JOURNAL OF DIFFERENTIAL EQUATIONS 90, 1–30 (1991)

Regularity and Existence of Solutions of Elliptic Equations with p,q-Growth Conditions

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Received March 21, 1989

1. INTRODUCTION

We are here mainly interested in the regularity of weak solutions of elliptic equations of the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i(x, Du) = b(x), \qquad x \in \Omega, \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^n$ $(n \geq 2)$ and where a^i satisfy some non-standard growth conditions (that we call briefly *p,q-growth conditions*) like, for example,

$$\sum_{i,j} a^i_{\xi_j}(x, \xi)\, \lambda_i \lambda_j \geq m(1 + |\xi|^2)^{(p-2)/2}\, |\lambda|^2, \qquad \forall \xi, \lambda \in \mathbb{R}^n, \text{ a.e } x \in \Omega, \tag{1.2}$$

$$|a^i_{\xi_j}(x, \xi)| \leq M(1 + |\xi|^2)^{(q-2)/2}, \qquad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega, \forall i, j, \tag{1.3}$$

for some positive constants m, M, and for exponents $q \geq p \geq 2$.

Under (1.2), (1.3), and some other assumptions, by assuming also that the quotient q/p is sufficiently close to one in dependence on n (precisely, if $q/p < n/(n-2)$), then we prove that every weak solution to (1.1) of class $W^{1,q}_{\text{loc}}(\Omega)$ is locally Lipschitz-continuous in Ω. Moreover, there are positive constants β, c, and $\theta \geq 1$ such that

$$\|(1 + |Du|^2)^{1/2}\|_{L^\infty(B_\rho)} \leq c \left(\frac{1}{(R-\rho)^\beta} \|(1 + |Du|^2)^{1/2}\|_{L^q(B_R)} \right)^\theta \tag{1.4}$$

for every ρ, R $(0 < \rho < R \leq \rho + 1)$ such that the balls B_ρ, B_R of radii respectively ρ and R (and with the same center) are compactly contained in Ω.

The previous regularity result can be applied, for example, to equations

0022-0396/91 $3.00

studied in the setting of Orlicz spaces (see, for example, [3, 5, 8, 17]) of the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} (a(x)\, g(|Du|^2)\, u_{x_i}) = b(x), \tag{1.5}$$

where $a(x)$ is a Lipschitz-continuous function in Ω bounded from below by a positive constant, b is bounded in Ω and where g is the derivative of an N-function (see [9]) that, if it is not a power, then it can be typified by

$$g(t) = \frac{d}{dt}\, ((1+t)^{p/2} \log(1+t)). \tag{1.6}$$

If we pose $a^i(x, \xi) = a(x)\, g(|\xi|^2)\, \xi_i$, then for every $\varepsilon > 0$ there are constants m and $M(=M(\varepsilon))$ such that a^i satisfy (1.2), (1.3) with $q = p + \varepsilon$.

Similar results hold for the Euler's equation of the functional F, of the type recently studied by Zhikov [19], given by

$$F(u) = \int_{\Omega} (1 + |Du|^2)^{\alpha(x)}\, dx. \tag{1.7}$$

If $\alpha(x)$ is continuous in Ω then, locally in every ball B_R with radius R sufficiently small, (1.2) and (1.3) are satisfied, again with $q = p + \varepsilon$.

Another example of application of the regularity results of this paper is to elliptic equations of the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} (a(x)\, u_{x_i}) + \frac{\partial}{\partial x_n} (|u_{x_n}|^{q-2}\, u_{x_n}) = b(x) \tag{1.8}$$

with $b(x)$ locally bounded in Ω, $a(x)$ locally Lipschitz-continuous in Ω bounded from below by a positive constant and with $q \geqslant 2$ sufficiently close to 2. Note that p.d.e. of the type (1.8) have been considered by J. L. Lions [12, Chap. 2, Sects. 1.7 and 2.3] who showed the applicability of the existence theory of monotone operators to this case (see also [11, Remark 5]).

A second type of results that we will give in this paper is about the existence of solutions to the Eq. (1.1), satisfying some given Dirichlet boundary conditions. We will apply the a priori regularity results stated previously to the existence of weak (and classical) solutions.

First of all we will show that the solutions of our Dirichlet problems are a priori bounded in $W^{1,p}(\Omega)$. Thus it is natural to ask for an estimate of the type of (1.4) with the L^q-norm replaced by the L^p-norm. To this aim it is useful the well-known interpolation inequality

$$\|v\|_{L^q} \leqslant \|v\|_{L^p}^{p/q} \cdot \|v\|_{L^\infty}^{1-p/q}, \qquad \text{with} \quad v = (1 + |Du|^2)^{1/2}. \tag{1.9}$$

From (1.4), (1.9) we can derive formally an estimate of the L^∞-norm of the gradient of u in terms of its L^p-norm:

$$\|(1+|Du|^2)^{1/2}\|_{L^\infty} \leqslant c\, \|(1+|Du|^2)^{1/2}\|_{L^q}^{\theta}$$
$$\leqslant c\, \|(1+|Du|^2)^{1/2}\|_{L^p}^{\theta(p/q)} \cdot \|(1+|Du|^2)^{1/2}\|_{L^\infty}^{\theta(1-p/q)}.$$

If $\theta(1-p/q)<1$ then (up to the technical difficulty due to the different radii ρ and R) formally we obtain

$$\|(1+|Du|^2)^{1/2}\|_{L^\infty}^{1-\theta(1-p/q)} \leqslant c\, \|(1+|Du|^2)^{1/2}\|_{L^p}^{\theta(p/q)}. \tag{1.10}$$

It is clear that an a priori estimate like (1.10) is useful in the existence theory; we will prove this estimate in Theorem 3.1.

In order to test the condition $\theta(1-p/q)<1$, in Section 2 we give an explicit expression of θ (see (2.8)) from which we deduce that the exponent in the left hand side of (1.10) is positive if $q/p<(n+2)/n$.

We mention explicitly that this paper (except for Corollary 2.2 and its consequences) is self contained. Even in the known and important case $p=q$ we propose a complete proof of the local boundness of the gradient, partially new and partially similar to the first proof by Ladyzhenskaya and Ural'tseva (see [10, Chap. 4, Sect. 3]). Related regularity results on the local boundness and on the Hölder-continuity of the gradient for solutions of certain degenerate elliptic equations and systems of special form have been given by Uhlenbeck [18], Evans [4], and Di Benedetto [2].

In this paper we use for the gradient the method by iterations that Moser has introduced in [16] to infer the local boundness of solutions in the linear case. This method has been also applied by Giusti [7] to obtain the local boundness of the gradient in the case $p=q=2$.

If $p\neq q$ the existence and regularity results presented here seem to be new. We continue a research started by the author in [15]. We improve the regularity results stated in [15, Theorems B and C] in several directions: (1) we do not impose the variational condition $a^i_{\xi_j}=a^j_{\xi_i}$; (2) we allow a^i to depend also on x, other than on ξ; (3) we consider general exponents p, q greater than 2 (instead of $p=2$); (4) we obtain an explicit estimate of the L^∞-norm of the gradient in terms of its L^q-norm; in particular we obtain an explicit expression for the exponent θ in (1.4); (5) the condition $q/p<(n+2)/n$ described previously is less restrictive than the corresponding condition in [15, Theorem C]. On the contrary, Theorem A of [15] is a regularity result specific for the situation considered in the appendix of this paper.

We have already noted that scalar problems with exponents $p\neq q$ have already been considered in the mathematical literature. In the vectorial case vectorial problems with $p\neq q$ naturally arise in nonlinear elasticity (see, for example, [13]); for this reason it would be interesting to extend to strongly elliptic systems some of the results obtained here for elliptic equations.

2. Regularity

In this section we consider the equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i(x, Du) = b(x), \qquad x \in \Omega, \tag{2.1}$$

and we assume that $a^i(x, \xi)$, for $i = 1, 2, ..., n$, are locally Lipschitz-continuous functions in $\Omega \times \mathbb{R}^n$ $(n \geqslant 2)$.

We consider exponents p and q such that

$$2 \leqslant p \leqslant q < \frac{n}{n-2} p \tag{2.2}$$

$(2 \leqslant p \leqslant q$, if $n = 2)$. About the derivatives with respect to ξ, we assume that there are positive constants m, M such that, for every ξ, $\lambda \in \mathbb{R}^n$ and for a.e. $x \in \Omega$:

$$\sum_{i,j} a^i_{\xi_j}(x, \xi)\, \lambda_i \lambda_j \geqslant m(1 + |\xi|^2)^{(p-2)/2} |\lambda|^2; \tag{2.3}$$

$$|a^i_{\xi_j}(x, \xi)| \leqslant M(1 + |\xi|^2)^{(q-2)/2}, \qquad \forall i, j; \tag{2.4}$$

$$|a^i_{\xi_j}(x, \xi) - a^j_{\xi_i}(x, \xi)| \leqslant M(1 + |\xi|^2)^{(p+q-4)/4}, \qquad \forall i, j. \tag{2.5}$$

About the derivatives with respect to x we assume that, for every $\xi \in \mathbb{R}^n$ and for a.e. $x \in \Omega$:

$$|a^i_{x_s}(x, \xi)| \leqslant M(1 + |\xi|^2)^{(p+q-2)/4}, \qquad \forall i, s. \tag{2.6}$$

Under the previous assumptions, by a *weak solution of class* $W^{1,q}_{\mathrm{loc}}(\Omega)$ *to Eq.* (2.1) we mean a function $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ such that, for every $\Omega' \subset\subset \Omega$,

$$\int_{\Omega} \left\{ \sum_{i=1}^{n} a^i(x, Du)\, \phi_{x_i} + b(x)\phi \right\} dx = 0, \qquad \forall \phi \in W^{1,q}_0(\Omega'). \tag{2.7}$$

Let us define θ by

$$\theta = \frac{2q}{np - (n-2)q}, \qquad \text{if} \quad n > 2 \tag{2.8}$$

and, in the case $n = 2$, let θ be any number strictly greater than q/p, if $q/p > 1$, and let $\theta = 1$ if $q/p = 1$.

Let us denote by B_ρ, B_R balls compactly contained in Ω, of radii respectively ρ, R and with the same center.

THEOREM 2.1. *Let $b \in L^\infty_{\mathrm{loc}}(\Omega)$ and let* (2.2), (2.3), (2.4), (2.5), (2.6) *hold. Then every weak solution $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ to Eq.* (2.1) *is of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$. Moreover there are positive numbers c and β (independent of u) such that*

$$\sup_{x \in B_\rho} (1 + |Du(x)|^2)^{1/2} \leqslant c \left(\frac{1}{(R-\rho)^\beta} \, \|(1 + |Du|^2)^{1/2}\|_{L^q(B_R)} \right)^\theta \qquad (2.9)$$

for every ρ and R such that $0 < \rho < R \leqslant \rho + 1$.

In a standard way, for example, as in [10, Chap. 4, Sect. 6] or as in [7, Chap. V, Sect. 8] (see also [15, Theorem D]), from Theorem 2.1 we can deduce the following:

COROLLARY 2.2. *Let* (2.2) *to* (2.6) *hold. Let us assume also that, for $i = 1, 2, ..., n$, $a^i \in C^{k,\alpha}_{\mathrm{loc}}(\Omega \times \mathbb{R}^n)$ and $b \in C^{k-1,\alpha}_{\mathrm{loc}}(\Omega)$ for some $k \geqslant 1$. Then, if $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ is a weak solution to Eq.* (2.1), *then $u \in C^{k+1,\alpha}_{\mathrm{loc}}(\Omega)$.*

Remark 2.3. Independently of the results in the other sections of this paper, the previous regularity results can be applied, for example, to the equation

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{q-2} \frac{\partial u}{\partial x_i} + b^i(x, Du) \right) = b(x) \qquad (2.10)$$

with $b \in L^\infty_{\mathrm{loc}}(\Omega)$ and

$$\sum_{i,j} b^i_{\xi_j}(x, \xi) \, \lambda_i \lambda_j \geqslant m(1 + |\xi|^2)^{(p-2)/2} \, |\lambda|^2,$$

$$|b^i_{\xi_j}(x, \xi)| \leqslant M(1 + |\xi|^2)^{(p-2)/2}, \qquad |b^i_{x_s}(x, \xi)| \leqslant M(1 + |\xi|^2)^{(p-1)/2},$$

with p, q satisfying (2.2). Note that, if $q \neq p$, then $a^i(x, \xi) = |\xi|^{q-2} \xi_i + b^i(x, \xi)$ does not satisfy the ellipticity condition $\sum_{i,j} a^i_{\xi_j}(x, \xi) \, \lambda_i \lambda_j \geqslant m \, |\xi|^{q-2} \, |\lambda|^2$.

We will dedicate all this Section 2 to the proof of Theorem 2.1, through several lemmas.

LEMMA 2.4. *Under the assumptions* (2.3), (2.4), (2.5) *there is a constant c_1 such that, for every ξ, λ, $\eta \in \mathbb{R}^n$ and for a.e. $x \in \Omega$,*

$$\left| \sum_{i,j} a^i_{\xi_j}(x, \xi) \, \lambda_j \eta_i \right| \leqslant c_1 \left(\sum_{i,j} a^i_{\xi_j}(x, \xi) \, \lambda_i \lambda_j \right)^{1/2} (1 + |\xi|^2)^{(q-2)/4} \, |\eta|.$$

Proof. Let us denote by (b_{ij}) and (c_{ij}) the matrices defined by

$$b_{ij}(x) = \tfrac{1}{2}(a^i_{\xi_j} + a^j_{\xi_i}); \qquad c_{ij}(x) = \tfrac{1}{2}(a^i_{\xi_j} - a^j_{\xi_i}).$$

Since (b_{ij}) is a positive definite symmetric matrix, by the Cauchy–Schwarz inequality, by the fact that $\sum b_{ij}\lambda_i\lambda_j = \sum a^i_{\xi_j}\lambda_i\lambda_j$ and by (2.4) we obtain

$$\begin{aligned}\Big|\sum_{i,j} b_{ij}\lambda_j\eta_i\Big| &\leqslant \Big(\sum_{i,j} b_{ij}\lambda_i\lambda_j\Big)^{1/2}\Big(\sum_{i,j} b_{ij}\eta_i\eta_j\Big)^{1/2} \\ &\leqslant \sqrt{M}\Big(\sum_{i,j} a^i_{\xi_j}\lambda_i\lambda_j\Big)^{1/2}(1+|\xi|^2)^{(q-2)/4}\,|\eta|. \qquad (2.11)\end{aligned}$$

Moreover, by (2.5) and (2.3), we have

$$\begin{aligned}\Big|\sum_{i,j} c_{ij}\lambda_j\eta_i\Big| &\leqslant \frac{M}{2}(1+|\xi|^2)^{(p+q-4)/4}\,|\lambda|\,|\eta| \\ &\leqslant \frac{M}{2\sqrt{m}}\Big(\sum_{i,j} a^i_{\xi_j}\lambda_i\lambda_j\Big)^{1/2}(1+|\xi|^2)^{(q-2)/4}\,|\eta|. \qquad (2.12)\end{aligned}$$

Since $a^i_{\xi_j} = b_{ij} + c_{ij}$, we deduce our result from (2.11), (2.12).

LEMMA 2.5. *Under the assumptions* (2.3), (2.6) *there is a constant c_2 such that, for a.e. $x \in \Omega$, for every ξ, $\lambda \in \mathbb{R}^n$ and for $s = 1, 2, ..., n$:*

$$\Big|\sum_{i=1}^n a^i_{x_s}(x,\xi)\,\lambda_i\Big| \leqslant c_2\Big(\sum_{i,j} a^i_{\xi_j}(x,\xi)\,\lambda_i\lambda_j\Big)^{1/2}(1+|\xi|^2)^{q/4}.$$

Proof. By (2.6) and (2.3) we have

$$\begin{aligned}\Big|\sum_{i=1}^n a^i_{x_s}\lambda_i\Big| &\leqslant \Big(\sum_{i=1}^n |a^i_{x_s}|^2\Big)^{1/2}|\lambda| \leqslant \sqrt{n}M(1+|\xi|^2)^{(p+q-2)/4}\,|\lambda| \\ &\leqslant M\sqrt{\frac{n}{m}}\Big(\sum_{i,j} a^i_{\xi_j}\lambda_i\lambda_j\Big)^{1/2}(1+|\xi|^2)^{q/4}.\end{aligned}$$

For $\alpha \geqslant 2$ and $k > 0$ let $g_{\alpha,k}\colon \mathbb{R} \to \mathbb{R}$ be the function defined for $|t| \leqslant k$ by

$$g_{\alpha,k}(t) = t(1+t^2)^{(\alpha-2)/2} \qquad (2.13)$$

and extended to $\mathbb{R}$ linearly for $|t| > k$ as a function of class $C^1(\mathbb{R})$. Let us also define

$$G_{\alpha,k}(t) = g^2_{\alpha,k}(t)/g'_{\alpha,k}(t), \qquad (2.14)$$

where $g'_{\alpha,k}$ is the derivative of $g_{\alpha,k}$.

LEMMA 2.6. *The following estimates hold*:

(i) *For every* $\alpha \geqslant 2$ *and* $k > 0$ *there is a constant* $c_{\alpha,k}$ *such that*

$$G_{\alpha,k}(t) \leqslant c_{\alpha,k}(1+t^2), \qquad \forall t \in \mathbb{R}.$$

(ii) *For every* $\alpha \geqslant 2$ *and* $k > 0$ *we have*

$$G_{\alpha,k}(t) \leqslant 2\left(\frac{1+k^2}{k^2}\right)^{(\alpha-2)/2}(1+t^2)^{\alpha/2}, \qquad \forall t \in \mathbb{R}.$$

Proof. (i) follows from the fact that $g_{\alpha,k}$ is linear and $g'_{\alpha,k}$ is constant for $t > k$ and $t < -k$. To prove (ii) let us first observe that, if $|t| \leqslant k$, then, since $\alpha \geqslant 2$,

$$G_{\alpha,k}(t) = \frac{t^2(1+t^2)^{\alpha-2}}{(1+t^2)^{(\alpha-4)/2}\,[(\alpha-1)\,t^2+1]} \leqslant (1+t^2)^{\alpha/2}, \qquad \text{if} \quad |t| \leqslant k.$$

For $|t| \geqslant k$, again since $\alpha \geqslant 2$, we have

$$\begin{aligned} |g_{\alpha,k}(t)| &= k(1+k^2)^{(\alpha-2)/2} + (1+k^2)^{(\alpha-4)/2}\,[(\alpha-1)\,k^2+1](|t|-k) \\ &\leqslant (1+k^2)^{(\alpha-2)/2}\,[(\alpha-1)\,|t| - (\alpha-2)k]. \end{aligned}$$

Thus, since $g'_{\alpha,k}(t) \geqslant (1+k^2)^{(\alpha-2)/2}$, we obtain

$$\frac{G_{\alpha,k}(t)}{(1+t^2)^{\alpha/2}} \leqslant (1+k^2)^{(\alpha-2)/2}\left(\frac{(\alpha-1)\,|t| - (\alpha-2)k}{|t|^{\alpha/2}}\right)^2. \tag{2.15}$$

By a computation we can see that the maximum with respect to $|t|$ of the right hand side of (2.15) is assumed for $|t| = \alpha k/(\alpha-1)$ and its value is

$$4\left(\frac{1+k^2}{k^2}\right)^{(\alpha-2)/2}\left(\frac{\alpha-1}{\alpha}\right)^{-\alpha} < \frac{4}{e}\left(\frac{1+k^2}{k^2}\right)^{(\alpha-2)/2}$$

Fixed $s \in \{1, 2, ..., n\}$ we denote by e_s the unit coordinate vector in the x_s direction and we define the difference quotient Δ_h in the direction e_s (we do not denote explicitly the dependence on s) by $\Delta_h v(x) = [v(x+he_s) - v(x)]/h$. The function $\Delta_h v$ is defined in $\Omega_h = \{x \in \Omega :$ $\text{dist}(x, \partial\Omega) < h\}$ and, if $v \in W^{1,q}(\Omega)$, then $\Delta_h v \in W^{1,q}(\Omega_h)$.

We state in the following lemma the properties of the difference quotient that we will use in this paper.

LEMMA 2.7. *Let* Ω' *be an open set compactly contained in* Ω *and let* $h_0 = \text{dist}(\Omega', \Omega^c)$. *The following properties hold*:

(i) *If* $v \in W^{1,q}(\Omega)$ *for some* $q \geqslant 1$ *then for every* $h \leqslant h_0$: $\int_{\Omega'} |\Delta_h v|^q \, dx \leqslant \int_\Omega |v_{x_s}|^q \, dx$.

(ii) *If* $v \in L^q(\Omega)$ *for some* $q > 1$ *and if there is a constant* c *such that* $\|\Delta_h v\|_{L^q(\Omega')} \leqslant c$ *for every* $h \leqslant h_0$, *then* $v_{x_s} \in L^q(\Omega')$ *and* $\|v_{x_s}\|_{L^q(\Omega')} \leqslant c$.

(iii) *If* $v \in W^{1,q}(\Omega)$ *for some* $q > 1$, *then for every* $s = 1, 2, ..., n$, $\Delta_h v$ *converges to* v_{x_s} *strongly in* $L^q(\Omega')$.

Proof. The properties stated in (i), (ii) are well known and can be found, for example, in [1, Proposition IX.3]. Also the property (iii) can be proved with the argument of [1] in the following way: first, if $v \in W^{1,q}(\Omega)$ then $\Delta_h v$ is bounded in $L^q(\Omega')$ independently of h. Since $q > 1$, by a compactness argument we can show that, as $h \to 0$, $\Delta_h v$ converges to v_{x_s} in the weak topology of $L^q(\Omega')$. By the properties (i) and (ii) the L^q-norm of $\Delta_h v$ converges, as $h \to 0$, to the L^q-norm of v_{x_s}. This implies that $\Delta_h v$ converges to v_{x_s} in the norm topology.

Let $\Omega' \subset\subset \Omega$. Let η be a nonnegative function of class $C_0^1(\Omega')$. If h is sufficiently small it is well defined in Ω' the function

$$\phi = \Delta_{-h}(\eta^2 g_{\alpha,k}(\Delta_h u)). \tag{2.16}$$

Since $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ and since $g_{\alpha,k}$ is Lipschitz-continuous on $\mathbb{R}$, it is easy to see that $\phi \in W_0^{1,q}(\Omega')$. By using ϕ as test function in the weak form (2.7) of our equation, with simple computations we obtain

$$\int_\Omega \sum_{i=1}^n \Delta_h a^i(x, Du)(\eta^2 g'_{\alpha,k} \Delta_h u_{x_i} + 2\eta\eta_{x_i} g_{\alpha,k}) \, dx = \int_\Omega b(x) \, \Delta_{-h}(\eta^2 g_{\alpha,k}) \, dx.$$

Let us compute $\Delta_h a^i(x, Du)$:

$$\Delta_h a^i(x, Du) = \frac{1}{h} \int_0^1 \frac{d}{dt} a^i(x + the_s, Du + th \, \Delta_h \, Du) \, dt$$
$$= \int_0^1 \left(a^i_{x_s} + \sum_{j=1}^n a^i_{\xi_j} \Delta_h u_{x_j} \right) dt.$$

It follows that

$$\int_\Omega \int_0^1 \eta^2 g'_{\alpha,k} \sum_{i,j} a^i_{\xi_j} \Delta_h u_{x_i} \Delta_h u_{x_j} \, dx \, dt \tag{2.17}$$

$$= -\int_\Omega \int_0^1 \eta^2 g'_{\alpha,k} \sum_{i=1}^n a^i_{x_s} \Delta_h u_{x_i} \, dx \, dt \tag{2.18}$$

$$-\int_\Omega \int_0^1 2\eta g_{\alpha,k} \sum_{i=1}^n \left(a^i_{x_s} + \sum_{j=1}^n a^i_{\xi_j} \Delta_h u_{x_j} \right) \eta_{x_i} \, dx \, dt \tag{2.19}$$

$$+\int_\Omega b(x) \, \Delta_{-h}(\eta^2 g_{\alpha,k}) \, dx. \tag{2.20}$$

Let us estimate separately the terms in the right hand side. Let us start with (2.18); by Lemma 2.5 and by the inequality $|ab| \leqslant \varepsilon a^2 + b^2/(4\varepsilon)$, valid for every a, $b \in \mathbb{R}$ and every $\varepsilon > 0$, we have

$$\begin{aligned}
&\left| \int_\Omega \int_0^1 \eta^2 g'_{\alpha,k} \sum_{i=1}^n a^i_{x_s} \Delta_h u_{x_i} \, dx \, dt \right| \\
&\quad \leqslant c_2 \int_\Omega \int_0^1 \eta^2 g'_{\alpha,k} \left(\sum_{i,j} a^i_{\xi_j} \Delta_h u_{x_i} \Delta_h u_{x_j} \right)^{1/2} (1 + |Du + th \, \Delta_h \, Du|^2)^{q/4} \, dx \, dt \\
&\quad \leqslant \varepsilon c_2 \int_\Omega \int_0^1 \eta^2 g'_{\alpha,k} \sum_{i,j} a^i_{\xi_j} \Delta_h u_{x_i} \Delta_h u_{x_j} \, dx \, dt \\
&\qquad + \frac{c_2}{4\varepsilon} \int_\Omega \int_0^1 \eta^2 g'_{\alpha,k} (1 + |Du + th \, \Delta_h \, Du|^2)^{q/2} \, dx \, dt. \qquad (2.21)
\end{aligned}$$

About the term (2.19), we have the following estimates (2.22) and (2.23), the first of them being a consequence of the assumption (2.6):

$$\begin{aligned}
&\left| \int_\Omega \int_0^1 \eta g_{\alpha,k} \sum_{i=1}^n a^i_{x_s} \eta_{x_i} \, dx \, dt \right| \\
&\quad \leqslant nM \int_\Omega \int_0^1 \eta \, |D\eta| \, |g_{\alpha,k}| \, (1 + |Du + th \, \Delta_h \, Du|^2)^{(p+q-2)/4} \, dx \, dt \\
&\quad \leqslant nM \int_\Omega \int_0^1 \eta \, |D\eta| \, |g_{\alpha,k}| \, (1 + |Du + th \, \Delta_h \, Du|^2)^{(q-1)/2} \, dx \, dt. \qquad (2.22)
\end{aligned}$$

By Lemma 2.4 and by using the definition of $G_{\alpha,k}$ in (2.14) we have also

$$\begin{aligned}
&\left| \int_\Omega \int_0^1 \eta g_{\alpha,k} \sum_{i,j} a^i_{\xi_j} \Delta_h u_{x_j} \eta_{x_i} \, dx \, dt \right| \\
&\quad \leqslant c_1 \int_\Omega \int_0^1 \left(\eta^2 g'_{\alpha,k} \sum_{i,j} a^i_{\xi_j} \Delta_h u_{x_i} \Delta_h u_{x_j} \right)^{1/2} \\
&\qquad \cdot (G_{\alpha,k} \, (1 + |Du + th \, \Delta_h \, Du|^2)^{(q-2)/2} \, |D\eta|^2)^{1/2} \, dx \, dt \\
&\quad \leqslant \varepsilon c_1 \int_\Omega \int_0^1 \eta^2 g'_{\alpha,k} \sum_{i,j} a^i_{\xi_j} \Delta_h u_{x_i} \Delta_h u_{x_j} \, dx \, dt \\
&\qquad + \frac{c_1}{4\varepsilon} \int_\Omega \int_0^1 G_{\alpha,k} (1 + |Du + th \, \Delta_h \, Du|^2)^{(q-2)/2} \, |D\eta|^2 \, dx \, dt. \qquad (2.23)
\end{aligned}$$

About the term (2.20), by using property (i) of Lemma 2.7 (with $\Omega' \supset \operatorname{supp} \eta$), we obtain

$$\begin{aligned}
&\left| \int_\Omega b(x)\, \Delta_{-h}(\eta^2 g_{\alpha,k})\, dx \right| \\
&\quad \leqslant \|b\|_{L^\infty(\Omega')} \int_\Omega \left| \frac{\partial}{\partial x_s} (\eta^2 g_{\alpha,k}) \right| dx \\
&\quad \leqslant \|b\|_{L^\infty(\Omega')} \int_\Omega (2\eta\, |\eta_{x_s}|\, |g_{\alpha,k}| + \eta^2 g'_{\alpha,k}\, |\Delta_h u_{x_s}|)\, dx \\
&\quad \leqslant \|b\|_{L^\infty(\Omega')} \left\{ \int_\Omega 2\eta\, |D_\eta|\, |g_{\alpha,k}|\, dx \right. \\
&\qquad \left. + \varepsilon \int_\Omega \eta^2 g'_{\alpha,k}\, |\Delta_h u_{x_s}|^2\, dx + \frac{1}{4\varepsilon} \int_\Omega \eta^2 g'_{\alpha,k}\, dx \right\}.
\end{aligned} \tag{2.24}$$

Finally, to estimate the left hand side (2.17) we use the ellipticity assumption (2.3):

$$\begin{aligned}
&\int_\Omega \int_0^1 \eta^2 g'_{\alpha,k} \sum_{i,j} a^i_{\xi_j}\, \Delta_h u_{x_i}\, \Delta_h u_{x_j}\, dx\, dt \\
&\quad \geqslant m \int_\Omega \int_0^1 \eta^2 g'_{\alpha,k} (1 + |Du + th\, \Delta_h u|^2)^{(p-2)/2}\, |\Delta_h Du|^2\, dx\, dt.
\end{aligned} \tag{2.25}$$

By the relations from (2.17) to (2.25), by choosing ε sufficiently small, we deduce that there is a positive constant c_3 (depending on the L^∞-norm of $b(x)$) such that the following estimate holds (note in particular (2.24), whose ε term goes in (2.26) and whose $1/(4\varepsilon)$ term goes in (2.27)):

$$\frac{1}{c_3} \int_\Omega \int_0^1 \eta^2 g'_{\alpha,k} (1 + |Du + th\, \Delta_h\, Du|^2)^{(p-2)/2}\, |\Delta_h\, Du|^2\, dx\, dt \tag{2.26}$$

$$\leqslant \int_\Omega \int_0^1 \eta^2 g'_{\alpha,k} (1 + |Du + th\, \Delta_h\, Du|^2)^{q/2}\, dx\, dt \tag{2.27}$$

$$+ \int_\Omega \int_0^1 \eta\, |D\eta|\, |g_{\alpha,k}|\, (1 + |Du + th\, \Delta_h\, Du|^2)^{(q-1)/2}\, dx\, dt \tag{2.28}$$

$$+ \int_\Omega \int_0^1 |D\eta|^2\, G_{\alpha,k} (1 + |Du + th\, \Delta_h\, Du|^2)^{(q-2)/2}\, dx\, dt. \tag{2.29}$$

LEMMA 2.8. *Under the previous assumptions $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$. Moreover there is a constant c_4 such that*

$$\int_\Omega \eta^2 \sum_{s=1}^n (1+|u_{x_s}|^2)^{(\alpha+p-4)/2} |Du_{x_s}|^2 \, dx$$
$$\leqslant c_4(\alpha-1) \int_\Omega (\eta^2+|D\eta|^2) \sum_{s=1}^n (1+|u_{x_s}|^2)^{(\alpha+q-2)/2} \, dx$$

for every $\alpha \geqslant 2$ such that the right hand side is finite.

Proof. To estimate (2.28) and (2.29) we will use the inequalities

$$|g_{\alpha,k}| \, (1+|Du+th\,\Delta_h\,Du|^2)^{(q-1)/2}$$
$$\leqslant \frac{1}{q} |g_{\alpha,k}|^q + \frac{q-1}{q} (1+|Du+th\,\Delta_h\,Du|^2)^{q/2};$$
$$G_{\alpha,k}(1+|Du+th\,\Delta_h\,Du|^2)^{(q-2)/2}$$
$$\leqslant \frac{2}{q} G_{\alpha,k}^{q/2} + \frac{q-2}{q} (1+|Du+th\,\Delta_h\,Du|^2)^{q/2}.$$

Let us first consider the case $\alpha = 2$. For every $k > 0$ and $t \in \mathbb{R}$ we have

$$g_{2,k}(t) = t, \qquad g'_{2,k}(t) = 1, \qquad G_{2,k}(t) = t^2.$$

By the previous estimates, by (2.26)–(2.29), and by taking $\eta = 1$ on $\Omega' \subset\subset \Omega$, we deduce that the integral

$$\int_{\Omega'} |\Delta_h\,Du|^2 \, dx$$

is bounded by a constant independent of h (here we use the assumption $p \geqslant 2$). Thus it is sufficient to apply the property (ii) of Lemma 2.7 to obtain $u \in W^{2,2}(\Omega')$.

Now we go to the limit as $h \to 0$. Let Ω' such that supp $\eta \subset \Omega' \subset\subset \Omega$. Since $u \in W^{1,q}(\Omega')$, by Lemma 2.7(iii) the difference quotient $\Delta_h u$ converges to u_{x_s} in $L^q(\Omega')$. Moreover

$$Du + th\,\Delta_h\,Du = (1-t)\,Du(x) + t\,Du(x+he_s)$$

converges, as $h \to 0$, to Du in $L^q(\Omega')$, by the continuity in L^q of the translation.

Let us recall the definition of $g_{\alpha,k}$ in (2.13) and Lemma 2.6(i); since, for $|t| > k$, $g_{\alpha,k}$ is linear and $g_{\alpha,k}$ is linear and $G_{\alpha,k}$ is quadratic, then as $h \to 0$,

$$g_{\alpha,k}(\Delta_h u) \to g_{\alpha,k}(u_{x_s}) \quad \text{in} \quad L^q(\Omega');$$
$$G_{\alpha,k}(\Delta_h u) \to G_{\alpha,k}(u_{x_s}) \quad \text{in} \quad L^{q/2}(\Omega').$$

By using the inequalities written at the beginning of the proof of this lemma we see that we can go to the limit as $h \to 0$ in the integrals in (2.28), (2.29). Since $g'_{\alpha,k}$ is bounded in $\mathbb{R}$, we can go to the limit as $h \to 0$ also in the integral in (2.27). Finally, we go to the limit in the left hand side (2.26) since the integral is lower semicontinuous. We obtain an estimate similar to (2.26)–(2.29), where the difference quotient is replaced by the partial derivative with respect to x_s, where $h=0$ and without the integrals with respect to t.

Then we use the relations (see Lemma 2.6(ii)):

$$|g_{\alpha,k}(t)| \leqslant (1+t^2)^{(\alpha-1)/2},$$
$$g'_{\alpha,k}(t) \leqslant (\alpha-1)(1+t^2)^{(\alpha-2)/2},$$
$$G_{\alpha,k}(t) \leqslant 2\left(\frac{1+k^2}{k^2}\right)^{(\alpha-2)/2}(1+t^2)^{\alpha/2},$$

and also the fact that $\lim_{k\to+\infty} g'_{\alpha,k}(t) \geqslant (1+t^2)^{(\alpha-2)/2}$. By Fatou's lemma we can go to the limit as $k \to +\infty$. We obtain

$$\frac{1}{c_3}\int_\Omega \eta^2(1+|u_{x_s}|^2)^{(\alpha-2)/2}(1+|Du|^2)^{(p-2)/2}|Du_{x_s}|^2\,dx$$
$$\leqslant (\alpha-1)\int_\Omega \eta^2(1+|u_{x_s}|^2)^{(\alpha-2)/2}(1+|Du|^2)^{q/2}\,dx$$
$$+\int_\Omega \eta\,|D\eta|\,(1+|u_{x_s}|^2)^{(\alpha-1)/2}(1+|Du|^2)^{(q-1)/2}\,dx$$
$$+2\int_\Omega |D\eta|^2(1+|u_{x_s}|^2)^{\alpha/2}(1+|Du|^2)^{(q-2)/2}\,dx.$$

First we sum up with respect to $s=1, 2, ..., n$. Then we note that there is a constant c_5, which depends only on n and q, such that the quantities

$$(1+|Du|^2)^{q/2}, \qquad (1+|Du|^2)^{(q-1)/2}, \qquad (1+|Du|^2)^{(q-2)/2}$$

are respectively less than or equal to

$$c_5\sum_{s=1}^n (1+|u_{x_s}|^2)^{q/2}, \qquad c_5\sum_{s=1}^n (1+|u_{x_s}|^2)^{(q-1)/2},$$
$$c_5\sum_{s=1}^n (1+|u_{x_s}|^2)^{(q-2)/2}.$$

Now the conclusion of the proof of lemma 2.8 follows easily by the inequality stated in the next lemma, with $y_s = 1+|u_{x_s}|^2$.

LEMMA 2.9. *Let $y_s \geqslant 0$ for $s=1,2,\ldots,n$ and let $a,b>0$. Then*

$$\sum_{s=1}^{n} y_s^a \cdot \sum_{s=1}^{n} y_s^b \leqslant \left(1+\frac{n(n-1)}{2}\right) \sum_{s=1}^{n} y_s^{a+b}.$$

Proof.

$$\begin{aligned}\sum_{s=1}^{n} y_s^a \cdot \sum_{s=1}^{n} y_s^b &= \sum_{s=1}^{n} y_s^{a+b} + \sum_{i\neq j} [y_i^a y_j^b + y_j^a y_i^b] \\ &\leqslant \sum_{s=1}^{n} y_s^{a+b} + \sum_{i\neq j} \left[\left(\frac{a}{a+b}\, y_i^{a+b} + \frac{b}{a+b}\, y_j^{a+b}\right)\right. \\ &\quad \left. + \left(\frac{a}{a+b}\, y_j^{a+b} + \frac{b}{a+b}\, y_i^{a+b}\right)\right] \\ &= \sum_{s=1}^{n} y_s^{a+b} + \binom{n}{2} \sum_{s=1}^{n} y_s^{a+b}.\end{aligned}$$

Let us denote by B_R and B_ρ balls compactly contained in Ω, of radii respectively R, ρ and with the same center.

LEMMA 2.10. *There is a constant c_6 such that, for every R and ρ $(0<\rho<R\leqslant\rho+1)$ and for every $\alpha\geqslant 2$, then*

$$\begin{aligned}&\left(\int_{B_\rho} \sum_{s=1}^{n} (1+|u_{x_s}|^2)^{(\alpha+p-2)\,2^*/4}\,dx\right)^{2/2^*} \\ &\qquad \leqslant \frac{c_6\alpha^3}{(R-\rho)^2} \int_{B_R} \sum_{s=1}^{n} (1+|u_{x_s}|^2)^{(\alpha+q-2)/2}\,dx,\end{aligned}$$

where $2^*=2n/(n-2)$ if $n>2$, while 2^* is any fixed number greater than $2q/p$, if $n=2$.

Proof. By computing the gradient of $(1+|u_{x_s}|^2)^{(\alpha+p-2)/4}$ we obtain the estimate

$$\begin{aligned}&|D[\eta(1+|u_{x_s}|^2)^{(\alpha+p-2)/4}]|^2 \\ &\qquad \leqslant \frac{(\alpha+p-2)^2}{2}\,\eta^2(1+|u_{x_s}|^2)^{(\alpha+p-4)/2}\,|Du_{x_s}|^2 \\ &\qquad\quad + 2\,|D\eta|^2\,(1+|u_{x_s}|^2)^{(\alpha+p-2)/2}.\end{aligned}$$

Since $p\leqslant q$, from Lemma 2.8 we deduce that there is a constant c_7 such that

$$\int_\Omega \sum_{s=1}^n |D[\eta(1+|u_{x_s}|^2)^{(\alpha+p-2)/4}]|^2\,dx$$
$$\leqslant c_7\alpha^3 \int_\Omega (\eta^2+|D\eta|^2) \sum_{s=1}^n (1+|u_{x_s}|^2)^{(\alpha+q-2)/2}\,dx. \tag{2.30}$$

By Sobolev's inequality, for every $s=1, 2, ..., n$, we have

$$\left(\int_\Omega [\eta(1+|u_{x_s}|^2)^{(\alpha+p-2)/4}]^{2^*}\,dx\right)^{2/2^*}$$
$$\leqslant c_8 \int_\Omega |D[\eta(1+|u_{x_s}|^2)^{(\alpha+p-2)/4}]|^2\,dx. \tag{2.31}$$

By using the inequality $\sum_{s=1}^n y_s^a \leqslant (\sum_{s=1}^n y_s)^a$ with $a=2^*/2>1$, and Minkowski's inequality with exponent $2^*/2$, from (2.30), (2.31) we obtain

$$\left(\int_\Omega \eta^{2^*} \sum_{s=1}^n (1+|u_{x_s}|^2)^{(\alpha+p-2)\,2^*/4}\,dx\right)^{2/2^*}$$
$$\leqslant \left(\int_\Omega \left[\eta^2 \sum_{s=1}^n (1+|u_{x_s}|^2)^{(\alpha+p-2)/2}\right]^{2^*/2} dx\right)^{2^*/2}$$
$$\leqslant \sum_{s=1}^n \left(\int_\Omega [\eta(1+|u_{x_s}|^2)^{(\alpha+p-2)/4}]^{2^*}\,dx\right)^{2/2^*}$$
$$\leqslant c_9\alpha^3 \int_\Omega (\eta^2+|D\eta|^2) \sum_{s=1}^n (1+|u_{x_s}|^2)^{(\alpha+q-2)/2}\,dx.$$

We obtain the conclusion of the proof of Lemma 2.10 by taking as test function η such that $\eta\in C_0^1(B_R)$, $\eta\geqslant 0$ in B_R, $\eta=1$ on B_ρ and $|D\eta|\leqslant 2/(R-\rho)$.

We define by induction a sequence α_k in the following way:

$$\alpha_1=2; \qquad \alpha_{k+1}=(\alpha_k+p-2)\,\frac{2^*}{2}-(q-2), \qquad \forall k\geqslant 1. \tag{2.32}$$

LEMMA 2.11. *If α_k is the sequence defined in* (2.32), *then the following representation formulas hold*:

$$\alpha_k=2+\left(p\,\frac{2^*}{2}-q\right)\sum_{i=0}^{k-2}\left(\frac{2^*}{2}\right)^i, \qquad \forall k\geqslant 2; \tag{2.33}$$

$$\alpha_k=2+\frac{p(2^*/2)-q}{(2^*/2)-1}\left[\left(\frac{2^*}{2}\right)^{k-1}-1\right], \qquad \forall k\geqslant 1. \tag{2.34}$$

Proof. Of course for every $k \geqslant 2$ the representation formulas (2.33), (2.34) are equivalent to each other. We prove (2.33) by induction. For $k=2$ the right hand side of (2.33) is equal to $2+p(2^*/2)-q$, like in (2.32). If we assume that (2.33) holds for some k, then, by (2.32):

$$\begin{aligned}\alpha_{k+1} &= (\alpha_{k+1}+q-2)+2-q=(\alpha_k+p-2)\frac{2^*}{2}+2-q \\ &= (\alpha_k-2)\frac{2^*}{2}+p\frac{2^*}{2}+2-q \\ &= 2+\left(p\frac{2^*}{2}-q\right)\sum_{i=0}^{k-2}\left(\frac{2^*}{2}\right)^{i+1}+\left(p\frac{2^*}{2}-q\right) \\ &= 2+\left(p\frac{2^*}{2}-q\right)\sum_{i=0}^{k-1}\left(\frac{2^*}{2}\right)^{i}.\end{aligned}$$

For $0<\rho_0<R_0\leqslant\rho_0+1$ let us define $R_k=\rho_0+(R_0-\rho_0)\,2^{-k}$, $\forall k\geqslant 1$. Let us insert in the estimate of Lemma 2.10 $R=R_k$ and $\rho=R_{k+1}$ (thus $R-\rho=(R_0-\rho_0)\,2^{-(k+1)}$). Let us also define

$$A_k=\left(\int_{B_{R_k}}\sum_{s=1}^{n}(1+|u_{x_s}|^2)^{(\alpha_k+q-2)/2}\,dx\right)^{1/(\alpha_k+q-2)}, \qquad \forall k\geqslant 1. \tag{2.35}$$

Thus, under our notations, the estimate of Lemma 2.10 can be written in the form $(\forall k\geqslant 1)$:

$$A_{k+1}\leqslant\left[\frac{c_6\alpha_k^3 4^{k+1}}{(R_0-\rho_0)^2}\right]^{1/(\alpha_k+p-2)}\cdot A_k^{(\alpha_k+q-2)/(\alpha_k+p-2)}. \tag{2.36}$$

LEMMA 2.12. *Let θ be defined by*

$$\theta=\prod_{i=1}^{\infty}\frac{\alpha_i+q-2}{\alpha_i+p-2}. \tag{2.37}$$

Then θ is finite and is given by

$$\theta=\frac{q}{p}\cdot\frac{(2^*/2)-1}{(2^*/2)-(q/p)}. \tag{2.38}$$

Proof. By using the definition of α_k in (2.32) we have

$$\prod_{i=1}^{k}\frac{\alpha_i+q-2}{\alpha_i+p-2}=q\left(\frac{2^*}{2}\right)^{k-1}\frac{1}{\alpha_k+p-2}.$$

By (2.34) we deduce that

$$\prod_{i=1}^{k} \frac{\alpha_i+q-2}{\alpha_i+p-2} = \frac{q(2^*/2)^{k-1}}{p+\frac{p(2^*/2)-q}{(2^*/2)-1}\left[\left(\frac{2^*}{2}\right)^{k-1}-1\right]}.$$

Since $2^*/2>1$, as $k \to +\infty$ we obtain (2.38).

Remark 2.13. Note in particular that $\theta \geqslant 1$ and that $\theta=1$ if and only if $p=q$.

If $n>2$, then the expression of θ in (2.38) is the same as in (2.8). Moreover, if $n=2$ and $p<q$, then $2^*/2$ is any number greater than q/p; thus we can choose $2^*/2$ so large that θ in (2.38) is as close to q/p as we like.

LEMMA 2.14. *There are positive constants β and c_{10} such that*

$$A_{k+1} \leqslant c_{10}\left(\frac{1}{(R_0-\rho_0)^{\beta}} A_1\right)^{\theta}, \qquad \forall k \geqslant 1.$$

Proof. Without loss of generality we can assume that $A_1 \geqslant 1$ and that $c_6 \geqslant 2^{-7}$. By iterating (2.36) we can easily see that

$$A_{k+1} \leqslant A_1^{\theta} \cdot \prod_{i=1}^{k} \left[\frac{c_6\alpha_i^3 4^{i+1}}{(R_0-\rho_0)^2}\right]^{\theta/(\alpha_i+p-2)} \leqslant A_1^{\theta} c_{10} (R_0-\rho_0)^{-2\theta \sum_{i=1}^{\infty} 1/(\alpha_i+p-2)}, \tag{2.39}$$

where c_{10} is the constant (the series is convergent since, by (2.34), α_i grows exponentially):

$$c_{10} = \exp\left(\theta \sum_{i=1}^{\infty} \frac{\log[c_6\alpha_i^3 4^{i+1}]}{\alpha_i+p-2}\right) < +\infty.$$

About the series in (2.39) we deduce from (2.34) that

$$\beta \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} \frac{2}{\alpha_i+p-2} \leqslant 2 \sum_{i=1}^{\infty} \left[\frac{p(2^*/2)-q}{(2^*/2)-1}\left(\frac{2^*}{2}\right)^{i-1}\right]^{-1} = \frac{2^*}{p(2^*/2)-q}. \tag{2.40}$$

Note that, if $p=q$, then (2.40) holds with equality.

The conclusion of Lemma 2.14 follows from (2.39) and (2.40).

Let us apply Lemma 2.14. To this aim let us recall the definition of A_k

in (2.35). Since $\rho_0 < R_k < R_0$, for every $s = 1, 2, ..., n$ and every $k \geqslant 2$ we have

$$\left(\int_{B_{\rho_0}} (1 + |u_{x_s}|^{(\alpha_k + q - 2)/2}\, dx\right)^{1/(\alpha_k + q - 2)}$$
$$\leqslant \frac{c_{10}}{(R_0 - \rho_0)^{\beta\theta}} \left(\int_{B_{R_0}} \sum_{i=1}^{n} (1 + |u_{x_i}|^2)^{q/2}\, dx\right)^{\theta/q}.$$

Since $(\alpha_k + q - 2) \to +\infty$, as $k \to +\infty$ the left hand side converges to the essential supremum of $(1 + |u_{x_s}|^2)^{1/2}$ in B_{ρ_0}. By adding up with respect to $s = 1, 2, ..., n$ we obtain (2.9) and thus we conclude the proof of Theorem 2.1.

3. Interpolation

In this section we utilize the interpolation inequality

$$\|v\|_{L^q} \leqslant \|v\|_{L^p}^{p/q} \cdot \|v\|_{L^\infty}^{1-p/q} \tag{3.1}$$

(consequence of the pointwise inequality $|v(x)|^q \leqslant |v(x)|^p \, \|v\|_{L^\infty}^{q-p}$; see also Brezis [1], *Commentaires sur le chapitre* IX) to deduce from the results of the previous section new estimates of the essential supremum of the modulus of the gradient of weak solutions in terms of its L^p-norm.

Let us consider again Eq. (2.1) with $a^i(x, \xi)$ satisfying (2.3), (2.4), (2.5), (2.6) for some positive constants m, M and for exponents p, q related by

$$2 \leqslant p \leqslant q < \frac{n+2}{n}\, p. \tag{3.2}$$

Like in the previous section we denote by B_R, B_ρ balls compactly contained in Ω (open set of $\mathbb{R}^n$, $n \geqslant 2$) of radii respectively R, ρ and with the same center. Finally, let α and θ by defined by

$$\alpha = \frac{2p}{(n+2)\,p - nq}, \qquad \theta = \frac{2q}{np - (n-2)\,q}, \tag{3.3}$$

if $n > 2$; moreover, if $n = 2$ and $q/p > 1$, then let θ be any number such that $q/p < \theta < q/(q-p)$ and let α be defined by the following formula (3.6); finally, if $n = 2$ and $p = q$, then let $\alpha = \theta = 1$.

Theorem 3.1. *Let* $b \in L^\infty_{\text{loc}}(\Omega)$ *and let* (2.3), (2.4), (2.5), (2.6), *and* (3.2) *hold. Let* α, θ *be defined by* (3.3). *There are positive numbers* c *and* β *such that*

$$\|(1+|Du|^2)^{1/2}\|_{L^q(B_\rho)} \leqslant c\left(\frac{1}{(R-\rho)^{\beta(q-p)/p}}\,\|(1+|Du|^2)^{1/2}\|_{L^p(B_R)}^{1/\theta}\right)^{\alpha} \quad (3.4)$$

$$\|(1+|Du|^2)^{1/2}\|_{L^\infty(B_\rho)} \leqslant c\left(\frac{1}{(R-\rho)^{\beta q/p}}\,\|(1+|Du|^2)^{1/2}\|_{L^p(B_R)}\right)^{\alpha} \quad (3.5)$$

for every weak solution u of class $W^{1,q}_{\mathrm{loc}}(\Omega)$ *to Eq.* (2.1) *and for every* ρ *and* R *such that* $0<\rho<R\leqslant\rho+1$.

Remark 3.2. By a direct computation we can see that

$$\alpha=\frac{\theta p/q}{1-\theta(1-p/q)}. \quad (3.6)$$

Thus the value of the exponent α in (3.5) is the same as that one in the inequality (1.10) in the introduction, that has been deduced formally.

Proof of Theorem 3.1. Let us apply the interpolation inequality (3.1) with $v=(1+|Du|^2)^{1/2}$. Let us define $\gamma=\theta(1-p/q)$. By the estimate (2.9) we obtain

$$\begin{aligned}\|(1+|Du|^2)^{1/2}\|_{L^q(B_\rho)} &\leqslant \|(1+|Du|^2)^{1/2}\|_{L^p(B\rho)}^{p/q}\cdot\|(1+|Du|^2)^{1/2}\|_{L^\infty(B\rho)}^{1-p/q}\\ &\leqslant c^{1-p/q}\,\|(1+|Du|^2)^{1/2}\|_{L^p(B\rho)}^{p/q}\\ &\quad\cdot\left(\frac{1}{(R-\rho)^{\beta}}\,\|(1+|Du|^2)^{1/2}\|_{L^q(B_R)}\right)^{\gamma}. \end{aligned} \quad (3.7)$$

For $R_0>\rho_0>0$ and for every $k\geqslant 1$ let us define $\rho_k=R_0-(R_0-\rho_0)\,2^{-k}$ (note that this subdivision of the interval $[\rho_0, R_0]$ is different from that one considered in Section 2; with the subdivision considered there we would not reach the conclusion here). Let us insert in (3.7) $\rho=\rho_k$ and $R=\rho_{k+1}$; then we have $R-\rho=(R_0-\rho_0)\,2^{-(k+1)}$. For $k=0, 1, 2, \ldots$ let us also define

$$B_k=\|(1+|Du|^2)^{1/2}\|_{L^q(B_{\rho_k})}. \quad (3.8)$$

With these notations, by (3.7) for every $k\geqslant 0$ we have

$$B_k\leqslant c^{1-p/q}\,\|(1+|Du|^2)^{1/2}\|_{L^p(B_{R_0})}^{p/q}\cdot\left(\frac{2^{\beta(k+1)}}{(R_0-\rho_0)^{\beta}}\,B_{k+1}\right)^{\gamma}.$$

By iterating the previous inequality we can see that for $k\geqslant 1$, we have

$$B_0\leqslant\left(\frac{c^{1-p/q}}{(R_0-\rho_0)^{\beta\gamma}}\,\|(1+|Du|^2)^{1/2}\|_{L^p(B_{R_0})}^{p/q}\right)^{\sum_{i=0}^{k-1}\gamma^i}\cdot 2^{\beta\sum_{i=0}^{k} i\gamma^i}\cdot(B_k)^{\gamma^k}.$$

The assumptions (3.2) implies that $\gamma < 1$. Thus the series previously written are convergent. Since B_k is bounded by

$$B_k \leqslant \|(1+|Du|^2)^{1/2}\|_{L^q(B_{R_0})}, \qquad \forall k \in \mathbb{N},$$

we can go to the limit as $k \to +\infty$ and we obtain (for some constant c_1),

$$B_0 \leqslant c_1 \left(\frac{1}{(R_0-\rho_0)^{\beta\gamma}} \|(1+|Du|^2)^{1/2}\|_{L^p(B_{R_0})}^{p/q} \right)^{1/(1-\gamma)}.$$

Since $\gamma/(1-\gamma) = ((q-p)/p)\alpha$ and $p/(q(1-\gamma)) = \alpha/\theta$, we have proved (3.4).

The estimate (3.5) can be proved either in the same way, or by combining (3.4) and (2.9). In fact, for example by (2.9), (3.4), if $\rho' = (R+\rho)/2$, we have

$$\begin{aligned}
&\|(1+|Du|^2)^{1/2}\|_{L^\infty(B_\rho)} \\
&\leqslant c \left(\frac{1}{(\rho'-\rho)^\beta} \|(1+|Du|^2)^{1/2}\|_{L^q(B_{\rho'})} \right)^\theta \\
&\leqslant c_2 \left(\frac{1}{(\rho'-\rho)^\beta} \cdot \frac{1}{(R-\rho')^{\frac{\beta(q-p)}{p}\alpha}} \|(1+|Du|^2)^{1/2}\|_{L^p(B_R)}^{\alpha/\theta} \right)^\theta.
\end{aligned}$$

Since $\rho' - \rho = R - \rho'$ and since $1 + ((q-p)/p)\alpha = (q/p)\cdot(\alpha/\theta)$, we have the conclusion (3.5).

4. Existence

In this section we consider the Dirichlet problem (4.1) in a bounded open set $\Omega \subset \mathbb{R}^n$ with $n \geqslant 2$:

$$\begin{cases} u(x) = u_0(x), & x \in \partial\Omega; \\ \displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i} a^i(x, Du) = b(x), & x \in \Omega. \end{cases} \tag{4.1}$$

The functions $a^i(x,\xi)$, for $i = 1, 2, \ldots, n$, are supposed to be locally Lipschitz-continuous in $\Omega \times \mathbb{R}^n$.

We will utilize the regularity and interpolation results proved in Sections 2 and 3. Thus, like in the previous section, we assume that (2.3), (2.4), (2.5), and (2.6) hold for some positive constants m, M and for exponents p, q satisfying

$$2 \leqslant p \leqslant q < \frac{n+2}{n} p. \tag{4.2}$$

For the existence theory we need also an assumption on a^i (other than on its derivatives), for example of the type

$$|a^i(x,0)| \leqslant M, \qquad \forall x \in \Omega,\ \forall i = 1, 2, ..., n. \tag{4.3}$$

Finally we assume that

$$b \in L^{p/(p-1)}(\Omega) \cap L^\infty_{\text{loc}}(\Omega); \qquad u_0 \in W^{1,r}(\Omega), \text{ with } r = p(q-1)/(p-1). \tag{4.4}$$

Under the previous assumptions, by a *weak solution of class* $W^{1,q}_{\text{loc}}(\Omega)$ *to the Dirichlet problem* (4.1) we mean a function u in the Sobolev class

$$u - u_0 \in W^{1,p}_0(\Omega) \cap W^{1,q}_{\text{loc}}(\Omega) \tag{4.5}$$

such that, for every $\Omega' \subset\subset \Omega$

$$\int_\Omega \left\{ \sum_{i=1}^n a^i(x, Du)\, \phi_{x_i} + b(x)\phi \right\} dx = 0, \qquad \forall \phi \in W^{1,q}_0(\Omega'). \tag{4.6}$$

THEOREM 4.1. *Let* (2.3), (2.4), (2.5), (2.6), (4.2), (4.3), *and* (4.4) *hold. Then there exists a weak solution* u *of class* $W^{1,q}_{\text{loc}}(\Omega)$ *to the Dirichlet problem* (4.1). *Moreover the* $W^{1,p}(\Omega)$*-norm of* u *is bounded by a quantity that depends only on* n, m, M, p, q, $\|b\|_{L^{p/(p-1)}}$, $\|Du_0\|_{L^r}$. *Finally,* $u \in W^{1,\infty}_{\text{loc}}(\Omega) \cap W^{2,2}_{\text{loc}}(\Omega)$ *and for every* $\Omega' \subset\subset \Omega$ *there is a constant* c *such that, for* α *and* θ *given by* (3.3), *then*

$$\|(1 + |Du|^2)^{1/2}\|_{L^\infty(\Omega')} \leqslant c\, \|(1 + |Du|^2)^{1/2}\|^{\alpha}_{L^p(\Omega)}, \tag{4.7}$$

$$\|D^2 u\|_{L^2(\Omega')} \leqslant c\, \|(1 + |Du|^2)^{1/2}\|^{q\alpha/2\theta}_{L^p(\Omega)}. \tag{4.8}$$

Remark 4.2. If $u \in W^{1,q}(\Omega)$ then, by the usual method of monotonicity, it is easy to show that, under our assumptions, the Dirichlet problem (4.1) has at most one solution in the class $u_0 + W^{1,q}_0(\Omega)$. Thus the problem of uniqueness is related to the a priori regularity of weak solutions up to the boundary. We do not discuss the boundary regularity in this paper.

By Theorem 4.1, Corollary 2.2, and by integrating by parts in (4.6) we deduce the following:

COROLLARY 4.3. *Let the assumptions of Theorem* 4.1 *hold. Let us assume also that* $a^i \in C^{1,\alpha}_{\text{loc}}(\Omega \times \mathbb{R}^n)$ *for* $i = 1, 2, ..., n$ *and that* $b \in C^{0,\alpha}_{\text{loc}}(\Omega)$. *Then there exists a solution to the Dirichlet problem*

$$\begin{cases} u \in (u_0 + W^{1,p}_0(\Omega)) \cap C^{2,\alpha}_{\text{loc}}(\Omega) \\ \displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i} a^i(x, Du) = b(x), \qquad \forall x \in \Omega \end{cases} \tag{4.9}$$

(*the equation is satisfied for every* $x \in \Omega$ *in the classical sense*). *More regularity holds like in Corollary* 2.2.

The proof of Theorem 4.1 will follow through some lemmas.

LEMMA 4.4. *Under the assumptions* (2.3), (2.4), *and* (4.3) *there is a constant* c_1 *such that, for every* ξ, $\eta \in \mathbb{R}^n$ *and for every* $x \in \Omega$,

$$|\xi|^p \leqslant c_1 \left\{(1+|\eta|^2)^{p(q-1)/(2(p-1))} + \sum_{i=1}^n a^i(x, \xi)(\xi_i - \eta_i)\right\}.$$

Proof. For ξ, $\eta \in \mathbb{R}^n$ let us define

$$f(t) = \sum_{i=1}^n a^i(x, t\xi + (1-t)\eta)(\xi_i - \eta_i), \qquad \forall t \in [0, 1].$$

By (2.3) and by Jensen's inequality we obtain

$$\begin{aligned} &\sum_{i=1}^n [a^i(x, \xi) - a^i(x, \eta)](\xi_i - \eta_i) \\ &\quad = f(1) - f(0) = \int_0^1 f'(t)\, dt \\ &\quad = \int_0^1 \sum_{i,j} a^i_{\xi_j}(x, \eta + t(\xi - \eta))(\xi_i - \eta_i)(\xi_j - \eta_j)\, dt \\ &\quad \geqslant m\, |\xi - \eta|^2 \int_0^1 (1 + |\eta + t(\xi - \eta)|^2)^{(p-2)/2}\, dt \\ &\quad \geqslant m\, |\xi - \eta|^2 \left(1 + \left|\int_0^1 \{\eta + t(\xi - \eta)\}\, dt\right|^2\right)^{(p-2)/2} \\ &\quad = m\, |\xi - \eta|^2 \left(1 + \left|\frac{\xi + \eta}{2}\right|^2\right)^{(p-2)/2}. \end{aligned} \tag{4.10}$$

There are constants c_2 and c_3 such that, for every $\varepsilon > 0$,

$$\begin{aligned} |\xi|^p &= |\xi|^2\, |\xi|^{p-2} \leqslant c_2(|\xi - \eta|^2 + |\eta|^2)(|\xi + \eta|^{p-2} + |\eta|^{p-2}) \\ &\leqslant c_3(|\xi - \eta|^2\, |\xi + \eta|^{p-2} + |\xi|^2\, |\eta|^{p-2} + |\xi|^{p-2}\, |\eta|^2 + |\eta|^p) \\ &\leqslant c_3 \left\{|\xi - \eta|^2\, |\xi + \eta|^{p-2} + \left(\frac{2\varepsilon^{p/2}}{p} + \frac{(p-2)\, \varepsilon^{p/(p-2)}}{p}\right) |\xi|^p \right. \\ &\qquad \left. + \left(\frac{p-2}{p\varepsilon^{p/(p-2)}} + \frac{2}{p\varepsilon^{p/2}} + 1\right) |\eta|^p \right\}. \end{aligned} \tag{4.11}$$

By choosing ε sufficiently small, by (4.10) and (4.11) we deduce the existence of a constant c_4 such that

$$|\xi|^p \leqslant c_4 \left\{ |\eta|^p + \sum_{i=1}^n [a^i(x,\xi) - a^i(x,\eta)](\xi_i - \eta_i) \right\}. \tag{4.12}$$

For every fixed $i = 1, 2, ..., n$ and $\eta \in \mathbb{R}^n$ let us define $g(t) = a^i(x, t\eta)$, $\forall t \in [0, 1]$. By (2.4) and (4.3) we have

$$\begin{aligned} |a^i(x,\eta)| &\leqslant |a^i(x,0)| + \int_0^1 |g'(t)|\, dt \\ &\leqslant M + \int_0^1 \left| \sum_{j=1}^n a^i_{\xi_j}(x, t\eta)\, \eta_j \right| dt \leqslant (n+1)\, M(1+|\eta|^2)^{(q-1)/2}. \end{aligned} \tag{4.13}$$

By (4.12), (4.13) there is a constant c_5 such that, for every $\varepsilon > 0$,

$$\begin{aligned} |\xi|^p \leqslant c_5 \Bigg\{ &|\eta|^p + \sum_{i=1}^n a^i(x,\xi)(\xi_i - \eta_i) + \frac{\varepsilon^p}{p}(|\xi|^p + |\eta|^p) \\ &+ \frac{p-1}{p\varepsilon^{p/(p-1)}}(1+|\eta|^2)^{(q-1)/2 \cdot p/(p-1)} \Bigg\}. \end{aligned}$$

We obtain the conclusion of the proof of lemma 4.4 choosing ε sufficiently small.

For every $\varepsilon \in (0, 1]$ let us consider the Dirichlet problem

$$\begin{cases} u - u_0 \in W_0^{1,q}(\Omega) \\ \displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i} [a^i(x, Du) + \varepsilon(1 + |Du|^2)^{(q-2)/2} u_{x_i}] = b(x). \end{cases} \tag{4.14}$$

By (4.10) the differential operator associated to $\{a^i\}$ is monotone. We can apply the theory of monotone operators (see, for example, [11, 12]) to infer the existence, for every $\varepsilon \in (0, 1]$, of a (unique) solution $u_\varepsilon \in W^{1,q}(\Omega)$ to the Dirichlet problem (4.14).

LEMMA 4.5. *Under the assumptions* (2.3), (2.4), (4.3), (4.4) *there is a constant* c_6 (*independent of* ε) *such that*

$$\|u_\varepsilon\|_{W^{1,p}(\Omega)} \leqslant c_6, \qquad \forall \varepsilon \in (0, 1]. \tag{4.15}$$

Proof. Let us use the notation

$$a^i_\varepsilon(x,\xi) = a^i(x,\xi) + \varepsilon(1+|\xi|^2)^{(q-2)/2}\, \xi_i. \tag{4.16}$$

Then, for $0<\varepsilon\leqslant 1$, a^i_ε satisfies the same assumptions as a^i with constants $m'=m$ and $M'=M+(q-1)$. Thus all the previous estimates hold for u_ε with constants independent of $\varepsilon\in(0,1]$.

In particular, if we apply Lemma 4.4 to a^i_ε with $\xi=Du_\varepsilon$ and with $\eta=Du_0$, then for all $\delta>0$ we have

$$\int_\Omega |Du_\varepsilon|^p\,dx\leqslant c_1\left\{\int_\Omega (1+|Du_0|^2)^{p(q-1)/(2(p-1))}\,dx-\int_\Omega b(u_\varepsilon-u_0)\,dx\right\}$$

$$\leqslant c_1\left\{\int_\Omega (1+|Du_0|^2)^{p(q-1)/(2(p-1))}\,dx\right.$$

$$\left.+\frac{\delta^p}{p}\int_\Omega |u_\varepsilon-u_0|^p\,dx+\frac{p-1}{p\delta^{p/(p-1)}}\int_\Omega |b|^{p/(p-1)}\,dx\right\}.$$

We obtain the conclusion of Lemma 4.5 by using assumption (4.4), Sobolev's inequality and by choosing δ sufficiently small.

LEMMA 4.6. *Under the assumptions of Theorem* 4.1, *for every $\Omega'\subset\subset\Omega$ there is a constant c_7 such that*

$$\int_{\Omega'} |D^2u_\varepsilon|^2\,dx\leqslant c_7\,\|(1+|Du_\varepsilon|^2)^{1/2}\|^{(\alpha/\theta)q}_{L^p(\Omega)},\qquad \forall\varepsilon\in(0,1].\tag{4.17}$$

Proof. Let us use Lemma 2.8 with $\alpha=2$. Since $p\geqslant 2$, we deduce that, for some constant c_8:

$$\int_\Omega \eta^2\,|D^2u_\varepsilon|^2\,dx\leqslant\int_\Omega \eta^2\sum_{s=1}^n (1+|(u_\varepsilon)_{x_s}|^2)^{(p-2)/2}\,|D(u_\varepsilon)_{x_s}|^2\,dx$$

$$\leqslant c_8\int_\Omega (\eta^2+|D\eta|^2)(1+|Du_\varepsilon|^2)^{q/2}\,dx.$$

Now the thesis of Lemma 4.6 follows easily from the interpolation inequality (3.4).

LEMMA 4.7. *Under the assumptions of Theorem* 4.1, *for every $\Omega'\subset\subset\Omega$ there is a constant c_9 such that*

$$\|(1+|Du_\varepsilon|^2)^{1/2}\|_{L^\infty(\Omega')}\leqslant c_9\,\|(1+|Du_\varepsilon|^2)^{1/2}\|^{\alpha}_{L^p(\Omega)},\qquad \forall\varepsilon\in(0,1].\tag{4.18}$$

Proof. Is a consequence of the interpolation inequality (3.5).

We are ready to go to the limit as $\varepsilon\to 0$. By Lemmas 4.5 and 4.6 the sequence u_ε is bounded in $W^{2,2}_{\text{loc}}(\Omega)$; by Lemmas 4.5 and 4.7 the sequence u_ε is bounded in $W^{1,\infty}_{\text{loc}}(\Omega)$. Thus we can extract a sequence, that we will continue to denote by u_ε, that, as $\varepsilon \to 0$, converges in the strong topology of $W^{1,2}_{\text{loc}}(\Omega)$ to a function u in the Sobolev class

$$u \in (u_0 + W_0^{1,p}(\Omega)) \cap W^{1,\infty}_{\text{loc}}(\Omega) \cap W^{2,2}_{\text{loc}}(\Omega).$$

By extracting a subsequence, we can assume that Du_ε converges to Du almost everywhere in Ω.

Let $\Omega' \subset\subset \Omega$ and let $\phi \in W_0^{1,q}(\Omega')$. Since $|Du_\varepsilon(x)|$ is pointwise bounded in Ω' independently of ε, we can go to the limit as $\varepsilon \to 0$ in the integral identity

$$\int_{\Omega'} \left\{ \sum_{i=1}^{n} a_\varepsilon^i(x, Du_\varepsilon)\, \phi_{x_i} + b(x)\phi \right\} = 0$$

and we obtain that u is a weak solution (of class $W^{1,\infty}_{\text{loc}}(\Omega)$) to the Dirichlet problem (4.1).

Finally (4.7), (4.8) hold for u, other than for u_ε, by (4.15), (4.17), (4.18) and since the lower semicontinuity of the norms.

5. Some More on Regularity, Interpolation, and Existence

In this section we consider again the elliptic Eq. (1.1) in an open set $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) and we assume that, for some positive constants m, M, for every ξ, $\lambda \in \mathbb{R}^n$ and for a.e. $x \in \Omega$:

$$\sum_{i,j} a^i_{\xi_j}(x, \xi)\, \lambda_i \lambda_j \geq m(1 + |\xi|^2)^{(p-2)/2}\, |\lambda|^2; \tag{5.1}$$

$$|a^i_{\xi_j}(x, \xi)| \leq M(1 + |\xi|^2)^{(q-2)/2}, \qquad \forall i, j; \tag{5.2}$$

$$|a^i_{x_s}(x, \xi)| \leq M(1 + |\xi|^2)^{(q-1)/2}, \qquad \forall i, s. \tag{5.3}$$

Theorem 5.1. *Let $b \in L^\infty_{\text{loc}}(\Omega)$ and let* (5.1), (5.2), (5.3) *hold with exponents p, q related by*

$$2 \leq p \leq q < \frac{n-1}{n-2}\, p \tag{5.4}$$

($2 \leq p \leq q$, *if* $n = 2$). *Then every weak solution $u \in W^{1,2q-p}_{\text{loc}}(\Omega)$ to Eq.* (2.1) *is of class $W^{1,\infty}_{\text{loc}}(\Omega)$; the estimate* (2.9) *holds with q replaced by $2q-p$ and θ given by*

$$\theta = \frac{2q-p}{(n-1)p - (n-2)q}, \qquad \text{if} \quad n > 2, \tag{5.5}$$

while, if $n = 2$, then θ is any number strictly greater than $(2q-p)/p$ if $q > p$ and $\theta = 1$ if $q = p$.

Proof. Let $r=2q-p$. Then (5.2), (5.3) can be written respectively in the form

$$|a^i_{\xi_j}(x,\xi)|\leqslant M(1+|\xi|^2)^{(p+r-4)/4}, \qquad \forall i,j;$$
$$|a^i_{x_s}(x,\xi)|\leqslant M(1+|\xi|^2)^{(p+r-2)/4}, \qquad \forall i,s.$$

Moreover (5.4), in terms of p and r, is equivalent to $2\leqslant p\leqslant r<(n/(n-2))\,p$.

Thus all the assumptions of theorem 2.1 are satisfied with q replaced by r. In particular (2.4) holds with q replaced by r since $r\geqslant q$. Then the conclusion of Theorem 2.1 holds with $\theta=2r/[np-(n-2)r]$ (if $n>2$), that corresponds to (5.5).

By starting from the previous theorem, instead of Theorem 2.1, by the interpolation inequality (3.1) and with the same proof of Section 3 we obtain:

THEOREM 5.2. *Let* $b\in L^\infty_{\mathrm{loc}}(\Omega)$ *and let* (5,1), (5.2), (5.3) *hold for some exponents* p, q *related by*

$$2\leqslant p\leqslant q<\frac{n+2+\sqrt{n^2+4}}{2n}\,p. \tag{5.6}$$

Then the estimates (3.4), (3.5) *of Theorem* 3.1 *hold with* α, θ *given by*

$$\alpha=\frac{p(2q-p)}{-nq^2+(n+2)\,pq-p^2}, \qquad \theta=\frac{2q-p}{(n-1)p-(n-2)q}, \tag{5.7}$$

if $n>2$; *while, if* $n=2$ *and* $q>p$, *then let* θ *be any number such that* $(2q-p)/p<\theta<q/(q-p)$ *and let* α *be defined by* (3.6); *finally, if* $n=2$ *and* $q=p$, *then let* $\alpha=\theta=1$.

By using the regularity and interpolation results of Theorems 5.1 and 5.2, with the same method of Section 4 we can prove the following:

THEOREM 5.3. *Let* (4.3), (5.1), (5.2), (5.3), *and* (5.6) *hold. Let us also assume that*

$$b\in L^{p/(p-1)}(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega); \qquad u_0\in W^{1,r}(\Omega),\ \text{with } r=\frac{p(2q-p-1)}{p-1}. \tag{5.8}$$

Then there is a solution of class $W^{1,q}_{\mathrm{loc}}(\Omega)$ *to the Dirichlet problem* (4.1). *Moreover the estimates of Theorem* 4.1 *hold with* α, θ *given by* (5.7).

Remark 5.4. Under the assumptions made in the present section, more regularity holds like in Corollaries 2.2 and 4.3.

Remark 5.5. Let us consider in a bounded open set $\Omega \subset \mathbb{R}^n$ the p.d.e.

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} (\alpha(x)(1+|Du|^2)^{\beta(x)} u_{x_i}) = b(x). \tag{5.9}$$

If $\beta(x) = \alpha(x) - 1$ and $b(x) = 0$, then (5.9) is the Euler's equation associated to the integral (1.7), of the type considered by Zhikov [19].

Let us assume that $\alpha(x)$ and $\beta(x)$ are Lipschitz-continuous functions in $\bar{\Omega}$ such that $\alpha(x) \geqslant m > 0$ and $\beta(x) \geqslant 0$ for every $x \in \bar{\Omega}$. Moreover we assume that the oscillation of $\beta(x)$ in Ω is sufficiently small; precisely

$$\frac{1+\sup\{\beta(x) : x \in \Omega\}}{1+\inf\{\beta(x) : x \in \Omega\}} < \frac{n+2+\sqrt{n^2+4}}{2n}. \tag{5.10}$$

With the position $a^i(x, \xi) = \alpha(x)(1+|\xi|^2)^{\beta(x)} \xi_i$, we note in particular that, for every $\xi, \lambda \in \mathbb{R}^n$ and for $x \in \Omega$,

$$\sum_{i,j} a^i_{\xi_j}(x, \xi)\, \lambda_i \lambda_j \geqslant m(1+|\xi|^2)^{\beta(x)} |\lambda|^2;$$

$$a^i_{x_s}(x, \xi) = (\alpha_{x_s} + \alpha\beta_{x_s} \log(1+|\xi|^2))(1+|\xi|^2)^{\beta(x)} \xi_i.$$

If we take exponents p, q such that $p = 2(1+\inf\{\beta(x) : x \in \Omega\})$ and

$$2(1+\sup\{\beta(x) : x \in \Omega\}) < q < \frac{n+2+\sqrt{n^2+4}}{n} (1+\inf\{\beta(x) : x \in \Omega\})$$

then all the assumptions of Theorem 5.3 are satisfied. Thus the Dirichlet problem associated to (5.9), with data satisfying (5.8), has a weak solution with all the regularity stated previously. In particular, if $b \in C^{0,\alpha}_{\text{loc}}(\Omega)$ and α, $\beta \in C^{1,\alpha}_{\text{loc}}(\Omega)$, then there is a function $u \in C^{2,\alpha}_{\text{loc}}(\Omega)$ that assumes the boundary datum in the sense of $W^{1,p}(\Omega)$ and that is a classical solution in Ω to Eq. (5.9).

6. Discontinuous Unbounded Solutions

In this section we consider equations of the type

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i(Du) = 0, \qquad \text{in } \Omega, \tag{6.1}$$

where $a^i(\xi)$ satisfy, for every ξ, $\lambda \in \mathbb{R}^n$, for some m, $M>0$ and $q \geqslant p > 1$, the conditions

$$\sum_{i=1}^{n} a^i(\xi)\,\xi_i \geqslant m\,|\xi|^p; \qquad |a^i(\xi)| \leqslant M(1+|\xi|^{q-1});$$
$$\sum_{i,j} a^i_{\xi_j}(\xi)\,\lambda_i\lambda_j \geqslant 0; \qquad a^i_{\xi_j} = a^j_{\xi_i}; \tag{6.2}$$

$$\text{if} \quad p \leqslant 2 \leqslant q, \qquad \sum_{i,j} a^i_{\xi_j}(\xi)\,\lambda_i\lambda_j \geqslant m(1+|\xi|^2)^{(p-2)/2}\,|\lambda|^2;$$
$$|a^i_{\xi_j}(\xi)| \leqslant M(1+|\xi|^2)^{(q-2)/2}. \tag{6.3}$$

As a generalization of Giaquinta [6] and Marcellini [14] we will show that, for some exponents p, q, the elliptic equation (6.1) may have discontinuous weak solutions (*thanks to Francesco Leonetti for having checked and revised this section*). Of course, if Eq. (6.1) has a discontinuous weak solution in the Sobolev class $W^{1,p}(\Omega)$, then necessarily p is less than or equal to n.

THEOREM 6.1. *Let* $n>2$, $1<p<n-1$ *and*

$$q > \frac{(n-1)p}{n-1-p}. \tag{6.4}$$

Then there are functions a^i *locally Lipschitz-continuous in* $\mathbb{R}^n$ *for* $i=1, 2, ..., n$, *satisfying* (6.2), (6.3), *such that the corresponding Eq.* (6.1) *admits unbounded weak solutions.*

Remark 6.2. Let $n>2$ and $p>1$ such that $2(n-2)/n<p<2(n-1)/n$. Then it is possible to consider exponents p, q satisfying (6.4) and such that $2 \leqslant q < np/(n-2)$. Thus, in particular, from the previous result we deduce that Theorem 2.1 does not hold (with the assumptions $1 \leqslant q/p < n/(n-2)$) if we drop the condition $p \geqslant 2$.

Proof of Theorem 6.1. The first part of the proof is a generalization of a similar result given in [6, 14] (see also [15]). First of all, by a computation we can see that the function

$$u = c x_n^{q/(q-p)} \left(\sum_{i=1}^{n-1} x_i^2 \right)^{-p/(2(q-p))}, \tag{6.5}$$

for a particular choice of the constant $c \neq 0$ (here we use the condition $q>(n-2)p/(n-1-p)$), is a classical solution to the equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left(\sum_{s=1}^{n-1} u_{x_s}^2 \right)^{(p-2)/2} u_{x_i} \right) + \frac{\partial}{\partial x_n} \left(|u_{x_n}|^{q-2}\, u_{x_n} \right) = 0 \tag{6.6}$$

for $\sum_{i=1}^{n-1} x_i^2 > 0$, $x_n > 0$. Let $c_1 > 0$ and let $\Omega \subset \{x \in \mathbb{R}^n : x_n > c_1\}$, Ω bounded. Then there is a constant c_2 such that

$$u_{x_n} > c_2, \sum_{s=1}^{n-1} u_{x_s}^2 > c_2, \qquad \forall x \in \Omega : \sum_{i=1}^{n-1} x_i^2 > 0.$$

Let us consider the functions $g(t) = t^{(p-2)/2}$, $h(t) = t^{q-2}$ for $t > c_2$ and let us extend them to $\mathbb{R}$ as even functions with the constant value $g(t) = (c_2)^{(p-2)/2}$, $h(t) = (c_2)^{q-2}$ for $|t| < c_2$.

Then of course u is a classical solution also to

$$\sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} \left(g\left(\sum_{s=1}^{n-1} u_{x_s}^2 \right) u_{x_i} \right) + \frac{\partial}{\partial x_n} (h(u_{x_n}) u_{x_n}) = 0 \tag{6.7}$$

and (6.7) is an elliptic equation of the type (6.1), (6.2), (6.3).

By using (6.4) we can see that

$$u_{x_i} \in L^p(\Omega), \forall i = 1, 2, ..., n-1; \qquad u_{x_n} \in L^q(\Omega). \tag{6.8}$$

Let us note that, if n is sufficiently large, then $u \in W^{1,q}(\Omega)$, too.

By adapting a well-known argument by De Giorgi (see, for example. Giusti [7, Chap. VI, Sect. 1]; see also [14] for the details) and by using the condition $p < n-1$, we can conclude that u is a weak solution to (6.6) (or equivalently to (6.7)).

APPENDIX: A Simple Existence Theorem

We think it is of interest to give an existence theorem for a class of Dirichlet problems associated to some nonlinear p.d.e., already considered by Leray and J. L. Lions (see [11, Remark 5; 12, Chap. 2, Sects. 1.7 and 2.3]; see also [5, 17]), whose proof is a direct application of the theory of monotone operators and for which the previous regularity results apply. Let us also mention that [15, Theorem A] is a regularity result, specific for the situation considered here.

We consider the p.d.e. (1.1) with $a^i(x, \xi)$ Caratheodory functions satisfying, for some constants m, $M > 0$, for exponents $q_i > 1$, $\forall i = 1, 2, ..., n$, for a.e. $x \in \Omega$ (open bounded set of $\mathbb{R}^n$) and for every ξ, $\eta \in \mathbb{R}^n$ with $\xi \neq \eta$,

$$\sum_{i=1}^{n} (a^i(x, \xi) - a^i(x, \eta))(\xi_i - \eta_i) > 0; \tag{7.1}$$

$$\sum_{i=1}^{n} a^i(x, \xi)\, \xi_i \geqslant m \sum_{i=1}^{n} |\xi_i|^{q_i}; \tag{7.2}$$

$$|a^i(x, \xi)| \leqslant M \left(1 + \sum_{j=1}^{n} |\xi_j|^{q_j} \right)^{(1-1/q_i)}, \qquad \forall i = 1, 2, ..., n. \tag{7.3}$$

Note that the growth condition (7.3) is very natural if $a^i(x, \xi) = f_{\xi_i}(x, \xi)$, with $f(x, \xi)$ Caratheodory function, convex with respect to ξ and such that

$$|f(x, \xi)| \leqslant c\left(1 + \sum_{j=1}^{n} |\xi_j|^{q_j}\right) \tag{7.4}$$

for some constant c and for a.e. $x \in \Omega$, $\xi \in \mathbb{R}^n$. In fact it is possible to show [15, Lemma 2.1] that, under our positions, (7.4) implies (7.3).

We look for weak solutions to (1.1) in the Sobolev class

$$V = \{v \in W^{1,1}(\Omega) : v_{x_i} \in L^{q_i}(\Omega),\ \forall i = 1, 2, ..., n\}. \tag{7.5}$$

Let us denote by V' the dual space of V_0, where $V_0 = V \cap W_0^{1,1}(\Omega)$.

THEOREM 7.1. *Let* (7.1), (7.2), (7.3) *hold. Then, for every* $u_0 \in V$ *and* $b \in V'$ *there is a unique* $u \in u_0 + V_0$ *such that*

$$\int_\Omega \sum_{i=1}^{n} a^i(x, Du)\, \phi_{x_i}\, dx + \langle b, \phi \rangle = 0, \qquad \forall \phi \in V_0. \tag{7.6}$$

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