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Existence of positive solutions for nonlinear fractional functional differential equation[☆]

Yulin Zhao[a,*], Haibo Chen[b], Li Huang[a]

[a] School of Science, Hunan University of Technology, Zhuzhou, 412008, PR China

[b] Department of Mathematics, Central South University, Changsha, 410075, PR China

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ABSTRACT

In this paper, the existence of positive solutions for the nonlinear Caputo fractional functional differential equation in the form

$$\begin{cases} D_{0+}^q y(t) + r(t)f(y_t) = 0, \quad \forall t \in (0,1), q \in (n-1, n], \\ y^{(i)}(0) = 0, \quad 0 \le i \le n-3, \\ \alpha y^{(n-2)}(t) - \beta y^{(n-1)}(t) = \eta(t), \quad t \in [-\tau, 0], \\ \gamma y^{(n-2)}(t) + \delta y^{(n-1)}(t) = \xi(t), \quad t \in [1, 1+a] \end{cases}$$

is studied. By constructing a special cone and using Krasnosel'skii's fixed point theorem, various results on the existence of at least one or two positive solutions to the fractional functional differential equation are established. The main results improve and generalize the existing results.

1. Introduction

The purpose of this paper is to establish the conditions for the existence of positive solutions for the following nonlinear Caputo fractional functional differential equation of the form

$$\begin{cases} D_{0+}^q y(t) + r(t)f(y_t) = 0, \quad \forall t \in (0,1), q \in (n-1, n], \\ y^{(i)}(0) = 0, \quad 0 \le i \le n-3, \\ \alpha y^{(n-2)}(t) - \beta y^{(n-1)}(t) = \eta(t), \quad t \in [-\tau, 0], \\ \gamma y^{(n-2)}(t) + \delta y^{(n-1)}(t) = \xi(t), \quad t \in [1, 1+a], \end{cases} \tag{1.1}$$

where

(H0) $\tau, a, \alpha, \beta, \gamma$ and δ are nonnegative constants satisfying $0 \le \tau + a \le 1$ and $\rho^{-1} = \alpha\gamma + \alpha\delta + \beta\gamma > 0$.
(H1) $y_t = y(t+\theta), \theta \in [-\tau, a]$; $\eta \in C([-\tau, 0], \mathbf{R})$, $\xi \in C([1, b], \mathbf{R})$, $\eta(0) = \xi(1) = 0$, where $\mathbf{R} = (-\infty, +\infty)$, $b = 1 + a$.

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* Corresponding author.
E-mail addresses: zhaoylch@sina.com (Y. Zhao), math_chb@mail.csu.edu.cn (H. Chen), hl6207@sina.com (L. Huang).

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Let $C = C^{n-2}([-\tau, a], \mathbf{R})$ be a Banach space with $\|\varphi\|_C = \sup_{-\tau \le \theta \le a} |\varphi^{(n-2)}(\theta)|$ for $\varphi \in C$, and $C^+ = \{\varphi \in C : \varphi(\theta) \geq 0, -\tau \le \theta \le a\}$.

Define

$$E = \{t \in [0, 1] : 0 \le t + \theta \le 1, -\tau \le \theta \le a\}.$$

Obviously, E possesses nonzero measure from the assumption that $0 \le t + \theta \le 1$.

Fractional differential equations have gained importance due to their numerous applications in many fields of science and engineering including fluid flow, rheology, diffusive transport akin to diffusion, electrical networks, probability, etc. For details, see [1–4] and the references therein. In recent years, there are some papers dealing with the existence of the solutions of initial value problems or linear boundary value problems for fractional differential equations by means of techniques of nonlinear analysis (fixed point theorems, Leray–Schauder theory, lower and upper solutions method, etc); see for e.g. [5–13].

In [8], Bai and Lü investigate the following two point boundary value problem of fractional differential equations

$$\begin{cases} D_{0+}^q u(t) + r(t) f(t, u(t)) = 0, & 0 < t < 1,\ 1 < q \le 2, \\ u(0) = u(1) = 0, \end{cases}$$

where D_{0+}^q is the Riemann–Liouville fractional derivative.

In [12], by means of Amann theorem and the method of upper and lower solutions, Liu and Jia study the existence and multiplicity of positive solutions for nonlinear fractional differential equations

$$\begin{cases} D_{0+}^q u(t) + f(t, u(t), u'(t)) = 0, & 0 < t < 1, \\ g_0(u(0), u'(0)) = 0 = g_1(u(1), u'(1)), \\ u''(0) = u'''(0) = \cdots = u^{(n-1)}(0) = 0, \end{cases}$$

where $n - 1 < q \le n$ is a real number and D_{0+}^q is the standard Caputo fractional derivative.

In [13], Rehman and Khan investigate the existence and uniqueness of solutions for the following fractional differential equations

$$\begin{cases} D_{0+}^q u(t) + f(t, u(t), D_{0+}^\nu u(t)) = 0, & 1 < q \le 2, 0 < \nu < 1, 0 < t < 1, \\ u(0) = 0, \qquad D_{0+}^\nu u(1) - \displaystyle\sum_{i=1}^{m-2} a_i D_{0+}^\nu u(\xi_i) = u_0, \end{cases}$$

where $0 < \xi_i < 1$, $a_i \in [0, +\infty)$, $(i = 1, 2, \ldots, m-2)$ and D_{0+}^q is the standard Riemann–Liouville fractional derivative. By means of the Schauder fixed point theorem and the Banach contraction principle, some results on the existence of solutions are obtained for the above fractional boundary value problems.

When $q = n$, the problem (1.1) is reduced to nth-order boundary value problem, which has been studied by Hong et al. [14]. For the situation that $\tau = a = 0$ and $1 < q \le 2$, the problem (1.1) becomes the two-point boundary value problem of fractional functional differential equations and has been investigated in the recent literature such as [8,10,15]. However, the results dealing with the existence of positive solutions for boundary value problem of fractional functional differential equations are relatively scare. Li et al. [16] investigate the existence of at least one positive solution for the following boundary value problem of fractional functional differential equations

$$\begin{cases} D_{0+}^q y(t) + f(t, y_t) = 0, & \forall t \in (0, 1),\ 1 < q \le 2, \\ \alpha y(t) - \beta y'(t) = \eta(t), & t \in [-\tau, 0], \\ \gamma y(t) + \delta y'(t) = \xi(t), & t \in [1, 1 + a], \end{cases}$$

where D_{0+}^q is the Caputo fractional order derivative.

Recently, in [17], Weng and Jiang have studied the existence of positive solutions for the boundary value problem of second order functional equations. In [18], Zhao and Chen discussed the existence of multiple positive solutions for the following second order functional differential equations

$$\begin{cases} y''(t) + f(t, y(t - \tau)) = 0, & \forall t \in (0, 1) \setminus \{\tau\}, \\ y(t) = \eta(t), & \forall t \in [-\tau, 0], \\ y(1) = 0, \end{cases}$$

where the nonlinearity f may be singular and takes negative values.

Motivated and inspired by the work above, we are concerned with the existence of positive solutions for the fractional functional equations (1.1) under suitable conditions on f. The main tool used in this paper is the theory of Krasnosel'skii's fixed point theorem in cones. Some sufficient conditions for the existence of at least one positive solution or at least two positive solutions of the BVP (1.1) are obtained, and the main results of this paper is to extend and supplement some results in [17,14,16].

Throughout this paper, we suppose that the following conditions are satisfied.

(H2) $f(\varphi)$ is a nonnegative continuous function defined on C^+.
(H3) $r(t)$ is a nonnegative measurable function defined on $(0, 1)$, and satisfies

$$0 < \int_E r(t)h(t)dt \le \int_0^1 r(t)h(t)dt < +\infty,$$

where $h(t)$ is defined as

$$h(s) = \frac{1}{\Gamma(q-n+2)}(\beta+\alpha s)[\gamma(1-s)+\delta(q-n+1)](1-s)^{q-n}.$$

In obtaining positive solutions of problem (1.1), we will need the following fixed point theorem in cones.

Lemma A (*Krasnosel'skii's [19]*)**.** *Let K be a cone in a Banach space E. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$. If $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ is a completely continuous operator such that either:*

(i) $\|Tx\| \le \|x\|, x \in K \cap \partial\Omega_1$, *and* $\|Tx\| \ge \|x\|, x \in K \cap \partial\Omega_2$, *or*
(ii) $\|Tx\| \ge \|x\|, x \in K \cap \partial\Omega_1$, *and* $\|Tx\| \le \|x\|, x \in K \cap \partial\Omega_2$,

then T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

2. Preliminaries

In this section, we introduce preliminary facts and properties which are used throughout this paper. The definitions on the fractional integral and the Caputo fractional derivative can be found in the recent literature [2–5,8].

Definition 2.1. The left sided Riemann–Liouville fractional integral of order $q > 0$ of a function $f : [0, +\infty) \to \mathbf{R}$ is given by

$$I_{0+}^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds.$$

Definition 2.2. The Caputo derivative of fractional order $q > 0$ of a function $f : [0, +\infty) \to \mathbf{R}$ is defined as

$$D_{0+}^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s)ds,$$

where $n - 1 < q \le n$.

Lemma 2.1. *Let $q > 0$. Then the fractional differential equation*

$$D_{0+}^q y(t) = 0$$

has a unique solution $y(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}$, $c_i \in \mathbf{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [q] + 1$.

Lemma 2.2. *Let $q > 0$. Then the following equality holds for $y \in L(0, 1)$, $D_{0+}^q y \in L(0, 1)$,*

$$I_{0+}^q D_{0+}^q y(t) = y(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

for some $c_i \in \mathbf{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [q] + 1$.

Definition 2.3. We say a function $y(t)$ is a solution of BVP (1.1) if

(1) $y(t)$ is continuous on $[0, b]$;
(2) $y(t) = y(-\tau; t)$ for $t \in [-\tau, 0]$ where $y(-\tau; t) : [-\tau, 0] \to [0, +\infty)$ satisfies

$$y^{(n-2)}(-\tau; t) = e^{\frac{\alpha}{\beta}t}\left(\frac{1}{\beta}\int_t^0 e^{-\frac{\alpha}{\beta}s}\eta(s)ds + y^{(n-2)}(0)\right); \tag{2.1}$$

(3) $y(t) = y(b; t)$ for $t \in [1, b]$, where $y(b; t) : [1, b] \to [0, +\infty)$ satisfies

$$y^{(n-2)}(b; t) = e^{-\frac{\gamma}{\delta}t}\left(\frac{1}{\delta}\int_1^t e^{-\frac{\gamma}{\delta}s}\xi(s)ds + e^{\frac{\gamma}{\delta}}y^{(n-2)}(1)\right); \tag{2.2}$$

(4) $D_{0+}^q y(t) = -r(t)f(y_t)$ for $t \in (0, 1)$ almost everywhere.

3. Main results and proofs

Lemma 3.1. *Suppose that* (H0)–(H2) *hold, and* $q \in (n-1, n]$, *the unique solution of BVP* (1.1) *is*

$$y(t) = \begin{cases} y(-\tau; t), & -\tau \le t \le 0, \\ \int_0^1 G(t,s) r(s) f(y(s+\theta)) ds, & 0 \le t \le 1, \\ y(b; t), & 1 \le t \le b, \end{cases}$$

where

$$G(t,s) = \begin{cases} -\frac{(t-s)^{q-1}}{\Gamma(q)} + G_0(t,s), & 0 \le s \le t \le 1, \\ G_0(t,s), & 0 \le t \le s \le 1, \end{cases}$$

and

$$G_0(t,s) = \frac{\rho[\beta(n-1) + \alpha t] t^{n-2}}{(n-1)! \Gamma(q-n+2)} [\gamma(1-s) + \delta(q-n+1)](1-s)^{q-n},$$

moreover, $y(-\tau; t)$ *and* $y(b; t)$ *satisfy* (2.1) *and* (2.2), *respectively.*

Proof. By $D_{0+}^q y(t) + z(t) = 0$, $t \in (0, 1)$, and the boundary conditions $y(0) = y'(0) = \cdots = y^{(n-3)}(0) = 0$, we have

$$\begin{aligned} y(t) &= -I_{0+}^q z(t) + y(0) + y'(0)t + \frac{y''(0)}{2!} t^2 + \cdots + \frac{y^{(n-2)}(0)}{(n-2)!} t^{n-2} + \frac{y^{(n-1)}(0)}{(n-1)!} t^{n-1} \\ &= -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} z(s) ds + \frac{y^{(n-2)}(0)}{(n-2)!} t^{n-2} + \frac{y^{(n-1)}(0)}{(n-1)!} t^{n-1}. \end{aligned}$$

By virtue of the proposition of the Caputo derivative, we have

$$y^{(n-2)}(t) = -\frac{1}{\Gamma(q-n+2)} \int_0^t (t-s)^{q-n+1} z(s) ds + y^{(n-2)}(0) + y^{(n-1)}(0) t$$

and

$$y^{(n-1)}(t) = -\frac{1}{\Gamma(q-n+1)} \int_0^t (t-s)^{q-n} z(s) ds + y^{(n-1)}(0).$$

Then

$$y^{(n-2)}(1) = -\frac{1}{\Gamma(q-n+2)} \int_0^1 (1-s)^{q-n+1} z(s) ds + y^{(n-2)}(0) + y^{(n-1)}(0)$$

and

$$y^{(n-1)}(1) = -\frac{1}{\Gamma(q-n+1)} \int_0^1 (1-s)^{q-n} z(s) ds + y^{(n-1)}(0).$$

According to boundary conditions $\alpha y^{(n-2)}(0) - \beta y^{(n-1)}(0) = \eta(0) = 0$, $\gamma y^{(n-2)}(1) + \delta y^{(n-1)}(1) = \xi(1) = 0$, and noting that $\Gamma(q-n+2) = (q-n+1)\Gamma(q-n+1)$, we get

$$\begin{cases} \alpha y^{(n-2)}(0) - \beta y^{(n-1)}(0) = 0, \\ \gamma y^{(n-2)}(0) + (\gamma + \delta) y^{(n-1)}(0) = \frac{1}{\Gamma(q-n+2)} \int_0^1 [\gamma(1-s) + \delta(q-n+1)](1-s)^{q-n} z(s) ds. \end{cases}$$

Hence

$$y^{(n-2)}(0) = \frac{\rho\beta}{\Gamma(q-n+2)} \int_0^1 [\gamma(1-s) + \delta(q-n+1)](1-s)^{q-n} z(s) ds$$

and

$$y^{(n-1)}(0) = \frac{\rho\alpha}{\Gamma(q-n+2)} \int_0^1 [\gamma(1-s) + \delta(q-n+1)](1-s)^{q-n} z(s) ds.$$

So we can easily obtain that

$$y(t) = -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}z(s)ds + \frac{\rho[\beta(n-1)+\alpha t]t^{n-2}}{(n-1)!\Gamma(q-n+2)}\int_0^1 [\gamma(1-s)+\delta(q-n+1)](1-s)^{q-n}z(s)ds$$
$$= \int_0^1 G(t,s)z(s)ds.$$

Then we complete the proof of Lemma 3.1. □

By direct computation, we have

$$\frac{\partial^{n-2}}{\partial t^{n-2}}G(t,s) = g(t,s), \quad t,s \in [0,1],$$

where

$$g(t,s) = \begin{cases} g_1(t,s), & 0 \le s \le t \le 1, \\ g_0(t,s), & 0 \le t \le s \le 1, \end{cases} \tag{3.1}$$

and $g_1(t,s) = -\frac{(t-s)^{q-n+1}}{\Gamma(q-n+2)} + g_0(t,s)$, $g_0(t,s) = \frac{\rho(\beta+\alpha t)}{\Gamma(q-n+2)}[\gamma(1-s)+\delta(q-n+1)](1-s)^{q-n}$.

Lemma 3.2. *The function $g(t,s)$ defined as in (3.1) has the following properties:*

(i) *$g(t,s)$ is continuous on $[0,1]\times[0,1]$;*
(ii) *for $\beta > \frac{n-q}{q-n+1}\alpha$, we have $g(t,s) > 0$ for any $t,s \in [0,1]$;*
(iii) *for $\beta > \frac{n-q}{q-n+1}\alpha$, we have $g(t,s) \le g(s,s)$ for $t,s \in (0,1)$;*
(iv) *there exists a positive number λ such that $\rho\lambda h(s) \le g(t,s) \le \rho h(s)$ for $t,s \in [0,1]$.*

Proof. It is easy to prove that (i) holds. First, we check that (ii) is true. For $0 \le s \le t \le 1$, we get

$$\frac{\partial g_1(t,s)}{\partial t} = -\frac{(t-s)^{q-n}}{\Gamma(q-n+1)} + \frac{\rho\alpha}{\Gamma(q-n+2)}[\gamma(1-s)+\delta(q-n+1)](1-s)^{q-n}$$

and

$$\frac{\partial^2 g_1(t,s)}{\partial t^2} = \frac{(n-q)(t-s)^{q-n-1}}{\Gamma(q-n+1)} \ge 0.$$

This implies that $\frac{\partial g_1(t,s)}{\partial t}$ is increasing on $t \in [s,1]$, so

$$\frac{\partial g_1(t,s)}{\partial t} \le \frac{\partial g_1(1,s)}{\partial t} = \frac{\rho\alpha\gamma(1-s)-(q-n+1)(1-\rho\alpha\delta)}{\Gamma(q-n+2)}(1-s)^{q-n}$$
$$\le \frac{\rho}{\Gamma(q-n+2)}[\alpha\gamma-(q-n+1)(\alpha+\beta)\gamma](1-s)^{q-n} \le 0$$

because of $\beta > \frac{n-q}{q-n+1}\alpha$. Then $g_1(t,s)$ is decreasing with respect to t on $[s,1]$, we get $g_1(1,s) \le g_1(t,s) \le g_1(s,s)$.

Furthermore, when $\beta > \frac{n-q}{q-n+1}\alpha$,

$$g_1(1,s) = -\frac{(1-s)^{q-n+1}}{\Gamma(q-n+2)} + \frac{\rho(\alpha+\beta)}{\Gamma(q-n+2)}[\gamma(1-s)+\delta(q-n+1)](1-s)^{q-n}$$
$$= \frac{\rho(1-s)^{q-n}}{\Gamma(q-n+2)}[-\rho^{-1}(1-s)+\gamma(\alpha+\beta)(1-s)+(\alpha+\beta)\delta(q-n+1)]$$
$$= \frac{\rho\delta(1-s)^{q-n}}{\Gamma(q-n+2)}[-\alpha+\alpha s+(\alpha+\beta)(q-n+1)]$$
$$\ge \frac{\rho\delta(1-s)^{q-n}}{\Gamma(q-n+2)}[\alpha(q-n)+\beta(q-n+1)] > 0.$$

When $0 \le t \le s \le 1$, we have

$$\frac{\partial g_0(t,s)}{\partial t} = \frac{\rho\alpha}{\Gamma(q-n+2)}[\gamma(1-s)+\delta(q-n+1)](1-s)^{q-n} \ge 0,$$

which implies $g_0(0,s) \le g_0(t,s) \le g_0(s,s) = \rho h(s)$. Obviously,

$$g_0(0,s) = \frac{\rho\beta}{\Gamma(q-n+2)}[\gamma(1-s)+\delta(q-n+1)](1-s)^{q-n} > 0.$$

From the proof of (ii), we can obtain $g(t,s) \le g(s,s)$ easily. Furthermore, we have

$$g^*(s) \le g(t,s) \le \rho h(s),$$

where

$$g^*(s) = \begin{cases} g_1(1,s), & 0 \le s < \dfrac{\beta\gamma - \alpha\delta(q-n)}{\alpha\delta + \beta\gamma}, \\ g_0(0,s), & \dfrac{\beta\gamma - \alpha\delta(q-n)}{\alpha\delta + \beta\gamma} \le s < 1. \end{cases}$$

Since

$$\begin{aligned} \inf_{0<s<1} \frac{g_1(1,s)}{g(s,s)} &= \inf_{0<s<1} \frac{\rho - (1-s)}{\rho(\beta+\alpha s)[\gamma(1-s)+\delta(q-n+1)]} + 1 \\ &\ge \frac{4\alpha\gamma\delta[\beta(q-n+1)+\alpha(q-n)]}{[\alpha\gamma - \beta\gamma + \alpha\delta(q-n+1)]^2 + 4\alpha\beta\gamma[\gamma+\delta(q-n+1)]} := \lambda_1 \end{aligned}$$

and

$$\begin{aligned} \inf_{0<s<1} \frac{g_0(0,s)}{g(s,s)} &= \inf_{0<s<1} \frac{\beta\gamma(1-s)+\beta\delta(q-n+1)}{\rho(\beta+\alpha s)[\gamma(1-s)+\delta(q-n+1)]} \\ &\ge \frac{4\alpha\beta\gamma\delta[\beta(q-n+1)+\alpha(q-n)]}{[\alpha\gamma - \beta\gamma + \alpha\delta(q-n+1)]^2 + 4\alpha\beta\gamma[\gamma+\delta(q-n+1)]} := \lambda_2. \end{aligned}$$

Take $\lambda = \min\{\lambda_1, \lambda_2\}$, then we get

$$\lambda\rho h(s) = \lambda g(s,s) \le g(t,s) \le g(s,s) = \rho h(s). \quad \square$$

Throughout this paper, assume that $x_0(t)$ is the solution of BVP (1.1) with $f \equiv 0$. Clearly, it satisfies

$$x_0^{(n-2)}(t) = \begin{cases} \dfrac{1}{\beta} e^{\frac{\alpha}{\beta}t} \displaystyle\int_t^0 e^{-\frac{\alpha}{\beta}s}\eta(s)ds, & -\tau \le t \le 0, \\ 0, & 0 \le t \le 1, \\ \dfrac{1}{\delta} e^{-\frac{\gamma}{\delta}t} \displaystyle\int_1^t e^{\frac{\gamma}{\delta}s}\xi(s)ds, & 1 \le t \le b. \end{cases}$$

Let $y(t)$ be a solution of BVP (1.1) and $x(t) = y(t) - x_0(t)$. Because of $x(t) \equiv y(t)$ for $0 \le t \le 1$, and $x(t)$ satisfies

$$x^{(n-2)}(t) = \begin{cases} e^{\frac{\alpha}{\beta}t} x^{(n-2)}(0), & -\tau \le t \le 0, \\ \displaystyle\int_0^1 g(t,s)r(s)f(x(s+\theta)+x_0(s+\theta))ds, & 0 \le t \le 1, \\ e^{-\frac{\gamma}{\delta}(t-1)} x^{(n-2)}(1), & 1 \le t \le b, \end{cases}$$

which implies

$$x(t) = \begin{cases} \left(\dfrac{\beta}{\alpha}\right)^{n-2} e^{\frac{\alpha}{\beta}t} x^{(n-2)}(0), & -\tau \le t \le 0, \\ \displaystyle\int_0^1 G(t,s)r(s)f(x(s+\theta)+x_0(s+\theta))ds, & 0 \le t \le 1, \\ \left(-\dfrac{\delta}{\gamma}\right)^{n-2} e^{-\frac{\gamma}{\delta}(t-1)} x^{(n-2)}(1), & 1 \le t \le b. \end{cases}$$

Define a cone K in the Banach space $X = C^{(n-2)}([-\tau, b], \mathbf{R})$ as follows

$$K := \{x \in X : x^{(n-2)}(t) \ge p(t)\|x\|_{[-\tau,b]}, t \in [-\tau, b]\},$$

where $\|x\|_{[-\tau,b]} := \sup\{|x^{(n-2)}(t)| : -\tau \le t \le b\}$, and

$$p(t) := \begin{cases} e^{\frac{\alpha}{\beta}t}, & -\tau \le t \le 0, \\ \lambda, & 0 \le t \le 1, \\ e^{-\frac{\gamma}{\delta}(t-1)}, & 1 \le t \le b. \end{cases}$$

Define a mapping $T : K \to K$ as

$$(Tx)(t) := \begin{cases} \left(\frac{\beta}{\alpha}\right)^{n-2} e^{\frac{\alpha}{\beta}t} x^{(n-2)}(0), & -\tau \le t \le 0, \\ \int_0^1 G(t,s)r(s)f(x(s+\theta)+x_0(s+\theta))ds, & 0 \le t \le 1, \\ \left(-\frac{\delta}{\gamma}\right)^{n-2} e^{-\frac{\gamma}{\delta}(t-1)} x^{(n-2)}(1), & 1 \le t \le b. \end{cases} \tag{3.2}$$

Thus

$$(Tx)^{(n-2)}(t) = \begin{cases} e^{\frac{\alpha}{\beta}t} \int_0^1 g(0,s)r(s)f(x(s+\theta)+x_0(s+\theta))ds, & -\tau \le t \le 0, \\ \int_0^1 g(t,s)r(s)f(x(s+\theta)+x_0(s+\theta))ds, & 0 \le t \le 1, \\ e^{-\frac{\gamma}{\delta}(t-1)} \int_0^1 g(1,s)r(s)f(x(s+\theta)+x_0(s+\theta))ds, & 1 \le t \le b. \end{cases} \tag{3.3}$$

Lemma 3.3. *$T : K \to K$ is completely continuous.*

Proof. It follows from (3.2) and (3.3) that we have for $-\tau \le t \le 0$

$$0 \le (Tx)^{(n-2)}(t) \le (Tx)^{(n-2)}(0)$$

and for $1 \le t \le b$

$$0 \le (Tx)^{(n-2)}(t) \le (Tx)^{(n-2)}(1).$$

Thus we get $\|Tx\|_{[-\tau,b]} = \|Tx\|_{[0,1]}$ and

$$(Tx)^{(n-2)}(t) \ge \begin{cases} e^{\frac{\alpha}{\beta}t}\|Tx\|_{[0,1]}, & -\tau \le t \le 0, \\ e^{-\frac{\gamma}{\delta}(t-1)}\|Tx\|_{[0,1]}, & 1 \le t \le b. \end{cases} \tag{3.4}$$

For $0 \le t \le 1$, $x \in K$, we obtain from Lemma 3.1(iii) and (iv)

$$\|Tx\|_{[-\tau,b]} = \|Tx\|_{[0,1]} \le \rho \int_0^1 h(s)r(s)f(x(s+\theta)+x_0(s+\theta))ds$$

and

$$(Tx)^{(n-2)}(t) \ge \rho\lambda \int_0^1 h(s)r(s)f(x(s+\theta)+x_0(s+\theta))ds,$$

which implies that $(Tx)^{(n-2)}(t) \ge \lambda\|Tx\|_{[0,1]} = \lambda\|Tx\|_{[-\tau,b]}$. This together with (3.4) shows that $(Tx)^{(n-2)}(t) \ge p(t)\|Tx\|_{[-\tau,b]}$. Therefore, we have $T(K) \subset K$.

Next we show that $T : K \to K$ is continuous. In fact, suppose that $x_m, x \in K$ with $\|x_m - x\|_{[-\tau,b]} \to 0$ as $n \to \infty$; then we have $\|x_m(t+\theta) - x(t+\theta)\|_{[-\tau,b]} = \sup_{-\tau\le\theta\le a} |x_m(t+\theta) - x(t+\theta)| \to 0$, $t \in [0,1]$. Thus for $t \in [-\tau, b]$, it can be seen from (H2) and (3.3) that

$$\|(Tx_m)^{(n-2)}(t) - (Tx)^{(n-2)}(t)\|_{[-\tau,b]} \le \rho \sup_{0\le t\le 1} |f(x_m(t+\theta)) - f(x(t+\theta))| \int_0^1 r(s)h(s)ds.$$

This implies that $\|Tx_n - Tx\|_{[-\tau,b]} \to 0$ as $n \to \infty$.

Let $B \subset K$ be a bounded subset of K and $M_0 > 0$ is a constant such that $\|x\|_{[-\tau,b]} \le M_0$. Define a set $S \in C^+$ as $S = \{\varphi \in C^+ : \|\varphi\|_C \le M_0\}$. Let $L = \max_{\varphi\in S} f(\varphi + \max_{t\in(0,1)} x_0(t+\theta))$.

Furthermore, we have for $-\tau \le t \le 0$

$$|(Tx)^{(n-1)}(t)| = \left|\frac{\alpha}{\beta}e^{\frac{\alpha}{\beta}t}\int_0^1 g(0,s)r(s)f(x(s+\theta)+x_0(s+\theta))ds\right| \le \frac{\alpha}{\beta}L\rho\int_0^1 r(s)h(s)ds =: L_1,$$

for $0 \le t \le 1$,

$$
\begin{aligned}
|(Tx)^{(n-1)}(t)| &= \Bigg|-\frac{q-n-1}{\Gamma(q-n+2)}\int_0^t (t-s)^{q-n} r(s) f(x(s+\theta)+x_0(s+\theta))ds \\
&\quad + \int_0^1 \frac{\alpha\rho}{\Gamma(q-n+2)}[\gamma(1-s)+\delta(q-n+1)](1-s)^{q-n} r(s) f(x(s+\theta)+x_0(s+\theta))ds\Bigg| \\
&\le \frac{q-n-1}{\Gamma(q-n+2)}\int_0^1 (1-s)^{q-n} r(s) f(x(s+\theta)+x_0(s+\theta))ds \\
&\quad + \int_0^1 \frac{\alpha\rho}{\Gamma(q-n+2)}[\gamma(1-s)+\delta(q-n+1)](1-s)^{q-n} r(s) f(x(s+\theta)+x_0(s+\theta))ds \\
&\le \frac{L}{\Gamma(q-n+2)}\int_0^1 [\alpha\delta\rho(1-s)+(q-n+1)(1+\alpha\delta\rho)](1-s)^{q-n} r(s)ds =: L_2,
\end{aligned}
$$

for $1 \le t \le b$,

$$
\begin{aligned}
|(Tx)^{(n-1)}(t)| &= \left|-\frac{\gamma}{\delta}e^{-\frac{\gamma}{\delta}}\int_0^1 g(1,s) r(s) f(x(s+\theta)+x_0(s+\theta))ds\right| \\
&\le \frac{\gamma}{\delta} L\rho \int_0^1 r(s)h(s)ds =: L_3.
\end{aligned}
$$

Thus, suppose that $x \in K$, $\forall \varepsilon > 0$, let $\delta_0 = \frac{\varepsilon}{\max\{L_1,L_2,L_3\}}$, for $t_1, t_2 \in [-\tau, b]$, $|t_1 - t_2| < \delta_0$, we get

$$
|(Tx)^{(n-2)}(t_1) - (Tx)^{(n-2)}(t_2)| \le \max\{L_1, L_2, L_3\}|t_1 - t_2| < \varepsilon.
$$

The proof of Lemma 3.2 is completed. □

Choose a $\sigma \in \left(0, \min\left\{e^{-\frac{\alpha}{\beta}\tau}, \lambda, e^{-\frac{\delta}{\gamma}a}\right\}\right)$ and let

$$
C^* = \{\varphi \in C^+ : 0 < \sigma\|\varphi\|_C \le \varphi^{(n-2)}(t), \quad \text{for } t \in [-\tau, a]\} \tag{3.5}
$$

and $E_\sigma = \{t \in E : \sigma \le t + \theta \le 1 - \sigma,\ \text{for } -\tau \le \theta \le a\}$.

Now, we can state and prove our main results.

Theorem 3.4. *Assume that* (H0)–(H3) *hold and* $\beta > \frac{n-q}{q-n+1}\alpha$. *Then BVP* (1.1) *has at least one positive solution if one of the following conditions is satisfied.*

(H4) $f_0^* > d_1$, *and* $f_\infty < d_2$, *or*
(H5) $f_0 < d_3, f_\infty^* > d_1$, *and* $\xi(t) \equiv \eta(t) \equiv 0$,

where

$$
f_0^* = \lim_{\varphi\in C^*, \|\varphi\|_C \downarrow 0} \frac{f(\varphi)}{\|\varphi\|_C}, \qquad f_0 = \lim_{\|\varphi\|_C \downarrow 0} \frac{f(\varphi)}{\|\varphi\|_C},
$$
$$
f_\infty^* = \lim_{\varphi\in C^*, \|\varphi\|_C \uparrow \infty} \frac{f(\varphi)}{\|\varphi\|_C}, \qquad f_\infty = \lim_{\|\varphi\|_C \uparrow \infty} \frac{f(\varphi)}{\|\varphi\|_C},
$$

and

$$
\begin{aligned}
d_1 &= \left[\lambda\rho\sigma\int_{E_\sigma} r(s)h(s)ds\right]^{-1}, \\
d_2 &= \left[2\rho(1+\|x_0\|_{[-\tau,b]})\int_0^1 r(s)h(s)ds\right]^{-1}, \\
d_3 &= \left[\rho\int_0^1 r(s)h(s)ds\right]^{-1}.
\end{aligned}
$$

Proof. By (H2) and (3.5), we have $0 < \int_{E_\sigma} r(t)h(t)dt < +\infty$.

If condition (H4) is satisfied, suppose that $f_0^* > d_1$. We can choose $\rho_1 > 0$ sufficiently small so that

$$
f(\varphi) \ge d_1\|\varphi\|_C, \quad \forall\varphi \in C^*, \qquad \|\varphi\|_C \le \rho_1.
$$

Set the first open subset of X by $\Omega_1 := \{x \in X : \|x\|_{[-\tau,b]} < \rho_1\}$. If $x \in \partial\Omega_1$, we have $\|x\|_{[-\tau,b]} = \rho_1$ for $t \in [0, 1]$. Furthermore, one has $x_t \in C^*$ for $t \in E_\sigma$ and

$$\|x\|_C \geq \sigma\|x\|_{[-\tau,b]} = \sigma\rho_1, \quad t \in E_\sigma. \tag{3.6}$$

Note that when $t \in E_\sigma$, we have $x_0(t + \theta) = 0 \in C$. Thus, one obtain

$$\begin{aligned}\|(Tx)^{(n-2)}(t)\|_{[-\tau,b]} &= \|(Tx)^{(n-2)}(t)\|_{[0,1]} = \left|\int_0^1 g(t,s)r(s)f(x(s+\theta)+x_0(s+\theta))ds\right| \\ &\geq \lambda\rho d_1\|x(s+\theta)+x_0(s+\theta)\|_C \int_{E_\sigma} r(s)h(s)ds \\ &\geq \lambda\rho d_1\sigma\rho_1 \int_{E_\sigma} r(s)h(s)ds = \rho_1 = \|x\|_{[-\tau,b]},\end{aligned} \tag{3.7}$$

which implies that $\|Tx\|_{[-\tau,b]} = \|Tx\|_{[0,1]} \geq \|x\|_{[-\tau,b]}, \forall x \in K \cap \partial\Omega_1$.

On the other hand, since $f_\infty < d_2$, there exists $M > \rho_1$ such that

$$f(\varphi) \leq d_2\|\varphi\|_C, \quad \varphi \in C^+, \qquad \|\varphi\|_C > M.$$

Choose $\rho_2 > 0$ sufficiently large so that

$$\rho_2 > 1 + \rho\max\{f(\varphi) : 0 \leq \|\varphi\|_C \leq M + \|x_0\|_{[-\tau,b]}\} \int_0^1 r(s)h(s)ds.$$

Set the second open subset of X by $\Omega_2 := \{x \in X : \|x\|_{[-\tau,b]} < \rho_2\}$. For $x \in \partial\Omega_2$, we have $\|x\|_{[-\tau,b]} = \rho_2$. It is easy to obtain, from the facts: $x_0^{(n-2)}(t) \geq 0$, $x^{(n-2)}(t) \geq 0$ for $t \in [-\tau, b]$, that for $s \in [0, 1]$

$$\|x^{(n-2)}(s+\theta) + x_0^{(n-2)}(s+\theta)\|_C \geq \|x^{(n-2)}(s+\theta)\|_C > M, \quad \text{if } \|x^{(n-2)}(s+\theta)\|_C > M,$$

and

$$\begin{aligned}\|x^{(n-2)}(s+\theta) + x_0^{(n-2)}(s+\theta)\|_C &\leq \|x^{(n-2)}(s+\theta)\|_C + \|x_0^{(n-2)}(s+\theta)\|_C \\ &\leq M + \|x_0\|_{[-\tau,b]}, \quad \text{if } \|x^{(n-2)}(s+\theta)\|_C \leq M.\end{aligned}$$

Hence

$$\begin{aligned}\|(Tx)^{(n-2)}(t)\|_{[-\tau,b]} &= \|(Tx)^{(n-2)}(t)\|_{[0,1]} = \int_0^1 g(t,s)r(s)f(x(s+\theta)+x_0(s+\theta))ds \\ &\leq \rho\int_{\|x^{(n-2)}(s+\theta)\|_C > M} r(s)h(s)f(x(s+\theta)+x_0(s+\theta))ds \\ &\quad + \rho\int_{\|x^{(n-2)}(s+\theta)\|_C \leq M} r(s)h(s)f(x(s+\theta)+x_0(s+\theta))ds \\ &\leq \rho\max\{d_2\|x(s+\theta)+x_0(s+\theta)\|_C, \max\{f(\varphi) : 0 \leq \|\varphi\|_C \leq M \\ &\quad + \|x_0\|_{[-\tau,b]}\}\}\int_0^1 r(s)h(s)ds \\ &\leq \max\left\{\frac{1}{2}\|x\|_{[-\tau,b]} + \frac{1}{2}, \rho\max\left\{f(\varphi)\int_0^1 r(s)h(s)ds : 0 \leq \|\varphi\|_C \leq M + \|x_0\|_{[-\tau,b]}\right\}\right\} \\ &\leq \frac{1}{2}\|x\|_{[-\tau,b]} + \frac{1}{2}\rho_2 < \rho_2 = \|x\|_{[-\tau,b]},\end{aligned}$$

which implies that

$$\|(Tx)(t)\|_{[-\tau,b]} = \|(Tx)(t)\|_{[0,1]} < \|x\|_{[-\tau,b]}, \quad \forall x \in K \cap \partial\Omega_2.$$

Therefore, by Lemma A, it follows that T has a fixed point $x^* \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ such that $0 < \rho_1 \leq \|x^*\|_{[-\tau,b]} = \|x^*\|_{[0,1]} \leq \rho_2$. So, the problem (1.1) has a positive solution $y(t) = x^*(t) + x_0(t)$ with $y \in [\rho_1 + M_0, \rho_2 + M_0]$, where $M_0 := \|x_0\|_{[-\tau,b]}$.

If condition (H5) is satisfied, obviously, one has $x_0 \equiv 0$. Suppose that $f_0 < d_3$, then there exists a $\rho_3 > 0$ such that

$$f(\varphi) \leq d_3\|\varphi\|_C, \quad \varphi \in C^+, \qquad \|\varphi\|_C \leq \rho_3.$$

Set the open subset of X by $\Omega_3 := \{x \in X : \|x\|_{[-\tau,b]} < \rho_3\}$. For $x \in \partial\Omega_3$, we deduce that $\|x^{(n-2)}(s+\theta) + x_0^{(n-2)}(s+\theta)\|_C \le \|x^{(n-2)}(s+\theta)\|_C \le \rho_3$ for $s \in [0, 1]$. Hence

$$\begin{aligned}\|(Tx)^{(n-2)}(t)\|_{[-\tau,b]} &= \|(Tx)^{(n-2)}(t)\|_{[0,1]} = \int_0^1 g(t,s)r(s)f(x(s+\theta)+x_0(s+\theta))ds \\ &\le \rho d_3 \int_0^1 r(s)h(s)\|x(s+\theta)+x_0(s+\theta)\|_C ds \\ &\le \rho d_3 \rho_3 \int_0^1 r(s)h(s)ds = \rho_3 = \|x\|_{[-\tau,b]},\end{aligned}$$

which leads to $\|(Tx)(t)\|_{[-\tau,b]} \le \|x\|_{[-\tau,b]}$, $\forall x \in K \cap \partial\Omega_3$. Next, since $f_\infty^* > d_1$, then for any $M^* > 0$, we can choose $\rho_4 > \rho_3$, so that

$$f(\varphi) \ge M^*\|\varphi\|_C, \quad \varphi \in C^*, \qquad \|\varphi\|_C \in (\sigma\rho_4, +\infty).$$

Set the open subset of X by $\Omega_4 := \{x \in X : \|x\|_{[-\tau,b]} < \rho_4\}$. For any $x \in K$ with $\|x\|_{[-\tau,b]} = \rho_4$, one can deduce that

$$\sigma\|x\|_{[-\tau,b]} \le p(t+\theta)\|x\|_{[-\tau,b]} \le x^{(n-2)}(t+\theta), \quad t \in E_\sigma,$$

which implies that $x_t \in C^*$ and

$$\|x(t+\theta)\|_C \ge \sigma\|x\|_{[-\tau,b]} = \sigma\rho_4, \quad \text{for } t \in E_\sigma.$$

Thus, similar to (3.7), we get

$$\begin{aligned}\|(Tx)^{(n-2)}(t)\|_{[-\tau,b]} &\ge \lambda\rho d_1 \|x(s+\theta)\|_C \int_{E_\sigma} r(s)h(s)ds \\ &\ge \lambda\rho d_1 \sigma\rho_4 \int_{E_\sigma} r(s)h(s)ds = \rho_4 = \|x\|_{[-\tau,b]},\end{aligned}$$

which implies that $\|Tx\|_{[-\tau,b]} \ge \|x\|_{[-\tau,b]}$, $\forall x \in K \cap \partial\Omega_4$. According to Lemma A, it follows that T has a fixed point $x^{**} \in K \cap (\bar{\Omega}_4 \setminus \Omega_3)$ such that $0 < \rho_3 \le \|x^{**}\|_{[-\tau,b]} = \|x^{**}\|_{[0,1]} \le \rho_4$. So, the problem (1.1) has a positive solution $y(t) = x^{**}(t)$ with $y \in [\rho_3, \rho_4]$. The proof of Theorem 3.4 is thus completed. □

Theorem 3.5. *Assume that* (H0)–(H3) *hold and* $\beta > \frac{n-q}{q-n+1}\alpha$. *Then BVP* (1.1) *has at least two positive solutions if the following conditions are satisfied.*

(C1) $f_0^* > d_1$, *and* $f_\infty^* > d_1$,

(C2) *there exist constant* $p_1 > 0$ *and* $\mu \in (0, \hbar)$ *such that*

$$f(\varphi) \le \mu p_1, \quad \forall \|\varphi\|_C \in (0, p_1 + p_0],$$

where

$$\hbar = \left[\rho\int_0^1 r(s)h(s)ds\right]^{-1}$$

and

$$p_0 = \max\left\{\max_{-\tau\le t\le 0}\frac{1}{\beta}e^{\frac{\alpha}{\beta}t}\int_t^0 e^{-\frac{\alpha}{\beta}s}\eta(s)ds,\ \max_{1\le t\le b}\frac{1}{\delta}e^{-\frac{\gamma}{\delta}t}\int_1^t e^{\frac{\gamma}{\delta}s}\xi(s)ds\right\}.$$

Proof. At first, in view of (C1), there exists a $p^* : 0 < p^* < p_1$ such that

$$f(\varphi) \ge d_1\|\varphi\|_C, \quad \text{for } \|\varphi\|_C \le p^*, \varphi \in C^*.$$

Set the first open subset of X by $\Omega_{p^*} = \{x : x \in X, \|x\|_{[-\tau,b]} < p^*\}$. Similar to (3.6), we have $x(s+\theta) \in C^*$ and $p^* \ge \|x(s+\theta)\|_C \ge \sigma\|x\|_{[-\tau,b]} = \sigma p^*$ for $s \in E_\sigma$. Furthermore, one can obtain $\|Tx\|_{[-\tau,b]} \ge p^* = \|x\|_{[-\tau,b]}$, which implies

$$\|Tx\|_{[-\tau,b]} = \|Tx\|_{[0,1]} \ge \|x\|_{[-\tau,b]}, \quad \forall x \in K \cap \partial\Omega_{p^*}.$$

On the other hand, since $f_\infty^* > d_1$, one can choose $d > p_1$ sufficiently large so that

$$f(\varphi) \ge d_1\|\varphi\|_C, \qquad \|\varphi\|_C \ge \sigma d, \quad \varphi \in C^*.$$

Set the second open subset of X by $\Omega_d = \{x : x \in X, \|x\|_{[-\tau,b]} < d\}$. For any $x \in K$ satisfies $\|x\|_{[-\tau,b]} = d$, one has $x(s+\theta) \in C^*$ and $\|x(s+\theta)\|_C \ge \sigma\|x\|_{[-\tau,b]} = \sigma d$ for $s \in E_\sigma$. Also one obtains an analogous inequality $\|Tx\|_{[-\tau,b]} \ge d = \|x\|_{[-\tau,b]}$, for all $x \in K \cap \partial\Omega_d$.

Finally, set the open subset of X by $\Omega_{p_1} = \{x : x \in X, \|x\|_{[-\tau,b]} < p_1\}$. For any $x \in K \cap \partial\Omega_{p_1}$, one has

$$\begin{aligned}\|(Tx)^{(n-2)}(t)\|_{[-\tau,b]} &= \|(Tx)^{(n-2)}(t)\|_{[0,1]} = \left|\int_0^1 g(t,s)r(s)f(x(s+\theta)+x_0(s+\theta))ds\right| \\ &\leq \rho\int_0^1 r(s)h(s)f(x(s+\theta)+x_0(s+\theta))ds \\ &\leq \rho\mu p_1\int_0^1 r(s)h(s)ds \leq p_1 = \|x\|_{[-\tau,b]},\end{aligned}$$

which yields

$$\|Tx\|_{[-\tau,b]} \leq \|x\|_{[-\tau,b]}, \quad \text{for } x \in K \cap \partial\Omega_{p_1}.$$

According to Lemma A, it follows that T has two fixed points x_1, x_2 such that $x_1 \in K \cap (\bar{\Omega}_{p_1} \setminus \Omega_{p^*})$, $x_2 \in K \cap (\bar{\Omega}_d \setminus \Omega_{p_1})$ with $0 < \|x_1\|_{[-\tau,b]} < p_1 < \|x_2\|_{[-\tau,b]}$. So, the problem (1.1) has at least two positive solutions $y_1(t) = x_1(t) + x_0(t)$, $y_2(t) = x_2(t) + x_0(t)$ with $0 < \|y_1\|_{[-\tau,b]} < p_1 + M_0 < \|y_2\|_{[-\tau,b]}$. The proof is complete. □

In the same way, we can prove the following corollaries.

Corollary 3.6. *The BVP (1.1) has at least one positive solution if* $f_0 = 0, f_\infty^* = \infty$, *and* $\xi(t) \equiv \eta(t) \equiv 0$.

Corollary 3.7. *The BVP (1.1) has at least one positive solution if* $f_0^* = \infty$ *and* $f_\infty = 0$.

Corollary 3.8. *Assume that* (H0)–(H3) *hold and* $\beta > \frac{n-q}{q-n+1}\alpha$. *Then BVP (1.1) has at least two positive solutions if the following conditions are satisfied.*

(C3) $f_0 < d_3, f_\infty < d_2$, *and* $\xi(t) \equiv \eta(t) \equiv 0$.

(C4) *there exist a constant* $p_2 > 0$ *and* $\mu_1 \in (d_1, +\infty)$ *such that*

$$f(\varphi) \geq \mu_1 p_2, \quad \forall \|\varphi\|_C \in [\sigma p_2, p_2],$$

where

$$d_1 = \left[\lambda\rho\sigma\int_{E_\sigma} r(s)h(s)ds\right]^{-1}.$$

Corollary 3.9. *Assume that* $f_0^* = f_\infty^* = \infty$, *and there exists a constant* $p_1 > 0$ *such that*

$$f(\varphi) \leq \mu p_1, \quad \forall \|\varphi\|_C \in (0, p_1 + p_0],$$

where

$$p_0 = \max\left\{\max_{-\tau\leq t\leq 0}\frac{1}{\beta}e^{\frac{\alpha}{\beta}t}\int_t^0 e^{-\frac{\alpha}{\beta}s}\eta(s)ds,\ \max_{1\leq t\leq b}\frac{1}{\delta}e^{-\frac{\gamma}{\delta}t}\int_1^t e^{\frac{\gamma}{\delta}s}\xi(s)ds\right\}.$$

and μ *is given in* (C2). *Then BVP (1.1) has at least two positive solutions* $y_1(t)$, *and* $y_2(t)$ *with* $0 < \|y_1\|_{[-\tau,b]} < p_1 + M_0 < \|y_2\|_{[-\tau,b]}$.

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