Journal of Computational and Applied Mathematics 233 (2009) 142–149

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space

M. Mursaleen, S.A. Mohiuddine *

Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India

ARTICLE INFO

Article history:
Received 7 November 2008
Received in revised form 1 July 2009

Keywords:
t-norm
t-conorm
Intuitionistic fuzzy normed spaces
Statistical convergence
Lacunary statistical convergence
Lacunary statistical Cauchy
Statistical completeness

ABSTRACT

The concept of statistical convergence was introduced by Fast [H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244] which was later on studied by many authors. In [J.A. Fridy, C. Orhan, Lacunary statistical convergence, Pacific J. Math. 160 (1993) 43–51], Fridy and Orhan introduced the idea of lacunary statistical convergence. Quite recently, the concept of statistical convergence of double sequences has been studied in intuitionistic fuzzy normed space by Mursaleen and Mohiuddine [M. Mursaleen, S.A. Mohiuddine, Statistical convergence of double sequences in intuitionistic fuzzy normed spaces, Chaos Solitons Fractals (2008), doi:10.1016/j.chaos.2008.09.018]. In this paper, we study lacunary statistical convergence in intuitionistic fuzzy normed space. We also introduce here a new concept, that is, statistical completeness and show that IFNS is statistically complete but not complete.

1. Introduction

In recent years, the fuzzy theory has emerged as the most active area of research in many branches of mathematics and engineering. This new theory was introduced in [1] in 1965 and since then a large number of research papers have appeared by using the concept of fuzzy set/numbers and fuzzification of many classical theories has also been made. It has also very useful application in various fields, e.g. population dynamics [2], chaos control [3], computer programming [4], nonlinear dynamical systems [5], fuzzy physics [6], fuzzy topology [7], etc. In [8], Park introduced the concept of intuitionistic fuzzy metric space and later on Saadati and Park [9] introduced the concept of intuitionistic fuzzy normed space. Recently, the concept of statistical convergence in intuitionistic fuzzy normed space was studied for single sequences in [10] and for double sequences in [11].

There are many situations where the norm of a vector is not possible to find and the concept of intuitionistic fuzzy norm seems to be more suitable in such cases, that is, we can deal with such situations by modelling the inexactness by intuitionistic fuzzy norm. Many authors have used the concept of fuzzy norm to deal with the inexactness of the norm in some situations (see [12,13]).

In this paper we shall study lacunary statistical convergence and lacunary statistical Cauchy in intuitionistic fuzzy normed space. We also introduce the concept of statistical completeness which would provide a more general framework to study the completeness of intuitionistic fuzzy normed spaces.

We outline the present work as follows. In Section 2, we recall some basic definitions related to the intuitionistic fuzzy normed space. In Section 3, we introduce lacunary statistical convergence in intuitionistic fuzzy normed space and prove our main results. Finally, Section 4 is devoted to introduce a new concept, i.e. (lacunary) statistical completeness and find its relation with completeness of intuitionistic fuzzy normed space.

* Corresponding author.
E-mail addresses: mursaleenm@gmail.com (M. Mursaleen), mohiuddine@gmail.com (S.A. Mohiuddine).

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doi:10.1016/j.cam.2009.07.005

2. Preliminaries

Definition 2.1 ([14]). A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a *continuous t-norm* if it satisfies the following conditions:

(a) $*$ is associative and commutative,
(b) $*$ is continuous,
(c) $a * 1 = a$ for all $a \in [0, 1]$,
(d) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for each $a, b, c, d \in [0, 1]$.

Definition 2.2 ([14]). A binary operation $\diamondsuit : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a *continuous t-conorm* if it satisfies the following conditions:

(a) $\diamondsuit$ is associative and commutative,
(b) $\diamondsuit$ is continuous,
(c) $a\diamondsuit 0 = a$ for all $a \in [0, 1]$,
(d) $a\diamondsuit b \le c\diamondsuit d$ whenever $a \le c$ and $b \le d$ for each $a, b, c, d \in [0, 1]$.

Using the continuous t-norm and t-conorm, Saadati and Park [9] have introduced the concept of intuitionistic fuzzy normed space as follows:

Definition 2.3. The five-tuple $(X, \mu, \nu, *, \diamondsuit)$ is said to be an *intuitionistic fuzzy normed space* (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamondsuit$ is a continuous t-conorm, and μ, ν fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions: For every $x, y \in X$ and $s, t > 0$,

(a) $\mu(x, t) + \nu(x, t) \le 1$,
(b) $\mu(x, t) > 0$,
(c) $\mu(x, t) = 1$ if and only if $x = 0$,
(d) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \ne 0$,
(e) $\mu(x, t) * \mu(y, s) \le \mu(x + y, t + s)$,
(f) $\mu(x, \cdot) : (0, \infty) \to [0, 1]$ is continuous,
(g) $\lim_{t\to\infty} \mu(x, t) = 1$ and $\lim_{t\to 0} \mu(x, t) = 0$,
(h) $\nu(x, t) < 1$,
(i) $\nu(x, t) = 0$ if and only if $x = 0$,
(j) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \ne 0$,
(k) $\nu(x, t)\diamondsuit\nu(y, s) \ge \nu(x + y, t + s)$,
(l) $\nu(x, \cdot) : (0, \infty) \to [0, 1]$ is continuous,
(m) $\lim_{t\to\infty} \nu(x, t) = 0$ and $\lim_{t\to 0} \nu(x, t) = 1$.

In this case (μ, ν) is called an *intuitionistic fuzzy norm*. For example, let $(X, \|.\|)$ be a normed space, and let $a * b = ab$ and $a\diamondsuit b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$. For all $x \in X$ and every $t > 0$, consider

$$\mu(x, t) := \frac{t}{t + \|x\|} \quad \text{and} \quad \nu(x, t) := \frac{\|x\|}{t + \|x\|}.$$

Then $(X, \mu, \nu, *, \diamondsuit)$ is an intuitionistic fuzzy normed space.

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are studied by Saadati and Park [9].

Definition 2.4. Let $(X, \mu, \nu, *, \diamondsuit)$ be an IFNS. Then, a sequence $x = (x_k)$ is said to be *convergent* to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - L, t) > 1 - \epsilon$ and $\nu(x_k - L, t) < \epsilon$ for all $k \ge k_0$. In this case we write (μ, ν)- $\lim x = L$ or $x_k \xrightarrow{(\mu,\nu)} L$ as $k \to \infty$.

Definition 2.5. Let $(V, \mu, \nu, *, \diamondsuit)$ be an IFNS. Then, $x = (x_k)$ is said to be *Cauchy sequence* with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - x_\ell, t) > 1 - \epsilon$ and $\nu(x_k - x_\ell, t) < \epsilon$ for all $k, \ell \ge k_0$.

Remark 2.1 ([9]). Let $(X, \|.\|)$ be a real normed linear space,

$$\mu(x, t) := \frac{t}{t + \|x\|} \quad \text{and} \quad \nu(x, t) := \frac{\|x\|}{t + \|x\|}$$

for all $x \in X$ and $t > 0$. Then $x_n \xrightarrow{\|.\|} x$ if and only if $x_n \xrightarrow{(\mu,\nu)} x$.

3. Lacunary statistical convergence in IFNS

In this section we study the concept of lacunary statistically convergent sequences in intuitionistic fuzzy normed space.

The idea of statistical convergence was first introduced in [15] and later on studied by many authors. The active researches on this topic were started after the paper of Fridy [16]. The concept of statistical convergence for fuzzy numbers has also been studied by various authors, e.g. [17,12,13,18].

Definition 3.1. Let N be a subset of $\mathbb{N}$, the set of natural numbers. Then the *asymptotic density* of N denoted by $\delta(N)$, is defined as

$$\delta(N) = \lim_n \frac{1}{n}|\{k \le n : k \in N\}|,$$

where the vertical bars denote the cardinality of the enclosed set.

A number sequence $x = (x_k)$ is said to be *statistically convergent* to the number L if for each $\epsilon > 0$, the set $K(\epsilon) = \{k \le n : |x_k - L| > \epsilon\}$ has asymptotic density zero, i.e.

$$\lim_n \frac{1}{n}|\{k \le n : |x_k - L| \ge \epsilon\}| = 0.$$

In this case we write $st\text{-}\lim x = L$ (see [15,16]).

Note that every convergent sequence is statistically convergent to the same limit, but converse need not be true.

Statistical convergence of double sequences $x = (x_{jk})$ has been defined and studied in [19]; and for fuzzy numbers in [18].

First we define the concept of θ-density:

Definition 3.2. By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r := k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r := (k_{r-1}, k_r]$, and the ratio k_r/k_{r-1} will be abbreviated by q_r.

Let $N \subseteq \mathbb{N}$. The number

$$\delta_\theta(N) = \lim_r \frac{1}{h_r}|\{k \in I_r : k \in N\}|$$

is said to be the θ-*density* of N, provided the limit exists.

Definition 3.3 (*[20,21]*). Let θ be a lacunary sequence. Then a sequence $x = (x_k)$ is said to S_θ-convergent to the number L if for every $\epsilon > 0$, the set $K(\epsilon)$ has θ-density zero, where

$$K(\epsilon) := \{k \in \mathbb{N} : |x_k - L| \ge \epsilon\}.$$

In this case we write $S_\theta\text{-}\lim x = L$ or $x_k \to L(S_\theta)$.

Now we define the S_θ-convergence with respect to IFNS.

Definition 3.4. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS and θ be a lacunary sequence. Then, a sequence $x = (x_k)$ is said to be S_θ-*convergent* to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if for every $\epsilon > 0$ and $t > 0$,

$$\delta_\theta(\{k \in \mathbb{N} : \mu(x_k - L, t) \le 1 - \epsilon \text{ or } \nu(x_k - L, t) \ge \epsilon\}) = 0, \tag{1}$$

or equivalently

$$\delta_\theta(\{k \in \mathbb{N} : \mu(x_k - L, t) > 1 - \epsilon \text{ and } \nu(x_k - L, t) < \epsilon\}) = 1. \tag{1'}$$

In this case we write $S_\theta^{(\mu,\nu)}\text{-}\lim x = L$ or $x_k \xrightarrow{(\mu,\nu)} L(S_\theta)$, where L is said to be $S_\theta^{(\mu,\nu)}\text{-}\lim x$ and we denote the set of all S_θ-convergent sequences with respect to the intuitionistic fuzzy norm (μ, ν) by $S_\theta^{(\mu,\nu)}$.

By using (1) and (1′), we easily get the following lemma.

Lemma 3.1. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS and θ be a lacunary sequence. Then, for every $\epsilon > 0$ and $t > 0$, the following statements are equivalent:*

(i) $S_\theta^{(\mu,\nu)}\text{-}\lim x = L$.
(ii) $\delta_\theta(\{k \in \mathbb{N} : \mu(x_k - L, t) \le 1 - \epsilon\}) = \delta_\theta(\{k \in \mathbb{N} : \nu(x_k - L, t) \ge \epsilon\}) = 0$.
(iii) $\delta_\theta(\{k \in \mathbb{N} : \mu(x_k - L, t) > 1 - \epsilon$ *and* $\nu(x_k - L, t) < \epsilon\}) = 1$.
(iv) $\delta_\theta(\{k \in \mathbb{N} : \mu(x_k - L, t) > 1 - \epsilon\}) = \delta_\theta(\{k \in \mathbb{N} : \nu(x_k - L, t) < \epsilon\}) = 1$.
(v) $S_\theta\text{-}\lim \mu(x_k - L, t) = 1$ *and* $S_\theta\text{-}\lim \nu(x_k - L, t) = 0$.

Theorem 3.2. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS and θ be a lacunary sequence. If a sequence $x = (x_k)$ is lacunary statistically convergent with respect to the intuitionistic fuzzy norms (μ, ν), then $S_\theta^{(\mu,\nu)}$-limit is unique.*

Proof. Suppose that $S_\theta^{(\mu,\nu)}$- $\lim x = L_1$, $S_\theta^{(\mu,\nu)}$- $\lim x = L_2$ and $L_1 \neq L_2$. Given $\epsilon > 0$ choose $s > 0$ such that $(1-s) * (1-s) > 1-\epsilon$ and $s \diamond s < \epsilon$. Then, for any $t > 0$, define the following sets as:

$$K_{\mu,1}(s,t) = \{k \in \mathbb{N} : \mu(x_k - L_1, t/2) \leq 1 - s\},$$
$$K_{\mu,2}(s,t) = \{k \in \mathbb{N} : \mu(x_k - L_2, t/2) \leq 1 - s\},$$
$$K_{\nu,1}(s,t) = \{k \in \mathbb{N} : \nu(x_k - L_1, t/2) \geq s\},$$
$$K_{\nu,2}(s,t) = \{k \in \mathbb{N} : \nu(x_k - L_2, t/2) \geq s\}.$$

Since $S_\theta^{(\mu,\nu)}$- $\lim x = L_1$, we have by Lemma 3.1

$$\delta_\theta(K_{\mu,1}(\epsilon,t)) = \delta_\theta(K_{\nu,1}(\epsilon,t)) = 0 \quad \text{for all } t > 0.$$

Furthermore, using $S_\theta^{(\mu,\nu)}$- $\lim x = L_2$, we get

$$\delta_\theta(K_{\mu,2}(\epsilon,t)) = \delta_\theta(K_{\nu,2}(\epsilon,t)) = 0 \quad \text{for all } t > 0.$$

Now let $K_{\mu,\nu}(\epsilon,t) = (K_{\mu,1}(\epsilon,t) \cup K_{\mu,2}(\epsilon,t)) \cap (K_{\nu,1}(\epsilon,t) \cup K_{\nu,2}(\epsilon,t))$. Then observe that $\delta_\theta(K_{\mu,\nu}(\epsilon,t)) = 0$ which implies $\delta_\theta(\mathbb{N} \setminus K_{\mu,\nu}(\epsilon,t)) = 1$. If $k \in \mathbb{N} \setminus K_{\mu,\nu}(\epsilon,t)$, then we have two possible cases. (a) $k \in \mathbb{N} \setminus (K_{\mu,1}(\epsilon,t) \cup K_{\mu,2}(\epsilon,t))$, and (b) $k \in \mathbb{N} \setminus (K_{\nu,1}(\epsilon,t) \cup K_{\nu,2}(\epsilon,t))$. We first consider that $k \in \mathbb{N} \setminus (K_{\mu,1}(\epsilon,t) \cup K_{\mu,2}(\epsilon,t))$. Then we have

$$\mu(L_1 - L_2, t) \geq \mu\left(x_k - L_1, \frac{t}{2}\right) * \mu\left(x_k - L_2, \frac{t}{2}\right) > (1-s) * (1-s).$$

Since $(1-s) * (1-s) > 1-\epsilon$, it follows that

$$\mu(L_1 - L_2, t) > 1 - \epsilon. \tag{2}$$

Since $\epsilon > 0$ was arbitrary, we get $\mu(L_1 - L_2, t) = 1$ for all $t > 0$, which yields $L_1 = L_2$. On the other hand, if $k \in \mathbb{N} \setminus (K_{\nu,1}(\epsilon,t) \cup K_{\nu,2}(\epsilon,t))$, then we may write

$$\nu(L_1 - L_2, t) \leq \nu\left(x_k - L_1, \frac{t}{2}\right) \diamond \nu\left(x_k - L_2, \frac{t}{2}\right) < s \diamond s.$$

Now using the fact that $s \diamond s < \epsilon$, we see that

$$\nu(L_1 - L_2, t) < \epsilon.$$

So we have $\nu(L_1 - L_2, t) = 0$ for all $t > 0$, which implies $L_1 = L_2$. Therefore, in all cases, we conclude that $S_\theta^{(\mu,\nu)}$-limit is unique.

This completes the proof of the theorem. □

Theorem 3.3. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS and θ be any lacunary sequence. If (μ, ν)- $\lim x = L$ then $S_\theta^{(\mu,\nu)}$- $\lim x = L$. But converse need not be true.*

Proof. Let (μ, ν)- $\lim x = L$. Then for every $\epsilon > 0$ and $t > 0$, there is a number $k_0 \in \mathbb{N}$ such that

$$\mu(x_k - L, t) > 1 - \epsilon \quad \text{and} \quad \nu(x_k - L, t) < \epsilon$$

for all $k \geq k_0$. Hence the set

$$\{k \in \mathbb{N} : \mu(x_k - L, t) \leq 1 - \epsilon \text{ or } \nu(x_k - L, t) \geq \epsilon\}$$

has finite number of terms. Since every finite subset of $\mathbb{N}$ has density zero and hence

$$\delta_\theta(\{k \in \mathbb{N} : \mu(x_k - L, t) \leq 1 - \epsilon \text{ or } \nu(x_k - L, t) \geq \epsilon\}) = 0,$$

that is, $S_\theta^{(\mu,\nu)}$- $\lim x = L$.

For converse, we construct the following example:

Example 3.1. Let $(\mathbb{R}, |.|)$ denote the space of all real numbers with the usual norm, and let $a * b = ab$ and $ab = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in \mathbb{R}$ and every $t > 0$, consider

$$\mu(x,t) := \frac{t}{t + |x|} \quad \text{and} \quad \nu(x,t) := \frac{|x|}{t + |x|}.$$

Then $(\mathbb{R}, \mu, \nu, *, \diamond)$ is an IFNS. Now we define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} k; & \text{for } k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r, r \in \mathbb{N} \\ 0; & \text{otherwise.} \end{cases}$$

Let for $\epsilon > 0, t > 0$

$$K_r(\epsilon, t) = \{k \in \mathbb{N} : \mu(x_k, t) \le 1 - \epsilon \text{ or } \nu(x_k, t) \ge \epsilon\}.$$

Then

$$\begin{aligned} K_r(\epsilon, t) &= \left\{k \in \mathbb{N} : \frac{t}{t + |x_k|} \le 1 - \epsilon \text{ or } \frac{|x_k|}{t + |x_k|} \ge \epsilon\right\}, \\ &= \left\{k \in \mathbb{N} : |x_k| \ge \frac{\epsilon t}{1 - \epsilon} > 0\right\}, \\ &= \{k \in \mathbb{N} : x_k = k\}, \\ &= \{k \in \mathbb{N} : k_r - [\sqrt{h_r}] + 1 \le k \le k_r, r \in \mathbb{N}\}, \end{aligned}$$

and so, we get

$$\frac{1}{h_r}|K_r(\epsilon, t)| \le \frac{1}{h_r}|\{k \in \mathbb{N} : k_r - [\sqrt{h_r}] + 1 \le k \le k_r, r \in \mathbb{N}\}| \le \frac{\sqrt{h_r}}{h_r},$$

which implies that $\lim_r \frac{1}{h_r}|K_r(t, \epsilon)| = 0$. Hence

$$\delta_\theta(K_r(t, \epsilon)) = \lim \frac{\sqrt{h_r}}{h_r} = 0 \quad \text{as } r \longrightarrow \infty$$

implies that $x_k \stackrel{(\mu,\nu)}{\longrightarrow} 0(S_\theta)$. On the other hand $x_k \stackrel{(\mu,\nu)}{\not\longrightarrow} 0$, since

$$\begin{aligned} \mu(x_k, t) &= \frac{t}{t + |x_k|} = \begin{cases} \frac{t}{t + k}, & \text{for } k_r - [\sqrt{h_r}] + 1 \le k \le k_r (r \in \mathbb{N}); \\ 1, & \text{otherwise}; \end{cases} \\ &\le 1, \end{aligned}$$

and

$$\begin{aligned} \nu(x_k, t) &= \frac{|x_k|}{t + |x_k|} = \begin{cases} \frac{k}{t + k}, & \text{for } k_r - [\sqrt{h_r}] + 1 \le k \le k_r (r \in \mathbb{N}); \\ 0, & \text{otherwise}; \end{cases} \\ &\ge 0. \end{aligned}$$

This completes the proof of the theorem. □

Theorem 3.4. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, for any lacunary sequence θ, $S_\theta^{(\mu,\nu)}$- $\lim x = L$ if and only if there exists a subset $K = \{j_1 < j_2 < \cdots\} \subseteq \mathbb{N}$ such that $\delta_\theta(K) = 1$ and (μ, ν)- $\lim_{n\to\infty} x_{j_n} = L$.*

Proof. *Necessity*. Suppose that $S_\theta^{(\mu,\nu)}$- $\lim x = L$. Let for any $t > 0$ and $s = 1, 2, \ldots$

$$M_{\mu,\nu}(s, t) = \left\{k \in \mathbb{N} : \mu(x_k - L, t) > 1 - \frac{1}{s} \text{ and } \nu(x_k - L, t) < \frac{1}{s}\right\}$$

and

$$K_{\mu,\nu}(s, t) = \left\{k \in \mathbb{N} : \mu(x_k - L, t) \le 1 - \frac{1}{s} \text{ or } \nu(x_k - L, t) \ge \frac{1}{s}\right\}.$$

Then $\delta_\theta(K_{\mu,\nu}(s, t)) = 0$ since $S_\theta^{(\mu,\nu)}$- $\lim x = L$. Also

$$M_{\mu,\nu}(s, t) \supset M_{\mu,\nu}(s + 1, t) \tag{3}$$

and

$$\delta_\theta(M_{\mu,\nu}(s, t)) = 1 \tag{4}$$

for $t > 0$ and $s = 1, 2, \ldots$.

Now we have to show that for $k \in M_{\mu,\nu}(s, t)$, $x_k \stackrel{(\mu,\nu)}{\longrightarrow} L$. Suppose that for some $k \in M_{\mu,\nu}(s, t)$, $x_k \stackrel{(\mu,\nu)}{\not\longrightarrow} L$. Therefore there is $\alpha > 0$ and a positive integer k_0 such that

$$\mu(x_k - L, t) \le 1 - \alpha \quad \text{or} \quad \nu(x_k - L, t) \ge \alpha$$

for all $k \geq k_0$. Let

$$\mu(x_k - L, t) > 1 - \alpha \quad \text{and} \quad \nu(x_k - L, t) < \alpha$$

for all $k < k_0$. Then

$$\delta_\theta(\{k \in \mathbb{N} : \mu(x_k - L, t) > 1 - \alpha \text{ and } \nu(x_k - L, t) < \alpha\}) = 0.$$

Since $\alpha > \frac{1}{s}$, we have

$$\delta_\theta(M_{\mu,\nu}(s, t)) = 0,$$

which contradicts (4). Therefore $x_k \xrightarrow{(\mu,\nu)} L$.

Sufficiency. Suppose that there exists a subset $K = \{j_1 < j_2 < \cdots\} \subseteq \mathbb{N}$ such that $\delta_\theta(K) = 1$ and (μ, ν)- $\lim_{n\to\infty} x_{k_n} = L$, i.e. there exists $N \in \mathbb{N}$ such that for every $\alpha > 0$ and $t > 0$

$$\mu(x_k - L, t) > 1 - \alpha \quad \text{and} \quad \nu(x_k - L, t) < \alpha.$$

Now

$$\begin{aligned} K_{\mu,\nu}(\alpha, t) &:= \{k \in \mathbb{N} : \mu(x_k - L, t) \leq 1 - \alpha \text{ or } \nu(x_k - L, t) \geq \alpha\} \\ &\subseteq \mathbb{N} - \{j_{N+1}, j_{N+2}, \ldots\}. \end{aligned}$$

Therefore $\delta_\theta(K_{\mu,\nu}(\alpha, t)) \leq 1 - 1 = 0$. Hence $S_\theta^{(\mu,\nu)}$- $\lim x = L$.

This completes the proof of the theorem. □

4. Lacunary statistically complete IFNS

In [5], Karakus et al. has defined the concept of statistically Cauchy sequences on intuitionistic fuzzy normed space. In this section we define lacunary statistically Cauchy sequences with respect to an intuitionistic fuzzy normed space and introduce a new concept of statistical completeness.

Definition 4.1. Let $(X, \mu, \nu, *, \diamond)$ be an IFNS and θ be a lacunary sequence. Then, a sequence $x = (x_k)$ is said to be *lacunary statistically Cauchy* (or S_θ-*Cauchy*) with respect to the intuitionistic fuzzy norm (μ, ν) if for every $\epsilon > 0$ and $t > 0$, there exists $N = N(\epsilon)$ such that

$$\delta_\theta(\{k \in \mathbb{N} : \mu(x_k - x_N, t) \leq 1 - \epsilon \text{ or } \nu(x_k - x_N, t) \geq \epsilon\}) = 0.$$

Theorem 4.1. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS and θ be any lacunary sequence. If a sequence $x = (x_k)$ is S_θ-convergent then it is S_θ-Cauchy with respect to the intuitionistic fuzzy norm (μ, ν).*

Proof. Let $x = (x_k)$ be S_θ-convergent to L with respect to the intuitionistic fuzzy norm (μ, ν), i.e., $S_\theta^{(\mu,\nu)}$- $\lim x = L$. For a given $\epsilon > 0$, choose $s > 0$ such that $(1 - \epsilon) * (1 - \epsilon) > 1 - s$ and $\epsilon \diamond \epsilon < s$. Then, for $t > 0$, we have

$$\delta_\theta(A(\epsilon, t)) = \delta_\theta(\{k \in \mathbb{N} : \mu(x_k - L, t/2) \leq 1 - \epsilon \text{ or } \nu(x_k - L, t/2) \geq \epsilon\}) = 0 \tag{5}$$

which implies that

$$\delta_\theta(A^C(\epsilon, t)) = \delta_\theta(\{k \in \mathbb{N} : \mu(x_k - L, t/2) > 1 - \epsilon \quad \text{and} \quad \nu(x_k - L, t/2) < \epsilon\}) = 1.$$

Let $q \in A^C(\epsilon, t)$. Then

$$\mu(x_q - L, t) > 1 - \epsilon \quad \text{and} \quad \nu(x_q - L, t) < \epsilon.$$

Now, let

$$B(\epsilon, t) = \{k \in \mathbb{N} : \mu(x_k - x_q, t) \leq 1 - s \text{ or } \nu(x_k - x_q, t) \geq s\}.$$

We need to show that $B(\epsilon, t) \subset A(\epsilon, t)$. Let $k \in B(\epsilon, t) \setminus A(\epsilon, t)$. Then we have

$$\mu(x_k - x_q, t) \leq 1 - s \quad \text{and} \quad \mu(x_k - L, t/2) > 1 - \epsilon,$$

in particular $\mu(x_q - L, t/2) > 1 - \epsilon$. Then

$$1 - s \geq \mu(x_k - x_q, t) \geq \mu(x_k - L, t/2) * \mu(x_q - L, t/2) > (1 - \epsilon) * (1 - \epsilon) > 1 - s,$$

which is not possible. On the other hand,

$$\nu(x_k - x_q, t) \geq s \quad \text{and} \quad \nu(x_k - L, t/2) < \epsilon,$$

in particular $\nu(x_q - L, t/2) < \epsilon$. Then

$$s \leq \nu(x_k - x_q, t) \leq \nu(x_k - L, t/2) * \nu(x_q - L, t/2) < \epsilon \diamond \epsilon < s,$$

which is not possible. Hence $B(\epsilon, t) \subset A(\epsilon, t)$. Therefore, by (5) $\delta_\theta(B(\epsilon, t)) = 0$. Hence x is S_θ-Cauchy with respect to the intuitionistic fuzzy norm (μ, ν).

This completes the proof of the theorem. □

Definition 4.2 (*[9]*). An intuitionistic fuzzy normed space $(X, \mu, \nu, *, \diamond)$ is said to be *complete* if every Cauchy sequence is convergent in $(X, \mu, \nu, *, \diamond)$.

We define the following:

Definition 4.3. An intuitionistic fuzzy normed space $(X, \mu, \nu, *, \diamond)$ is said to be *statistically* (S_θ-) *complete* if every statistically (S_θ-, respectively) Cauchy sequence with respect to intuitionistic fuzzy norm (μ, ν) is statistically (S_θ-, respectively) convergent with respect to intuitionistic fuzzy norm (μ, ν).

Theorem 4.2. *Let θ be any lacunary sequence. Then every intuitionistic fuzzy normed space $(X, \mu, \nu, *, \diamond)$ is S_θ-complete but not complete in general.*

Proof. Let $x = (x_k)$ be S_θ-Cauchy but not S_θ-convergent with respect to the intuitionistic fuzzy norm (μ, ν). For a given $\epsilon > 0$ and $t > 0$, choose $s > 0$ such that $(1 - \epsilon) * (1 - \epsilon) > 1 - s$ and $\epsilon \diamond \epsilon < s$. Now

$$\mu(x_k - x_N, t) \geq \mu(x_k - L, t/2) * \mu(x_N - L, t/2) > (1 - \epsilon) * (1 - \epsilon) > 1 - s$$

and

$$\nu(x_k - x_N, t) \leq \nu(x_k - L, t/2) \diamond \nu(x_N - L, t/2) < \epsilon \diamond \epsilon < s,$$

since x is not $S_\theta^{(\mu,\nu)}$-convergent. Therefore $\delta_\theta(E^C(\epsilon, t)) = 0$, where

$$E(\epsilon, t) = \{k \in \mathbb{N} : \nu_{x_k - x_N}(\epsilon) \leq 1 - r\}$$

and so $\delta_\theta(E(\epsilon, t)) = 1$, which is a contradiction, since x was S_θ-Cauchy with respect to intuitionistic fuzzy norm (μ, ν). So that x must be $S_\theta^{(\mu,\nu)}$-convergent with respect to intuitionistic fuzzy norm (μ, ν). Hence every IFNS is S_θ-complete.

To see that an IFNS is not complete in general, we have the following example:

Example 4.1 (*[9]*). Let $X = (0, 1]$ and

$$\mu(x, t) := \frac{t}{t + |x|} \quad \text{and} \quad \nu(x, t) := \frac{|x|}{t + |x|}.$$

Then $(X, \mu, \nu, \min, \max)$ is IFNS but not complete, since the sequence $(\frac{1}{n})$ is Cauchy with respect to (μ, ν) but not convergent with respect to the present (μ, ν).

This completes the proof of the theorem. □

From Theorems 3.4, 4.1 and 4.2, we can state the following:

Theorem 4.3. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS and θ be a lacunary sequence. Then, for any sequence $x = (x_{jk})$ in X, the following conditions are equivalent:*

(a) *x is a S_θ-convergent with respect to the intuitionistic fuzzy norm (μ, ν).*
(b) *x is a S_θ-Cauchy with respect to the intuitionistic fuzzy norm (μ, ν).*
(c) *Intuitionistic fuzzy normed space $(X, \mu, \nu, *, \diamond)$ is S_θ-complete.*
(d) *There exists an increasing index sequence $K = (k_n)$ of natural numbers such that $\delta_\theta(K) = 1$ and the subsequence (x_{k_n}) is a S_θ-Cauchy with respect to the intuitionistic fuzzy norm (μ, ν).*

Conclusion

Since every ordinary norm defines an intuitionistic fuzzy norm, our results are more general than the corresponding results in [20]. Furthermore, Definition 4.3 provides a new tool to study the completeness in the sense of statistical convergence.

Acknowledgements

The research of the first author is supported by the Department of Science and Technology, New Delhi, under grant number SR\ S4\ MS:505\07. The research of the second author is supported by the Department of Atomic Energy, Government of India under the NBHM-Post Doctoral Fellowship programme number 40/10/2008-R&D II/892.

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