_f f < 0\). So, \(f\) is of Type V.

**Case 2.3.** \(2i_f H_f - 3j_f f = 0\). Then \(H_f - \lambda f = 0\); i.e., \(H_f = \lambda f\). By Lemma 2.1(b), \(f = ax^2\) with \(x = \pm 1\) and \(a\) a quadratic binary form. Notice that \(D_{\psi} \neq 0\), if not by Lemma 2.1(a), \(H_f = \frac{1}{4}D_{\phi} = 0\) would imply \(\lambda = 0\).

If \(D_{\psi} > 0\), by Theorem 1.3, \(\psi\) can be transformed in \(\psi = \beta(x^2 + y^2)\). Consequently \(f = \alpha(x^2 + y^2)^2\) with \(H_f = \frac{1}{4}D_{\psi}(x^2 + y^2)^2 > 0\). Indeed, since \(i_f > 0\), from \(2i_f H_f - 3j_f f > 0\) we obtain \(a f > 0\) and \(f\) is of Type VI.

If \(D_{\psi} < 0\), by Theorem 1.3, \(\psi\) can be transformed into \(\psi = \sqrt{6} \beta xy\). Consequently \(f = 6\alpha x^2 y^2\) with \(H_f = 2D_{\psi} x^2 y^2 < 0\). Furthermore, since \(i_f > 0\), from \(2i_f H_f - 3j_f f < 0\) we obtain \(a f < 0\) and \(f\) is of Type VII.

**Case 3.** The resolvent has a triple root. That is, \(i_f = 0\), \(j_f = 0\), and this root is equal to zero. So \(\omega'(0) = 0\) and therefore \(H_f = \alpha l^4\) with \(x = \pm 1\) and \(l\) a linear form.

Now, assume \(H_f \neq 0\) and write \(H_f = ax^4\). Since \(i_f = 0\) by Proposition 1.2 \((H_f, f)^{(2)} = 8i_f f = 0\), \((H_f, f)^{(2)} = (ax^4, f)^{(2)} = 0\) implies \(a_2 = a_3 = a_4 = 0\);
HOMOGENEOUS CUBIC VECTOR FIELDS

i.e., $f$ is divisible by $x^3$ and by change of variables $f$ reduces to the form $f = 4x^3y$. So, $f$ is of Type VIII.

Now, assume $H_f = 0 \neq f$. By Lemma 2.2, $f = zm^n$ with $z = \pm 1$ and $m$ linear. So, $f$ can be reduced to the form $f = \alpha x^4$ and is of Type IX.

3. ALGEBRAIC CLASSIFICATION OF HOMOGENEOUS CUBIC VECTOR FIELDS

Let $X = (P, Q)$ be a homogeneous cubic vector field on the plane. Let $\dot{x} = P(x, y), \dot{y} = Q(x, y)$ be the system of differential equations associated to $X$. We shall write

$$
\dot{x} = P(x, y) = P_{111} x^3 + 3P_{112} x^2 y + 3P_{122} xy^2 + P_{222} y^3, \\
\dot{y} = Q(x, y) = Q_{111} x^3 + 3Q_{112} x^2 y + 3Q_{122} xy^2 + Q_{222} y^3. 
$$

Consider the fourth-order binary form

$$
F(x, y) = xQ(x, y) - yP(x, y).
$$

It is well known that the directions determined by the zeros of $F(x, y)$ are the only possible ones at which the orbits of $X$ come back or reach infinity (see, for instance, [Go] or [S]).

**Lemma 3.1.** Let $X = (P, Q)$ be a homogeneous cubic vector field on the plane. If $F(x, y) = a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 xy^3 + a_4 y^4$, then there exist $p_1, p_2, p_3$ such that system (3.1) becomes

$$
\dot{x} = (p_1 - a_1) x^3 + (p_2 - 3a_2) x^2 y + (p_3 - 3a_3) xy^2 - a_4 y^3, \\
\dot{y} = a_0 x^3 + (3a_1 + p_1) x^2 y + (3a_2 + p_2) xy^2 + (a_3 + p_3) y^3. 
$$

**Proof:** Let $p_1, p_2, p_3$ be defined by

$$
p_1 = \frac{3}{4} [P_{111} + P_{112}^2], \quad p_2 = \frac{3}{2} [P_{112} + P_{122}^2], \\
p_3 = \frac{3}{4} [P_{122} + P_{222}^2],
$$

and let $Q_{111}^i$ and $Q_{112}^j$ with $i, j, k \in \{1, 2\}$ be defined by

$$
P(x, y) - x(p_1 x^2 + p_2 xy + p_3 y^2) = Q_{111}^1 x^3 + 3Q_{112}^1 x^2 y + 3Q_{122}^1 xy^2 + Q_{222}^1 y^3, \\
Q(x, y) - y(p_1 x^2 + p_2 xy + p_3 y^2) = Q_{111}^2 x^3 + 3Q_{112}^2 x^2 y + 3Q_{122}^2 xy^2 + Q_{222}^2 y^3.
$$
A simple computation gives that
\[ Q_{111}^2 = P_{111}^2, \]
\[ Q_{111}^1 = -Q_{112}^1 = \frac{1}{4}[P_{111}^1 - 3P_{112}^2], \]
\[ Q_{112}^1 = -Q_{122}^2 = \frac{1}{4}[P_{112}^1 - P_{122}^2], \]
\[ Q_{122}^1 = -Q_{222}^2 = \frac{1}{4}[3P_{122}^1 - P_{222}^2], \]
\[ Q_{222}^1 = P_{222}^1, \] (3.3)
and
\[ F(x, y) = Q_{111}^2 x^4 + (3Q_{112}^2 - Q_{111}^1) x^3 y + (3Q_{122}^2 - 3Q_{112}^1) x^2 y^2 \]
\[ + (Q_{222}^2 - 3Q_{122}^1) xy^3 - Q_{222}^1 y^4. \]

Therefore, we obtain the relations
\[ Q_{111}^2 = a_0, \quad Q_{111}^1 = -a_1, \quad Q_{112}^1 = a_2, \quad Q_{112}^2 = -a_2, \] (3.4)
\[ Q_{122}^2 = a_3, \quad Q_{122}^1 = -a_3, \quad Q_{222}^2 = a_4, \quad Q_{222}^1 = -a_4, \]
and system (3.1) can be written as in (3.2).

**Theorem 3.2.** For each homogeneous vector field \( X = (P, Q) \) of degree 3, there exists some \( \sigma \in GL(2; \mathbb{R}) \) and a change of time scale which transforms \( X \) in one and only one of the following canonical forms:

I'. \[ \begin{align*}
\dot{x} &= p_1 x^3 + (p_2 - 3\mu) x^2 y + p_3 x y^2 - y^3, \\
\dot{y} &= x^3 + p_1 x^2 y + (p_2 + 3\mu) x y^2 + p_3 y^3,
\end{align*} \]
with \( \mu < -\frac{1}{3} \).

II'. \[ \begin{align*}
\dot{x} &= p_1 x^3 + (p_2 - 3\mu) x^2 y + p_3 x y^2 - \alpha y^3, \\
\dot{y} &= x^3 + p_1 x^2 y + (p_2 + 3\mu) x y^2 + p_3 y^3,
\end{align*} \]
with \( \alpha = \pm 1, \mu > -\frac{1}{3}, \mu \neq \frac{1}{3} \).

III'. \[ \begin{align*}
\dot{x} &= p_1 x^3 + (p_2 - 3\mu) x^2 y + p_3 x y^2 + y^3, \\
\dot{y} &= x^3 + p_1 x^2 y + (p_2 + 3\mu) x y^2 + p_3 y^3,
\end{align*} \]

IV'. \[ \begin{align*}
\dot{x} &= p_1 x^3 + (p_2 - 3\alpha) x^2 y + p_3 x y^2 - \alpha y^3, \\
\dot{y} &= p_1 x^2 y + (p_2 + 3\alpha) x y^2 + p_3 y^3,
\end{align*} \]
with \( \alpha = \pm 1 \).

V'. \[ \begin{align*}
\dot{x} &= p_1 x^3 + (p_2 - 3\alpha) x^2 y + p_3 x y^2 + \alpha y^3, \\
\dot{y} &= p_1 x^2 y + (p_2 + 3\alpha) x y^2 + p_3 y^3,
\end{align*} \]
with \( \alpha = \pm 1 \).
VI'. \begin{align*}
\dot{x} &= p_1 x^3 + (p_2 - \alpha) x^2 y + p_3 x y^2 - \alpha y^3, \\
\dot{y} &= \alpha x^3 + p_1 x^2 y + (p_2 + \alpha) x y^2 + p_3 y^3,
\end{align*}
with \( \alpha = \pm 1 \).

VII'. \begin{align*}
\dot{x} &= p_1 x^3 + (p_2 - 3\alpha) x^2 y + p_3 x y^2, \\
\dot{y} &= p_1 x^2 y + (p_2 + 3\alpha) x y^2 + p_3 y^3,
\end{align*}
with \( \alpha = \pm 1 \).

VIII'. \begin{align*}
\dot{x} &= (p_1 - 1) x^3 + p_2 x^2 y + p_3 x y^2, \\
\dot{y} &= p_1 x^2 y + (p_2 + 3\alpha) x y^2 + p_3 y^3,
\end{align*}
with \( \alpha = \pm 1 \).

IX'. \begin{align*}
\dot{x} &= p_1 x^3 + p_2 x^2 y + p_3 x y^2, \\
\dot{y} &= \alpha x^3 + p_1 x^2 y + p_2 x y^2 + p_3 y^3,
\end{align*}
with \( \alpha = \pm 1 \).

X'. \begin{align*}
\dot{x} &= x(p_1 x^2 + p_2 x y + p_3 y^2), \\
\dot{y} &= y(p_1 x^2 + p_2 x y + p_3 y^2).
\end{align*}

Proof. Let \( X = (P, Q) \) be a homogeneous cubic vector field and let \( f(x, y) = xQ(x, y) - yP(x, y) \). By Theorem 2.6 there exists some \( \sigma \in GL(2; \mathbb{R}) \) which transforms \( f \) in \( f_c \); i.e., \( f \circ \sigma = f_c \), where \( f_c \) is one of the 10 canonical forms of fourth-order binary form. Let

\[
A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
\]

be the matrix of \( \sigma \) in the canonical basis and let \( A = \alpha \delta - \beta \gamma \neq 0 \). We denote by \( (X, Y) \) the coordinates in the new basis. So \( f_c(X, Y) = (f \circ \sigma)(X, Y) = f(x, y) \) with \( X = (1/A)[\delta x - \beta y], Y = (1/A)[\gamma x + \alpha y] \). The system in the coordinates \( (X, Y) \) is

\[
\dot{X} = \frac{1}{A}[\delta P(\sigma(X, Y)) - \beta Q(\sigma(X, Y))],
\]
\[
\dot{Y} = \frac{1}{A}[\gamma P(\sigma(X, Y)) + \alpha Q(\sigma(X, Y))].
\]

Doing the change of time \( dt = A \, ds \) we obtain a vector field \( \vec{F} = (\vec{P}, \vec{Q}) \) with \( \vec{f}(X, Y) = X\vec{Q}(X, Y) - Y\vec{P}(X, Y) = (1/A)[(\alpha \delta - \beta \gamma)(xQ(x, y) - yP(x, y))] = f(x, y) = f_c(X, Y) \). By Theorem 2.6 and Lemma 3.1, the theorem follows.  

Consider system (3.1) and let \( F(x, y) = xQ(x, y) - yP(x, y) \). To study the infinite critical points of \( X \) we consider the induced vector field \( p(X) \) on the Poincaré two-sphere. We want to give a new classification of the homogeneous cubic vector field on the plane such that every canonical form has all the infinite critical points on the local chart \( U_i \) (for more details see [Go] or [S]).
Corollary 3.3. For each homogeneous cubic vector field $X = (P, Q)$ there exist some $\sigma \in \text{GL}(2; \mathbb{R})$ and a change of time scale which transforms $X$ in one and only one of the following canonical forms:

1. \[
\begin{align*}
\dot{x} &= p_1 x^3 + [p_2 + 3(1 + \mu^4)] x^2 y + p_3 xy^2 - 6 \mu^2 y^3, \\
\dot{y} &= 6 \mu^2 x^2 + p_1 x^2 y + [p_2 - 3(1 + \mu^4)] xy^2 + p_3 y^3,
\end{align*}
\]
with $\mu > 1$.

2. \[
\begin{align*}
\dot{x} &= p_1 x^3 + [p_2 - (\frac{\alpha}{2})] x^2 y + p_3 xy^2 + 3y^4, \\
\dot{y} &= p_1 x^2 y + [p_2 + (\frac{\alpha}{2})] xy^2 + p_3 y^3, \\
\end{align*}
\]
with $\alpha = \pm 1$.

3. \[
\begin{align*}
\dot{x} &= p_1 x^3 + p_2 x^2 y + p_3 xy^2 + \mu y^3, \\
\dot{y} &= \mu x^3 + p_1 x^2 y + p_2 xy^2 + p_3 y^3, \\
\end{align*}
\]
with $\mu \neq 0$.

4. \[
\begin{align*}
\dot{x} &= p_1 x^3 + (p_2 - 3\alpha) x^2 y + (p_3 + 6\alpha) xy^2 - 6xy^3, \\
\dot{y} &= p_1 x^2 y + (p_2 + 3\alpha) xy^2 + (p_3 - 6\alpha) y^3, \\
\end{align*}
\]
with $\alpha = \pm 1$.

5. \[
\begin{align*}
\dot{x} &= p_1 x^3 + p_2 x^2 y + (p_3 + 2) xy^2 - 4y^3, \\
\dot{y} &= p_1 x^2 y + p_2 xy^2 + (p_3 + 2) y^3. \\
\end{align*}
\]

6. \[
\begin{align*}
\dot{x} &= p_1 x^3 + (p_2 - 3\alpha) x^2 y + p_3 xy^2 - xy^3, \\
\dot{y} &= p_1 x^2 y + (p_2 + 3\alpha) xy^2 + p_3 y^3, \\
\end{align*}
\]
with $\alpha = \pm 1$.

7. \[
\begin{align*}
\dot{x} &= p_1 x^3 + p_2 x^2 y + p_3 xy^2 - xy^3, \\
\dot{y} &= p_1 x^2 y + p_2 xy^2 + p_3 y^3, \\
\end{align*}
\]
with $\alpha = \pm 1$.

8. \[
\begin{align*}
\dot{x} &= p_1 x^3 + (p_2 - 3\alpha \mu) x^2 y + p_3 xy^2 - xy^3, \\
\dot{y} &= \alpha x^3 + p_1 x^2 y + (p_2 + 3\alpha \mu) xy^2 + p_3 y^3, \\
\end{align*}
\]
with $\alpha = \pm 1, \mu > -\frac{1}{3}, \mu \neq \frac{1}{3}$.

9. \[
\begin{align*}
\dot{x} &= p_1 x^3 + (p_2 - \alpha) x^2 y + p_3 xy^2 - xy^3, \\
\dot{y} &= \alpha x^3 + p_1 x^2 y + (p_2 + \alpha) xy^2 + p_3 y^3, \\
\end{align*}
\]
with $\alpha = \pm 1$.

10. \[
\begin{align*}
\dot{x} &= p_1 x^3 + p_2 x^2 y + p_3 xy^2, \\
\dot{y} &= p_1 x^2 y + p_2 xy^2 + p_3 y^3. \\
\end{align*}
\]

Proof. Let $F_k$ be the fourth-order binary form associated to system $k$. 

Then

\[ F_1(x, y) = 6\mu^2 x^4 - 6(1 + \mu^4) x^3 y^2 + 6\mu^2 y^4 \]
with \( \mu > 1 \),

\[ F_2(x, y) = \alpha y^2(x^2 - y^2) \]
with \( \alpha = \pm 1 \),

\[ F_3(x, y) = \mu(x^4 - y^4) \]
with \( \mu \neq 0 \),

\[ F_4(x, y) = 6\alpha y^2(y - x)^2 \]
with \( \alpha = \pm 1 \),

\[ F_5(x, y) = 4(y - x) y^3 \]

\[ F_6(x, y) = \alpha y^2(6x^2 + y^2) \]
with \( \alpha = \pm 1 \),

\[ F_7(x, y) = \alpha y^4 \]
with \( \alpha = \pm 1 \),

\[ F_8(x, y) = \alpha(x^4 + 6\mu x^2 y^2 + y^4) \]
with \( \alpha = \pm 1, \mu > -\frac{1}{3}, \mu \neq \frac{1}{3} \),

\[ F_9(x, y) = \alpha(x^2 + y^2)^2 \]
with \( \alpha = \pm 1 \),

\[ F_{10}(x, y) = 0. \]

Through the algebraic characterization (see Theorem 2.6) we can see that \( F_1 \) is linearly equivalent to \( I \), \( F_2 \) to \( V \), \( F_3 \) to \( III \), \( F_4 \) to \( VII \), \( F_5 \) to \( VIII \), \( F_6 \) to \( IV \), \( F_7 \) to \( IX \), \( F_8 \) to \( II \), \( F_9 \) to \( VI \), and \( F_{10} \) to \( X \). Hence, by Lemma 3.1 the corollary follows.

4. Study of the Phase-Portraits of Homogeneous Vector Fields

\( X = (P, Q) \) with \( \text{degree}(P) = \text{degree}(Q) \) and \( P \) and \( Q \) without Common Factor

**Proposition 4.1.** Let \( X = (P, Q) \) be a homogeneous polynomial vector field in the plane with degree \( (P) = \text{degree}(Q) = n \) and assume that \( P \) and \( Q \) have no common factor. Assume that \( F(x, y) = xQ(x, y) - yP(x, y) \) has some real linear factor. Then the following hold.
(a) The linear factors of $F(x, y)$ are invariants by the flow of $X$.

(b) $X$ has no limit cycles.

(c) The critical points at infinity are all elementals and they are nodes, saddles, or saddle-nodes. An infinite critical point on the local chart $U_1, (y, z) = (λ_1, 0)$ shall be a saddle-node if and only if $λ_1$ is a root of $f(λ) = Q(1, λ) - λP(1, λ)$ of even multiplicity. Furthermore, the orbits in the Poincaré sphere near a saddle-node are drawn in Fig. 4.1.

(d) The behaviour of the flow of $P(X)$ in a neighbourhood of infinity determines the phase-portrait of $X$.

Proof. We can assume that all the infinite critical points are on the local chart $U_1$. This implies that the real linear factors of $F(x, y)$ are of the form $y = λx$. So $F(x, λx) = xQ(x, λx) - λxP(x, λx) - x^{n+1}[Q(1, λ) - λP(1, λ)] = 0$, or equivalently $Q(1, λ) - λP(1, λ) = 0$. Then $\dot{y}(x, λx) = Q(x, λx) = x^nQ(1, λ) = x^nλP(1, λ) = λP(x, λx) = λ\dot{x}(x, λx)$, and (a) follows.

Since $P(x, y)$ and $Q(x, y)$ are the product of $n$ straight lines (real or complex), the unique finite critical point of $X$ is $(0, 0)$. Since $F(x, y)$ has some real linear factor, by (a) $X$ has some invariant straight line which passes through the origin. Hence no limit cycle can surround $(0,0)$ and (b) is proved.

Now, let $λ_1 < λ_2 < \cdots < λ_k$ be the real roots of $f(λ) = 0$. By the Poincaré compactification (see [Go] or [S]), $(y, z) = (λ_i, 0)$ are the critical points of $P(X)$ in $U_1$. Each one of them has, as a linear part,

$$
\begin{pmatrix}
 f'(λ_i) & * \\
 0 & -P(1, λ_i)
\end{pmatrix}
$$

Fig. 4.1. The behaviour of the orbits of $P(X)$ near a saddle-node at infinity (we can reverse the orientation of the orbits).
with $P(1, \lambda_i) \neq 0$ (otherwise, $P(1, \lambda_i) = 0$ and $f(\lambda_i) = Q(1, \lambda_i) - \lambda_i P(1, \lambda_i) = 0$ implies $Q(1, \lambda_i) = 0$; therefore $P$ and $Q$ have the common factor $y - \lambda_i x$, which is a contradiction). So each one of the infinite critical points is elemental. Now, assume that $\lambda_i$ is a simple root of $f(\lambda)$; i.e., $f'(\lambda_i) \neq 0$. Then $(\lambda_i, 0)$ is a non-degenerate critical point, and since the equator of $S^2$ is invariant by the flow of $p(X)$ it shall be a node or a saddle. If $f'(\lambda_i) = 0$, by Theorem 65 of [ALGM], $(\lambda_i, 0)$ shall be a node, a saddle, or a saddle-node, and this last possibility occurs if and only if $\lambda_i$ is a root of $f$ of even multiplicity. Again from this theorem the equator separates a saddle-node $(0, \lambda_i)$ with $P(1, \lambda_i) \neq 0$ as in Fig. 4.2. Taking into account the Poincaré compactification we obtain Fig. 4.1.

To see (d) let $y - \lambda_i x = 0$ and $y - \lambda_{i+1} x = 0$, two consecutive invariant rays of $X$. They determine one sector at the origin. From the homogeneity of $P$ and $Q$ if $(x(t), y(t))$ is a solution of $X$, the same is true for $(\lambda x(t), \lambda y(t))$. So, we obtain the following (see Fig. 4.3):

- two consecutive parabolic sectors at infinity give a hyperbolic sector at $(0, 0)$;
- two consecutive hyperbolic sectors at infinity give an elliptic sector at $(0, 0)$; and
- one parabolic and one hyperbolic consecutive sectors at infinity give a parabolic sector at $(0, 0)$.

In short (d) follows.

**Proposition 4.2.** Let $X = (P, Q)$ be a homogeneous polynomial vector field in the plane with $\text{deg}(P) = \text{deg}(Q) = n$ and assume that $P$ and $Q$
have no common factor. Assume that \( F(x, y) = xQ(x, y) - yP(x, y) \) has no real linear factors and let \( I \) be defined by

\[
I = \int_{-\infty}^{\infty} \frac{P(1, y)}{F(1, y)} \, dy.
\]

Then, the phase portrait of \( X \) is determined by the flow near \((0, 0)\). It is a global center if and only if \( I = 0 \), and it is a global stable (resp. unstable) focus if and only if \( I < 0 \) (resp. \( I > 0 \)).

**Proof.** The expression of system \( \dot{x} = P(x, y), \dot{y} = Q(x, y) \) in polar coordinates is

\[
\begin{align*}
\dot{r} &= r'' \left[ \cos \theta P(\cos \theta, \sin \theta) + \sin \theta Q(\cos \theta, \sin \theta) \right] = r'' f(\theta), \\
\dot{\theta} &= r'^{-1} \left[ \cos \theta Q(\cos \theta, \sin \theta) - \sin \theta P(\cos \theta, \sin \theta) \right] - r'^{-1} g(\theta).
\end{align*}
\]

If we introduce a new time \( s \) via \( ds/dt = r'^{-1} \), then the above system becomes

\[
\begin{align*}
r' &= r f(\theta), \\
\theta' &= g(\theta),
\end{align*}
\]

where the prime denotes derivative with respect to \( s \). Let \( J \) be the positive \( x \)-axis. For \( x \in J \) we denote by \( r(\theta, x) \) the solution of (4.1) satisfying \( r(0, x) = x \). We denote by \( \gamma^+(x) \) the positive orbit of (4.1) through the point \( r = x, \theta = 0 \). We shall consider the return function \( x \to h(x) \) for \( x \in J \), where \( h(x) \) is the first crossing of \( \gamma^+(x) \) with \( J \) after \( x \). We adopt the convention that \( h(0) = 0 \). Notice that since \( F(x, y) \) has no linear real factors, the domain of definition of \( h(x) \) is \( J \) (use Proposition 4.1, the Poincaré–Bendixson theorem, and the local phase-portrait at a critical point of an analytical planar vector field; see [ALGM]). Let \( S(r, \theta) = r f(\theta)/g(\theta) \). From the equality

\[
\frac{d}{d\theta} \left( \frac{\partial r}{\partial x} (\theta, x) \right) = \frac{\partial S}{\partial r} (r(\theta, x), \theta) \frac{\partial r}{\partial x} (\theta, x) = \frac{f(\theta)}{g(\theta)} \frac{\partial r}{\partial x} (\theta, x),
\]

Fig. 4.3. The behaviour at infinity determines the phase-portrait of \( X \).
and calling \( v(\theta, x) = (\partial r / \partial x)(\theta, x) \), we consider the linear differential equation \( dv/d\theta = ( f(\theta)/g(\theta) ) v \). Its solution is \( v(\theta) = k \exp \int_0^\theta (f(x)/g(x)) \, dx \). Since \( (\partial r / \partial x)(0, x) = 1 = v(0) \), we have that \( k = 1 \), and so \( v(\theta) = \exp \int_0^\theta (f(x)/g(x)) \, dx \). Since \( h'(x) = (\partial r / \partial x)((\theta, x)|_{\theta = 2\pi} = v(2\pi) \) we obtain \( h'(x) = \exp \int_{2\pi}^0 (f(x)/g(x)) \, dx \).

Since \( h'(x) \) does not depend on \( x \), if we call \( c = h'(x) \), from the analyticity of the return map we deduce that \( h(x) = cx \). In short if \( c = 1 \), the origin is a center, and if \( c > 1 \) (resp. \( c < 1 \)) the origin is an unstable (resp. stable) focus. Notice that \( c = 1 \) (resp., \( c > 1 \) or \( c < 1 \)) if and only if \( I' = 0 \) (resp., \( I' > 0 \) or \( I' < 0 \)), where \( I' \) is defined by \( I' = \int_0^{2\pi} (f(x)/g(x)) \, dx \).

To end the proof we must show that the sign of \( I' \) is equal to the sign of \( I \). Since \( g(\theta) = 0 \) has no real roots, \( n \) is odd. So the function \( f(x)/g(x) \) has period \( \pi \). Doing the change \( y = \tan x \) we obtain

\[
\int_0^{2\pi} \frac{f(x)}{g(x)} \, dx = 2 \int_{-\pi/2}^{\pi/2} \frac{f(x)}{g(x)} \, dx = 2 \int_{-\infty}^{\infty} \frac{P(1, y) + yQ(1, y)}{(1 + y^2)[Q(1, y) - yP(1, y)]} \, dy.
\]

On the other hand

\[
\int_{-\infty}^{\infty} \frac{P(1, y) + yQ(1, y)}{(1 + y^2)[Q(1, y) - yP(1, y)]} \, dy - \int_{-\infty}^{\infty} \frac{P(1, y)}{Q(1, y) - yP(1, y)} \, dy
\]

\[
= \int_{-\infty}^{\infty} \frac{[P(1, y) + yQ(1, y)] - (1 + y^2) P(1, y)}{(1 + y^2)[Q(1, y) - yP(1, y)]} \, dy
\]

\[
= \int_{-\infty}^{\infty} \frac{y}{1 + y^2} \, dy = 0.
\]

Hence the proposition follows.

**Proposition 4.3.** Let \( X = (P, Q) \) be a homogeneous polynomial vector field in the plane with \( \deg(P) = \deg(Q) = n \) and assume that \( P \) and \( Q \) have no common factor. Let \( i \) be the index of \( X \) at \((0, 0)\). Then, \( |i| \leq n \).

**Proof.** Let \( C \) be a simple closed curve such that it surrounds the critical point \( O = (0, 0) \). Then, \( C \) meets \( P = 0 \) at most at \( 2n \) points \( M_k \). By computing the Poincaré index (see [ALGM]) taking \( d \) the direction \( x = 0 \), we have that \( p + q \leq 2n \), where \( p \) (resp. \( q \)) is the number of points \( M_k \) at which \( X \) crosses the direction of \( d \) in the counterclockwise (resp. clockwise) sense. Since \( i = (p - q)/2 \), from \( p + q \leq 2n \) we deduce that \( |i| \leq 2n/2 = n \).
homogeneous polynomial vector fields in the plane with \( P \) and \( Q \) relatively prime and degree \( \deg(P) = \deg(Q) \). Notice that Proposition 4.3 tells us that
\[
2(1 - n) \leq \sum_{i} \alpha_i \leq 2(1 + n),
\]
where \( \sum_{i} \alpha_i \) denotes the sum of the indices at the critical points which lie in the equator of the Poincaré sphere.

The study of the homogeneous polynomial vector fields \( X = (P, Q) \) with degree \( \deg(P) = \deg(Q) = n \) determines the behaviour at infinity of one “generic” family of vector fields. Let \( X_n \) be the set of polynomial fields \( X = (P, Q) \) defined on \( \mathbb{R}^2 \) such that the degree of \( P \) and \( Q \) is \( n \). Let \( \mathcal{G}_n \) be the set of vector fields \( X = (P, Q) \in X_n \) such that all the common points of \( P = 0, Q = 0 \) lie in the finite part of \( \mathbb{R}^2 \). More precisely, if \( \bar{P}(x, y, z) \) and \( \bar{Q}(x, y, z) \) are the homogeneous polynomials in the variables \( x, y, z \) with degree \( \deg(P) = \deg(Q) \) such that \( \bar{P}(x, y, 1) = P(x, y) \) and \( \bar{Q}(x, y, 1) = Q(x, y) \), \( X \in \mathcal{G}_n \) means that system \( \bar{P} = 0, \bar{Q} = 0 \), \( z = 0 \) has only the trivial solution \( x = y = z = 0 \). Notice that this condition is equivalent to saying that \( P_n \) and \( Q_n \), the homogeneous parts of maximal degree of \( P \) and \( Q \), respectively, are relatively prime. It is not hard to see that \( X_n \setminus \mathcal{G}_n \) is contained in an algebraic hypersurface of \( \mathcal{X}_n \) (for more details see [C] or [CL]).

**Theorem 4.4.** The behaviour at infinity of \( X = (P, Q) \in \mathcal{G}_n \) is determined by the homogeneous parts of maximal degree of \( P \) and \( Q \).

**Proof.** Let \( P_n \) and \( Q_n \) be the homogeneous parts of maximal degree of \( P \) and \( Q \), respectively. If \( X \) has no infinite critical points, i.e., \( F(x, y) = xQ_n(x, y) - yP_n(x, y) \) has no linear factors, then the infinity is a periodic orbit.

If \( X \) has infinite critical points, we can assume that all them are in the local chart \( U_1 \). The expression of \( p(X) \) in the local chart \( U_1 \) is given by
\[
\dot{y} = \left[ -yP_n(1, y) + Q_n(1, y) \right] + z\left[ -yP_{n-1}(1, y) + Q_{n-1}(1, y) \right] + \cdots + z^n\left[ -yP_0 + Q_0 \right],
\]
\[
\dot{z} = -P_n(1, y)z - P_{n-1}(1, y)z^2 - \cdots - P_0z^{n+1}.
\]
The critical points at infinity are \((y_j, 0)\) for \( j = 1, 2, ..., r \), where \( y_j \) is a root of \( f(y) = -yP_n(1, y) + Q_n(1, y) = 0 \) for each \( j = 1, 2, ..., r \). Without loss of generality we can assume that \( y_j = 0 \). So \( f(0) = Q_n(1, 0) = 0 \). Since \( X \in \mathcal{G}_n \), it follows that \( P_n(1, 0) \neq 0 \). The linear part of \((y, z) = (0, 0)\) is
\[
\begin{pmatrix}
    f'(0) & Q_{n-1}(1, 0) \\
    0 & -P_n(1, 0)
\end{pmatrix}.
\]
If \( f'(0) \neq 0 \), the critical point \((0, 0)\) shall be a node or a saddle depending on the sign of \(-P_n(1, 0) f'(0)\). Notice that

\[
P_n(1, 0) f'(0) = P_n(1, 0) \left[ -P_n(1, 0) + \frac{dQ_n(1, y)}{dy} \right]_{y = 0},
\]

only depends on \( P_n \) and \( Q_n \).

Now assume that \( f'(0) = 0 \). Then we shall apply Theorem 65 of [ALGM]. Consider the change of coordinates \( Y = -P_n(1, 0) y - Q_{n-1}(1, 0) z, \ Z = z \), and the change of time scale \( ds = -P_n(1, 0) dt \). In the new coordinates \( Y \) begins with terms of at least second degree and \( Z \) begins by \( Z \). The equation \( \dot{z} = 0 \) gives \( Z = 0 \). Hence, on \( Z = 0 \)

\[
\dot{Y} = \frac{-1}{P_n(1, 0)} \left[ \frac{1}{P_n(1, 0)} P_n \left( 1, \frac{-1}{P_n(1, 0)} Y \right) + Q_n \left( 1, \frac{-1}{P_n(1, 0)} Y \right) \right].
\]

Applying Theorem 65 of [ALGM] we deduce that the topological structure in a neighbourhood of \((0, 0)\) only depends on \( P_n \) and \( Q_n \).

Remark. Let \( X \in \mathcal{X}_n \). If \( X \) has infinite critical points, the knowledge of the behaviour of the orbits near each critical point determines the behaviour of the orbits in a neighbourhood at infinity. If \( X \) has not infinite critical points and

\[
P_n(1, y) \quad \text{is not zero, the equator is a periodic orbit stable or unstable depending on the sign of } I. \text{ In the case that } I = 0, \ P_n \text{ and } Q_n \text{ cannot determine the behaviour of the orbits near the equator (we need to compute the derivates of order superior of the return function).}
\]

5. Topological Classification of Homogeneous Cubic Vector Fields

In this section we return to the case \( n = 3 \).

**Theorem 5.1.** Let \( X = (P, Q) \) be a homogeneous cubic vector field in the plane. Assume that \( P \) and \( Q \) have no common factor. Then, the qualitative phase-portrait of \( X \) (up the orientation of the orbits) is of one of the first 17 types drawn in Fig. 5.1. Every phase-portrait is characterized for the algebraic conditions given in Table 5.2.
Fig. 5.1. The phase-portraits of the homogeneous cubic vector fields $X = (P, Q)$. The bold straight lines mean equilibrium lines.
FIG. 5.1—Continued
### TABLE 5.2
Algebraic Conditions by Characterizing the Homogeneous Cubic Vector Fields $X = (P, Q)$ in the Plane, Where $P$ and $Q$ Have No Common Factor.

<table>
<thead>
<tr>
<th>PP</th>
<th>S</th>
<th>R</th>
<th>Conditions on the coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>1</td>
<td>( \left{ \frac{-\mu, -1, 1}{\mu, \mu} \right} )</td>
<td>( (-1)^{i+1} P_i &gt; 0 ), for ( i = 1, 2, 3, 4 ).</td>
</tr>
<tr>
<td>(2)</td>
<td>1</td>
<td>( \left{ \frac{-\mu, -1, 1}{\mu, \mu} \right} )</td>
<td>( (-1)^{j+1} P_j &lt; 0 ) and ( (-1)^{i+1} P_i &gt; 0 ), ( \forall i \neq j ), for ( i = 1, 2, 3, 4 ).</td>
</tr>
<tr>
<td>(3)</td>
<td>1</td>
<td>( \left{ \frac{-\mu, -1, 1}{\mu, \mu} \right} )</td>
<td>( (-1)^{j+1} P_j &lt; 0 ), ( j \in { k, k+1 } ) and ( (-1)^{i+1} P_i &gt; 0 ), ( \forall i \neq j ), for ( k = 1, 2, 3 ).</td>
</tr>
<tr>
<td>(4)</td>
<td>1</td>
<td>( \left{ \frac{-\mu, -1, 1}{\mu, \mu} \right} )</td>
<td>( (-1)^{j+1} P_j &lt; 0 ), ( j \in { k, k+2 } ), and ( (-1)^{i+1} P_i &gt; 0 ), ( \forall i \neq j ).</td>
</tr>
<tr>
<td>(5)</td>
<td>1</td>
<td>( \left{ \frac{-\mu, -1, 1}{\mu, \mu} \right} )</td>
<td>( (-1)^{i+1} P_i &gt; 0 ), and ( (-1)^{j+1} P_j &lt; 0 ), ( \forall i \neq j ), for ( j = 1, 2, 3, 4 ).</td>
</tr>
<tr>
<td>(6)</td>
<td>2</td>
<td>{ -1, 0, 1 }</td>
<td>( xP_1 &lt; 0 ), ( xP_3 &gt; 0 ), ( xP_2 \neq 0 ).</td>
</tr>
<tr>
<td>(7)</td>
<td>2</td>
<td>{ -1, 0, 1 }</td>
<td>( -xP_1 &gt; 0 ), ( xP_3 &lt; 0 ), ( xP_2 &gt; 0 ) or ( -xP_1 &lt; 0 ), ( xP_3 &gt; 0 ), ( xP_2 &lt; 0 ).</td>
</tr>
<tr>
<td>(8)</td>
<td>2</td>
<td>{ -1, 0, 1 }</td>
<td>( -xP_1 &gt; 0 ), ( xP_2 &lt; 0 ), ( xP_3 &lt; 0 ) or ( xP_1 &lt; 0 ), ( xP_2 &gt; 0 ), ( xP_3 &gt; 0 ).</td>
</tr>
<tr>
<td>(9)</td>
<td>2</td>
<td>{ -1, 0, 1 }</td>
<td>( -xP_1 &lt; 0 ), ( xP_3 &lt; 0 ), ( xP_2 \neq 0 ).</td>
</tr>
<tr>
<td>(10)</td>
<td>3</td>
<td>{ -1, 1 }</td>
<td>( \mu P_1 &lt; 0 ), ( \mu P_2 &gt; 0 ).</td>
</tr>
<tr>
<td>(11)</td>
<td>3</td>
<td>{ 0, 1 }</td>
<td>( P_1 &gt; 0 ), ( P_2 &lt; 0 ).</td>
</tr>
<tr>
<td>(12)</td>
<td>3</td>
<td>{ -1, 1 }</td>
<td>( P_1 P_2 &gt; 0 ).</td>
</tr>
<tr>
<td>(13)</td>
<td>3</td>
<td>{ 0, 1 }</td>
<td>( P_1 P_2 &gt; 0 ).</td>
</tr>
<tr>
<td>(14)</td>
<td>3</td>
<td>{ 0, 1 }</td>
<td>( P_1 P_2 &lt; 0 ).</td>
</tr>
<tr>
<td>(15)</td>
<td>6</td>
<td>{ 0 }</td>
<td>( P_1 \neq 0 ).</td>
</tr>
<tr>
<td>(16)</td>
<td>7</td>
<td>{ 0 }</td>
<td>( P_1 \neq 0 ).</td>
</tr>
<tr>
<td>(17)</td>
<td>8</td>
<td>\emptyset</td>
<td>( \mu &gt; 1/3 ), and ( p_3 - p_1^2 = 0 ), or ( \mu \in (-1/3, 1/3) ), and ( P_1 + p_1 &lt; 0 ).</td>
</tr>
<tr>
<td>(18)</td>
<td>9</td>
<td>\emptyset</td>
<td>( p_1 + p_3 = 0 ).</td>
</tr>
<tr>
<td>(19)</td>
<td>8</td>
<td>\emptyset</td>
<td>( \mu &gt; 1/3 ), and ( p_3 - p_1^2 \neq 0 ), or ( \mu \in (-1/3, 1/3) ), and ( P_1 + p_1 \neq 0 ).</td>
</tr>
<tr>
<td>(20)</td>
<td>9</td>
<td>\emptyset</td>
<td>( p_1 + p_3 \neq 0 ).</td>
</tr>
</tbody>
</table>

**Note.** Here \( P_i = P(1, \lambda_i) \), where \( \lambda_1 < \lambda_2 < \cdots < \lambda_n \) are the roots of \( f(\lambda) = 0 \), PP means phase-portrait, S means system, and R means the set of roots of \( f(y) \).
**Proof.** First, consider the case that \( F(x, y) = -yP(x, y) + xQ(x, y) \) has two real simple roots and one real double root. From Corollary 3.3 we can assume that \( F(x, y) = \alpha y^2(x^2 - y^2) \) and \( X \) has the following canonical form:

\[
\begin{align*}
\dot{x} &= p_1 x^3 + (p_2 - \alpha/2) x^2 y + p_3 xy^2 + \alpha y^3, \\
\dot{y} &= p_1 x^2 y + (p_2 + \alpha/2) xy^2 + p_3 y^3.
\end{align*}
\]

All the critical points at infinity are in the local chart \( U_1 \) and their coordinates are \((-1, 0), (0, 0), \) and \((1, 0)\). Since \( y = 0 \) is a double root of \( f(y) = \alpha y^2(a - y^2) \), the critical point \((0, 0)\) shall be a saddle-node (s-n or n-s). If the hyperbolic sectors of a saddle-node of the equator of the Poincaré sphere (see Fig. 4.2) appear in counterclockwise before the parabolic sector then this saddle-node will be of type s-n, otherwise it will be of type n-s. Since \( y = 1 \) (resp. \(-1\)) is a simple root of \( f(y) \), the critical points \((\pm 1, 0)\) shall be nodes (n) or saddles (s). So we have the following possibilities:

1. (a) n, n-s, n and n, s-n, n which gives the phase-portrait (6);
2. (b) n, n-s, s and s, s-n, n which gives the phase-portrait (7);
3. (c) n, s-n, s and s, n-s, n which gives the phase-portrait (8);
4. (d) s, n-s, s and s, s-n, s which gives the phase-portrait (9).

Since the linear part of \((\lambda_i, 0)\) is

\[
\begin{pmatrix}
 f'(\lambda_i) & * \\
 0 & -P(1, \lambda_i)
\end{pmatrix},
\]

we know that if \(-f'(\lambda_i)P(1, \lambda_i) > 0\) (resp. \(< 0\)) \((\lambda_i, 0)\) shall be a node (resp. a saddle) for \(i = 1, 3\). Denoting \( P_i = P(1, \lambda_i) \) and \( \lambda_1 = -1, \lambda_2 = 0, \lambda_3 = 1 \) we have that \(-f'(1) P_1 = -2\alpha P_1 \) and \(-f'(1) P_3 = 2\alpha P_3 \). To study the root \( y = 0 \), notice that \( \alpha P_2 > 0 \) (resp. \( \alpha P_2 < 0 \)) gives a n-s (resp. s-n) counterclockwise on the equator of the Poincaré sphere. The phase-portrait (6) shall be realized in the case that \( \alpha P_1 > 0 \) and \( \alpha P_3 > 0 \) with \( \alpha P_2 \neq 0 \). The other inequalities are studied in a similar way. To complete the proof of Theorem 5.1 we should do a similar study for each system of Corollary 3.3.

**Example.** Consider the system \( \dot{x} = x^3 - 3x^2y - xy^2, \dot{y} = x^2y + 3xy^2 - y^3 \). Then \( F(x, y) = xQ(x, y) - yP(x, y) = 6x^2y^2 \). By Theorem 2.6 we know that \( F(x, y) \) is equivalent to \( F_\epsilon(x, y) = 6(y-x)^2y^2 \). By using the change \( x = X - Y \) and \( y = Y \), the system becomes \( \dot{X} = X^3 + X^2Y - 9XY^2 + 6Y^3 = \bar{P}(X, Y), \dot{Y} = X^2Y - 5XY^2 + 3Y^2 = \bar{Q}(X, Y) \). This system satisfies \( X\bar{Q} - Y\bar{P} = -(Y - X)^2Y^2 \). Doing the change of time scale \( ds = -dt \) we obtain
\[ \dot{x} = -X^3 - X^2 Y + 9XY^2 - 6Y^3, \quad \dot{y} = -X^2 Y + 5XY^2 - 3Y^2, \]

which is of the form (4) of Corollary 3.3. Here, \( P_1 = P(1, 0) = -1 \) and \( P_2 = P(1, 1) = 1 \). So \( P_1 P_2 < 0 \) and by Table 5.2 we know that the phase-portrait is given by Fig. 5.1 (14). Furthermore, by Theorem 4.3, every cubic system \( \dot{x} = P(x, y), \quad \dot{y} = Q(x, y) \) with \( P_1(x, y) = x^3 - 3x^2 y - xy^2, \quad Q_1(x, y) = x^2 y + 3xy^2 - y^3 \) has two saddle-nodes at infinity.

**Remark.** Let \( X = (P, Q) \) be a homogeneous cubic vector field and assume that \( P \) and \( Q \) have some factor in common. If \( P = R \bar{P} \) and \( Q = R \bar{Q} \) we can do the change of time \( ds = R \, dt \) to obtain \( \dot{x} = \bar{P}(x, y), \quad \dot{y} = \bar{Q}(x, y) \) with \( \bar{P} \) and \( \bar{Q} \) without common factor. The zeros of \( R = 0 \) are critical points of \( X \). The orbits of \((P, Q)\) are the same as the orbits of \( X \) in the region \( R > 0 \) while in the region \( R < 0 \) the orientation is reversed. From Propositions 4.1, 4.2, and 4.3 we can describe the phase-portraits for quadratic and linear systems and by adding the critical points \( R = 0 \) we obtain all the phase-portraits of \( X \). These phase-portraits are shown in Fig. 5.1.

**REFERENCES**


