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On the Remainder in the Approximation of Functions by Bernstein-Type Operators

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1. INTRODUCTION

Let $\{L_n\}$ be a sequence of linear operators on $C[0,1]$ into $C[0,1]$. Evaluation of the remainder

$$R_n(f,x) = f(x) - (L_n f)(x),$$

is useful in the investigation of the approximation properties of the operators L_n.

For the Bernstein polynomials, this remainder has been thoroughly investigated. First, some asymptotic formulae were given by Voronovskaya and by Bernstein (see [*10*], pp. 22–23) and later, the remainder was evaluated for different classes of functions by Popoviciu and by Lorentz (see [*10*], Th. 1.6.1 and Th. 1.6.2). More recently, O. Aramă [*3*] gave a representation of this remainder by means of divided differences. Using different methods of construction, L. Aramă [*1*] and Stancu [*14*] obtained independently another representation; but both restricted the functions $f(x)$ to be twice continuously differentiable in $[0,1]$. Furthermore, L. and O. Aramă [*2*], using L. Aramă's technique, obtained a similar representation of the remainder in the approximation of the above type of functions by generalized Bernstein polynomials (with some restrictions on the powers involved in the definition of these polynomials). We shall give here a representation of the remainder in the approximation of any continuous function on $[0,1]$ by generalized Bernstein polynomials. This representation will be expressed by means of divided differences. We shall also estimate the order of approximation of $f(x)$ by these operators.

Approximation operators resembling the Bernstein polynomials and known as Bernstein power-series were introduced by Meyer-König and Zeller [*12*]. Recently, Lupaş and Müller [*11*] showed that the remainder for Bernstein power-series has properties similar to those of the remainder for Bernstein polynomials. Operators generalizing the Bernstein power-series and resembling the generalized Bernstein polynomials were defined by Jakimovski and the author [*8*]. (Recently they have been redefined by Feller [*5*].) Representations and estimates of the remainders for these operators will also be provided.

2. Preliminaries

Let the sequence $\{\lambda_i\}$ $(i \geqslant 0)$ satisfy

$$0 \leqslant \lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots \uparrow \infty, \qquad \sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty. \tag{2.1}$$

Define

$$p_{nm}(x) = (-1)^{n-m} \lambda_{m+1} \cdot \ldots \cdot \lambda_n \sum_{i=m}^{n} x^{\lambda_i} / \omega'_{nm}(\lambda_i), \quad 0 \leqslant m < n = 1, 2, \ldots,$$

$$p_{nn}(x) = x^{\lambda_n}, \qquad n = 0, 1, 2, \ldots,$$

where

$$\omega_{nm}(t) = (t - \lambda_m) \cdot \ldots \cdot (t - \lambda_n), \qquad 0 \leqslant m < n = 1, 2, \ldots.$$

Also, set

$$\alpha_{nm} = \left[\left(1 - \frac{\lambda_1}{\lambda_{m+1}}\right) \cdot \ldots \cdot \left(1 - \frac{\lambda_1}{\lambda_n}\right) \right]^{1/\lambda_1}, \qquad 0 \leqslant m < n = 1, 2, \ldots,$$

$$\alpha_{nn} = 1, \qquad n = 0, 1, 2, \ldots,$$

and denote

$$q_{nm}(x) = \frac{\lambda_m}{\lambda_n} p_{nm}(x), \qquad \beta_{nm} = \alpha_{n-1, m-1}, \qquad 1 \leqslant m \leqslant n = 1, 2, \ldots.$$

The generalized Bernstein polynomials associated with the continuous function $f(x)$ were defined by Hirschman ahd Widder (see [*10*], §2.8) as

$$B_n(f, x) = \sum_{m=0}^{n} p_{nm}(x) f(\alpha_{nm}), \qquad 0 \leqslant x \leqslant 1, \qquad n = 0, 1, 2, \ldots. \tag{2.2}$$

A slight modification of [*10*], Th. 2.8.2, yields the following

Theorem A. *Let $f(x)$ be continuous in* $[0, 1]$.

(i) *If $\lambda_0 = 0$, then* $\lim_{n\to\infty} B_n(f, x) = f(x)$, *uniformly in* $0 \leqslant x \leqslant 1$.

(ii) *If $\lambda_0 > 0$, then* $\lim_{n\to\infty} B_n(f, x) = f(x)$ *for every* $0 < x \leqslant 1$, *uniformly in any interval* $[\delta, 1]$, $0 < \delta \leqslant 1$. *Moreover, since $B_n(f, 0) = 0$ for all $n \geqslant 0$, it follows that $B_n(f, 0) \to f(0)$ if and only if $f(0) = 0$.*

The generalized Bernstein power-series associated with a continuous function $f(x)$ are defined as

$$M_m(f, x) = \sum_{n=m}^{\infty} q_{nm}(x) f(\beta_{nm}), \qquad 0 < x \leqslant 1, \qquad m = 1, 2, \ldots. \tag{2.3}$$

Since $\lim_{x\to 0+} M_m(f, x) = f(0)$, it is convenient to define

$$M_m(f, 0) = f(0), \qquad m = 1, 2, \ldots. \tag{2.4}$$

The following result is stated in [8], Th. 4.1, and proved in [9], Th. 2.3. (See also [5].)

THEOREM B. *Let $f(x)$ be continuous in* $[0,1]$. *Then* $\lim_{m\to\infty} M_m(f,x) = f(x)$, *uniformly in* $0 \leqslant x \leqslant 1$.

We shall make use of the following results (see [10], p. 46, (10) and (11), and p. 47, (4)):

$$\left.\begin{aligned} &p_{nm}(x) \geqslant 0, \qquad 0 \leqslant x \leqslant 1, \qquad 0 \leqslant m \leqslant n = 0,1,2,\ldots; \\ \text{if} \quad &\lambda_0 = 0, \qquad \text{then} \sum_{m=0}^{n} p_{nm}(x) = 1, \qquad 0 \leqslant x \leqslant 1, \qquad n = 0,1,2,\ldots, \\ \text{and} \quad &\sum_{m=0}^{n} p_{nm}(x)\, \alpha_{nm}^{\lambda_1} = x^{\lambda_1}, \qquad 0 \leqslant x \leqslant 1, \qquad n = 0,1,2,\ldots. \end{aligned}\right\} \tag{2.5}$$

Also, by [9], (3.15),

$$\left.\begin{aligned} &\sum_{n=m}^{\infty} q_{nm}(x) = 1, \qquad 0 < x \leqslant 1, \qquad m = 1,2,\ldots, \\ \text{and} \quad &\sum_{n=m}^{\infty} q_{nm}(x)\, \beta_{mn}^{\lambda_1} = x^{\lambda_1}, \qquad 0 \leqslant x \leqslant 1, \qquad m = 1,2,\ldots. \end{aligned}\right\} \tag{2.6}$$

Let $\lambda > 0$ and define the divided differences of $f(x)$ in the following way. Let $x_0, x_1, \ldots$ be distinct points in the domain of definition of $f(x)$; define

$$[\lambda; x_0; f] = f(x_0),$$

$$[\lambda; x_0, \ldots, x_k; f] = \frac{[\lambda; x_0, \ldots, x_{k-1}; f] - [\lambda; x_1, \ldots, x_k; f]}{x_0^{\lambda} - x_k^{\lambda}}, \qquad k \geqslant 1.$$

For $\lambda = 1$, these are the ordinary divided differences. We shall need the general ones in order to describe the remainder for our operators in case $\lambda_1 \neq 1$. These divided differences are obtained from Popoviciu's general divided differences, [13], (22), by taking $g_i(x) = x^{i\lambda}$, $i = 0, 1, 2, \ldots$. We call a function $f(x)$ convex, non-concave, polynomial, non-convex, or concave of order s if the divided difference

$$[\lambda; x_0, \ldots, x_{s+1}; f],$$

is positive, non-negative, zero, non-positive, or negative, respectively, for all (distinct) $x_0, \ldots, x_{s+1}$ in the domain of definition of f.

3. REPRESENTATION OF THE REMAINDER

We establish, first, the following

THEOREM 1. *Let $\lambda_0 = 0$, and let $f(x)$ be continuous in* $[0,1]$. *Then for $n \geqslant 1$ and $0 \leqslant x \leqslant 1$,*

$$B_n(f,x) - B_{n+1}(f,x) = \sum_{m=1}^{n} p_{n+1,m}(x)\, \gamma_{nm}^2 [\lambda_1; \alpha_{n,m-1}, \alpha_{n+1,m}, \alpha_{nm}; f],$$

where

$$\gamma_{nm}^2 = \frac{\lambda_1{}^2 \alpha_{nm}^{2\lambda_1}}{\lambda_{n+1}} \left(\frac{1}{\lambda_m} - \frac{1}{\lambda_{n+1}}\right) > 0 \qquad \textit{for } 1 \leqslant m \leqslant n.$$

Thus, the sequence $\{B_n(f,x)\}(n \geqslant 1)$ *is decreasing, nonincreasing, stationary, nondecreasing, or increasing iff f is convex, non-concave, polynomial, non-convex, or concave of order* 1, *respectively.*

Theorem 1, for the ordinary Bernstein polynomials, was proved in [*3*].

Proof. It was proved by Hausdorff, [7], (*8*), that

$$p_{nm}(x) - p_{n+1,m}(x) = \frac{1}{\lambda_{n+1}} [\lambda_{m+1} p_{n+1,m+1}(x) - \lambda_m p_{n+1,m}(x)].$$

Therefore,

$$\begin{aligned} B_n(f,x) - B_{n+1}(f,x) = \sum_{m=0}^{n} & [p_{nm}(x) - p_{n+1,m}(x)] f(\alpha_{nm}) \\ & + \sum_{m=0}^{n} p_{n+1,m}(x)[f(\alpha_{nm}) - f(\alpha_{n+1,m})] \\ & - p_{n+1,n+1}(x) f(1), \end{aligned}$$

(since $\alpha_{n+1,n+1} = 1$)

$$\begin{aligned} &= \sum_{m=1}^{n} \frac{\lambda_m}{\lambda_{n+1}} p_{n+1,m}(x)[f(\alpha_{n,m-1}) - f(\alpha_{nm})] \\ &+ \sum_{m=1}^{n} p_{n+1,m}(x)[f(\alpha_{nm}) - f(\alpha_{n+1,m})], \end{aligned}$$

(since $\lambda_0 = 0$, $\alpha_{n0} = \alpha_{n+1,0} = 0$ and $\alpha_{nn} = 1$)

$$= I_1 + I_2, \text{ say.}$$

Now,

$$\alpha_{nm}^{\lambda_1} - \alpha_{n+1,m}^{\lambda_1} = \alpha_{nm}^{\lambda_1} \frac{\lambda_1}{\lambda_{n+1}} \qquad \text{and} \qquad \alpha_{nm}^{\lambda_1} - \alpha_{n,m-1}^{\lambda_1} = \alpha_{nm}^{\lambda_1} \frac{\lambda_1}{\lambda_m};$$

so

$$I_1 = -\sum_{m=1}^{n} \frac{\lambda_1}{\lambda_{n+1}} p_{n+1,m}(x)\, \alpha_{nm}^{\lambda_1} [\lambda_1; \alpha_{n,m-1}, \alpha_{nm}; f],$$

$$I_2 = \sum_{m=1}^{n} \frac{\lambda_1}{\lambda_{n+1}} p_{n+1,m}(x)\, \alpha_{mn}^{\lambda_1} [\lambda_1; \alpha_{nm}, \alpha_{n+1,m}; f].$$

Observing that

$$\alpha_{n+1,m}^{\lambda_1} - \alpha_{n,m=1}^{\lambda_1} = \lambda_1 \alpha_{nm}^{\lambda_1} \left(\frac{1}{\lambda_m} - \frac{1}{\lambda_{n+1}}\right),$$

if follows that

$$I_1 + I_2 = \sum_{m=1}^{n} p_{n+1,m}(x)\,\gamma_{nm}^2[\lambda_1; \alpha_{n,m-1}, \alpha_{n+1,m}, \alpha_{nm}; f].$$

This completes our proof.

We can establish now the following representation theorem.

THEOREM 2. *Let* $\lambda_0 = 0$. *Then for each* $n \geq 1$ *and each* $x_0 \in [0,1]$ *there are three distinct points* $\xi_1(n,x_0)$, $\xi_2(n,x_0)$, $\xi_3(n,x_0)$ *such that for every function* $f(x)$, *continuous in* $[0,1]$,

$$\begin{aligned} R_n(f,x_0) &= f(x_0) - B_n(f,x_0) \\ &= R_n(x^{2\lambda_1}, x_0)[\lambda_1; \xi_1(n,x_0), \xi_2(n,x_0), \xi_3(n,x_0); f]. \end{aligned}$$

Theorem 2, for the ordinary Bernstein polynomials, was proved in [*3*].

Proof. Given $n_0 \geq 1$ and $0 \leq x_0 \leq 1$, we shall prove that $R_{n_0}(f,x_0) \neq 0$ for every continuous function $f(x)$, convex of order 1. Then our theorem will follow by Popoviciu [*13*], Th. 5. Now, for such a function $f(x)$, the sequence $\{B_n(f,x_0)\}$ $(n \geq 1)$ is decreasing, by Theorem 1. Furthermore, by Theorem A, $B_n(f,x_0) \to f(x_0)$. Therefore

$$R_{n_0}(f,x_0) = f(x_0) - B_{n_0}(f,x_0) < 0.$$

This completes our proof.

Remark 1. In fact, $R_n(f,x_0)$ has degree of exactness 1 (see [*13*], §25), since by (2.5), $R_n(f,x_0)$ vanishes for the functions $1, x^{\lambda_1}$.

The following is an immediate consequence of Theorem 2.

COROLLARY 1. *If* $\lambda_0 = 0$ *and the function* $g(x) = f(x^{1/\lambda_1})$ *is twice continuously differentiable in* $(0,1)$, *then for each* $n \geq 1$ *and each* $x_0 \in [0,1]$ *there exists* $0 < \xi(n,x_0) < 1$ *such that*

$$R_n(f,x_0) = \tfrac{1}{2} R_n(x^{2\lambda_1}, x_0)\, g''(\xi). \tag{3.2}$$

For $\lambda_1 = 1$, we have $g(x) = f(x)$ and (3.2) was obtained in [*3*].

A representation of the remainder in the approximation of functions twice continuously differentiable in $[0,1]$ by generalized Bernstein polynomials in the case $\lambda_1 = 1$, which is much more precise than (3.2), follows immediately by Remark 1 and by Popoviciu [*13*], (84):

THEOREM 3. *Let* $\lambda_0 = 0$ *and* $\lambda_1 = 1$. *Then for each* $n \geq 1$, *each* $x_0 \in [0,1]$ *and every function* $f(x)$, *twice continuously differentiable in* $[0,1]$,

$$R_n(f,x_0) = \int_0^1 R_n(\phi_t, x_0) f''(t)\, dt,$$

where

$$\phi_t(x) = \frac{x - t + |x - t|}{2}.$$

Remark 2. Since $\phi_t(x)$ is non-concave of order 1, it follows by Theorem 1, that $R_n(\phi_t, x_0) \leqslant 0$ for all $0 \leqslant t \leqslant 1$. Also, $R_n(\phi_t, x_0)$ is continuous in $0 \leqslant t \leqslant 1$ and, in fact, infinitely differentiable for $t \neq x_0$ and $\neq \alpha_{nm}$, $0 \leqslant m \leqslant n$. Thus, applying the mean value theorem, it is possible to derive (3.2), in case $\lambda_1 = 1$, from Theorem 3.

Remark 3. Theorem 3 and Remark 2 were obtained in [*2*] by a long construction. It must be added, however, that the construction in [*2*] allows a wider class of Bernstein operators by assuming instead of (2.1) merely $\lambda_0 = 0$ and $1 = \lambda_1 < \lambda_k$ for $k > 1$. Estimate of this remainder is given in [*2*] only for sequences $\{\lambda_i\}$ satisfying a slight modification of (2.1), namely,

$$0 = \lambda_0 < 1 = \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \ldots \leqslant \lambda_n \leqslant \ldots \uparrow \infty, \sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty.$$

Theorem 3 and Remark 2, for the ordinary Bernstein polynomials, were obtained by L. Aramă [*1*] and by Stancu [*14*].

Similar results can be obtained for the generalized Bernstein power-series. We merely state them here, leaving their proofs to the reader.

THEOREM 4. *Let* $f(x)$ *be continuous in* $[0,1]$. *Then for every* $m \geqslant 1$ *and every* $x \in (0,1]$,

$$M_m(f,x) - M_{m+1}(f,x) = \sum_{n=m+1}^{\infty} q_{nm}(x)\,\gamma_{n-1,m}^2[\lambda_1; \beta_{nm}, \beta_{n+1,m+1}, \beta_{n,m+1}; f],$$

where $\gamma_{n-1,m}^2$ *is as in Theorem* 1. *Thus, for* $0 < x \leqslant 1$, *the sequence* $\{M_m(f,x)\}$ $(m \geqslant 1)$ *is decreasing, nonincreasing, stationary, nondecreasing, or increasing if f is convex, non-concave, polynomial, non-convex, or concave of order* 1, *respectively.*

Theorem 4, for the ordinary Bernstein power-series, was proved in [*11*]

THEOREM 5. *For each* $m \geqslant 1$ *and each* $x_0 \in [0,1]$ *there are three distinct points* $\zeta_1(m,x_0)$, $\zeta_2(m,x_0)$ $\zeta_3(m,x_0)$ *such that for every function* $f(x)$, *continuous in* $[0,1]$,

$$\begin{aligned}\mathscr{R}_m(f,x_0) &= f(x_0) - M_m(f,x_0) \\ &= \mathscr{R}_m(x^{2\lambda_1},x_0)[\lambda_1; \zeta_1(m,x_0), \zeta_2(m,x_0), \zeta_3(m,x_0); f].\end{aligned} \tag{3.4}$$

Theorem 5, for the ordinary Bernstein power-series, was proved in [*11*].

Remark 4. Like $R_n(f, x_0)$, also $\mathscr{R}_m(f, x_0)$ has degree of exactness 1, since, by (2.6), $\mathscr{R}_m(f, x_0)$ vanishes for the functions $1, x^{\lambda_1}$.

The following is an immediate consequence of Theorem 5.

COROLLARY 2. *If the function* $g(x) = f(x^{1/\lambda_1})$ *is twice continuously differentiable in* $[0,1]$, *then for each* $m \geqslant 1$ *and each* $x_0 \in [0,1]$ *there exists* $0 < \zeta(m, x_0) < 1$ *such that*

$$\mathscr{R}_m(f, x_0) = \tfrac{1}{2}\mathscr{R}_m(x^{2\lambda_1}, x_0)\, g''(\zeta). \tag{3.5}$$

Again, a more precise representation can be obtained in case $\lambda_1 = 1$ for functions twice continuously differentiable in $[0,1]$; it follows by Remark 4 and [*13*], (84).

THEOREM 6. *Let* $\lambda_1 = 1$. *Then for each* $m \geqslant 1$, *each* $x_0 \in [0,1]$ *and every function* $f(x)$, *twice continuously differentiable in* $[0,1]$,

$$\mathscr{R}_m(f, x_0) = \int_0^1 \mathscr{R}_m(\phi_t, x_0) f''(t)\, dt, \tag{3.6}$$

where $\phi_t(x)$ *is as in Theorem* 3.

Remark 5. $\mathscr{R}_m(\phi_t, x_0)$ has properties similar to those of $R_n(\phi_t, x_0)$, except that $\mathscr{R}_m(\phi_t, x_0)$ is infinitely differentiable for $t \neq x_0$ and $\neq \beta_{nm}$, $n \geqslant m$, if $0 < x_0 \leqslant 1$. If $x_0 = 0$, then $\mathscr{R}_m(\phi_t, 0) \equiv 0$.

4. ESTIMATE OF THE REMAINDER

The following is needed in the sequel.

LEMMA 1. *Let* $\lambda_0 = 0$. *Then there exists a constant* C_1 *such that for all* $n \geqslant 1$ *and all* $x_0 \in [0,1]$,

$$0 \leqslant -R_n(x^{2\lambda_1}, x_0) \leqslant C_1 \max_{1 \leqslant k \leqslant n} \left\{ \frac{1}{\lambda_k} \exp\left[-\lambda_1 \cdot \sum_{i=k}^{n} 1/\lambda_i \right] \right\}.$$

Proof. This lemma is a slight modification of estimates given by Gelfond [*6*], p. 417, and can be proved in the same way. It should be noted, however, that Gelfond's inequality (25),

$$\sum_{k+1}^{n} \frac{1}{\alpha_s^2} \prod_{k+1}^{n} \left(1 - \frac{\alpha_1}{\alpha_s}\right)^2 \leqslant \frac{K}{\alpha_{k+1}} \exp\left[-2\alpha_1 \sum_{k+1}^{n} 1/\alpha_s \right],$$

is incorrect. For example, it does not hold for the sequence $\alpha_i = i^{2/3}$. The above inequality holds for sequences $\{\alpha_i\}$ such that $\alpha_{i+1} - \alpha_i \geqslant a > 0$. One can prove without difficulty that for every $\epsilon > 0$ there exists a $K(\epsilon)$ such that

$$\sum_{k+1}^{n} \frac{1}{\alpha_s^2} \prod_{k+1}^{n} \left(1 - \frac{\alpha_1}{\alpha_s}\right)^2 \leqslant \frac{K(\epsilon)}{\alpha_{k+1}} \exp\left[-(2\alpha_1 - \epsilon) \cdot \sum_{k+1}^{n} 1/\alpha_s\right],$$

and this inequality, with $\epsilon = \alpha_1$, is used in proving our lemma.

LEMMA 2. *Let $\lambda_{m_0-1} \leqslant 2\lambda_1 < \lambda_{m_0}$. Then there exists a constant C_2 such that for all $m \geqslant m_0$ and all $x_0 \in [0,1]$,*

$$0 \leqslant -\mathscr{R}_m(x^{2\lambda_1}, x_0) \leqslant \frac{C_2}{\lambda_m}.$$

Proof. It follows by [9], (3.15), that for $m \geqslant m_0$ and $0 < x_0 \leqslant 1$,

$$x_0^{2\lambda_1} = \sum_{n=m}^{\infty} q_{nm}(x_0) \left(1 - \frac{2\lambda_1}{\lambda_m}\right) \cdot \ldots \cdot \left(1 - \frac{2\lambda_1}{\lambda_{n-1}}\right).$$

Hence

$$\mathscr{R}_m(x^{2\lambda_1}, x_0) = \sum_{n=m}^{\infty} q_{nm}(x_0) \left[\left(1 - \frac{2\lambda_1}{\lambda_m}\right) \cdot \ldots \cdot \left(1 - \frac{2\lambda_1}{\lambda_{n-1}}\right) - \beta_{nm}^{2\lambda_1}\right].$$

Now, by (2.6), if $0 < x_0 \leqslant 1$, then

$$\sum_{n=m}^{\infty} q_{nm}(x_0) = 1,$$

and modifying again Gelfond's estimates [6] as above, our lemma is proved for $0 < x_0 \leqslant 1$. For $x_0 = 0$, $\mathscr{R}_m(x^{2\lambda_1}, 0) = 0$.

Denote

$$\rho_n = \max_{1 \leqslant k \leqslant n} \left\{\frac{1}{\lambda_k} \exp\left[-\lambda_1 \cdot \sum_{i=k}^{n} 1/\lambda_i\right]\right\} n = 1, 2, \ldots. \tag{4.1}$$

(By (2.1), $\rho_n \to 0$ as $n \to \infty$.)

Our first estimate of the remainder is

THEOREM 7. *Let $\lambda_0 = 0$ and suppose that $g(x) = f(x^{1/\lambda_1})$ is twice continuously differentiable in* $[0,1]$. *Then*

$$|R_n(f, x)| \leqslant C_3 \rho_n, \qquad n = 1, 2, \ldots,$$

where $C_3 = \frac{1}{2} C_1 \max_{0 \leqslant x \leqslant 1} |g''(x)|$ and C_1 is taken from Lemma 1.

The proof follows immediately by (3.2) and Lemma 1. Similarly, we have

THEOREM 8. *Suppose that $g(x)=f(x^{1/\lambda_1})$ is twice continuously differentiable in* $[0,1]$. *Then*

$$|\mathscr{R}_m(f,x)| \leqslant \frac{C_4}{\lambda_m}, \qquad m \geqslant m_0,$$

where $C_4 = \frac{1}{2}C_2 \max_{0\leqslant x\leqslant 1}|g''(x)|$ and C_2 is taken from Lemma 2.

The proof is immediate.

Other methods of estimating the remainder, using the modulus of continuity of $f(x)$ or of $f'(x)$, if f is continuously differentiable, were developed by Popoviciu and Lorentz (see [*10*], Th. 1.6.1 and Th. 1.6.2). Since, if $\lambda_0 = 0$, we obtain by (2.5),

$$B_n((x^{\lambda_1} - x_0^{\lambda_1})^2, x_0) = R_n(x^{2\lambda_1}, x_0), \qquad n = 1, 2, \ldots,$$

it follows by a proof similar to that of [*10*], Th. 1.6.1, that we have

THEOREM 9. *Let $\lambda_0 = 0$ and let $\omega_g(\delta)$ denote the modulus of continuity of $g(x)=f(x^{1/\lambda_1})$. Then*

$$|R_n(f,x)| \leqslant C_5\, \omega_g(\rho_n^{1/2}), \qquad n = 1, 2, \ldots.$$

Also by (2.6),

$$M_m((x^{\lambda_1} - x_0^{\lambda_1})^2, x_0) = \circledR_m(x^{2\lambda_1}, x_0), \qquad m \geqslant m_0,$$

and also we have

THEOREM 10. *Let $\omega_g(\delta)$ be as in Theorem* 9. *Then*

$$|\mathscr{R}_m(f,x)| \leqslant C_6\, \omega_g(\lambda_m^{-1/2}), \qquad m \geqslant m_0.$$

Finally, the following result follows by a proof similar to that of [*10*], Th. 1.6.2.

THEOREM 11. *Suppose that $g(x)=f(x^{1/\lambda_1})$ is continuously differentiable in* $[0,1]$ *and let $\omega(g',\delta)$ be the modulus of continuity of g'.*

(i) *If $\lambda_0 = 0$, then*

$$|R_n(f,x)| \leqslant C_7\, \rho_n^{1/2}\, \omega(g', \rho_n^{1/2}), \qquad n = 1, 2, \ldots.$$

(ii) *For $m \geqslant m_0$,*

$$|\mathscr{R}_m(f,x)| \leqslant C_8\, \lambda_m^{-1/2}\, \omega(g', \lambda_m^{-1/2}).$$

For the ordinary Bernstein power-series, Theorems 10 and 11 (ii) were proved in [*11*]. (See also [*4*].)

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