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Exact conditions for no ruin for the generalised Ornstein–Uhlenbeck process

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The authors wish to dedicate this paper to the memory of Chris Heyde, mentor and friend

Abstract

For a bivariate Lévy process $(\xi_t, \eta_t)_{t\geq 0}$ the generalised Ornstein–Uhlenbeck (GOU) process is defined as

$$V_t := \mathrm{e}^{\xi_t}\left(z + \int_0^t \mathrm{e}^{-\xi_{s-}}\mathrm{d}\eta_s\right), \quad t \geq 0,$$

where $z \in \mathbb{R}$. We define necessary and sufficient conditions under which the infinite horizon ruin probability for the process is zero. These conditions are stated in terms of the canonical characteristics of the Lévy process and reveal the effect of the dependence relationship between ξ and η. We also present technical results which explain the structure of the lower bound of the GOU.

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1. Introduction and notation

For a bivariate Lévy process $(\xi, \eta) = (\xi_t, \eta_t)_{t\geq 0}$ the generalised Ornstein–Uhlenbeck (GOU) process $V = (V_t)_{t\geq 0}$, where $V_0 = z \in \mathbb{R}$, is defined as

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$$V_t := \mathrm{e}^{\xi_t}\left(z + \int_0^t \mathrm{e}^{-\xi_{s-}} \mathrm{d}\eta_s\right). \tag{1}$$

It is closely related to the stochastic integral process $Z = (Z_t)_{t\geq 0}$ defined as

$$Z_t := \int_0^t \mathrm{e}^{-\xi_{s-}} \mathrm{d}\eta_s. \tag{2}$$

The GOU is a time homogeneous strong Markov process. For an overview of its properties see Maller et al. [1] and Carmona et al. [2]. Applications are many, and include ones in option pricing (e.g. Yor [3]), financial time series (e.g. Klüppelberg et al. [4]), insurance, and risk theory (e.g. Paulsen [5], Nyrhinen [6]).

In this paper, we present some basic foundational results on the ruin probability for the GOU, in a very general setup. There are only a few papers dealing with this, or with passage-time problems for the GOU. Patie [7] and Novikov [8] give first-passage-time distributions for the special case where $\xi_t = \lambda t$ for $\lambda \in \mathbb{R}$, and η has no positive jumps. With regard to ruin probability, Nyrhinen [6] and Kalashnikov and Norberg [9] discretize the GOU into a stochastic recurrence equation. Under a variety of conditions, they produce some asymptotic equivalences for the infinite horizon ruin probability. Other work on the GOU ruin probability comes from Paulsen [5]. For the special case where ξ and η are independent, Paulsen gives conditions for certain ruin for the GOU, and a formula for the ruin probability under conditions which ensure that the integral process Z_t converges almost surely as $t \to \infty$.

Since these papers were written, the theory relating to the GOU, and to the process Z, has advanced. For the general case where dependence between ξ and η is allowed, Erickson and Maller [10] present necessary and sufficient conditions for the almost sure convergence of Z_t to a random variable Z_∞ as $t \to \infty$. Bertoin et al. [11] present necessary and sufficient conditions for continuity of the distribution of Z_∞ given that it exists. Lindner and Maller [12] show that strict stationarity of V is equivalent to convergence of an integral $\int_0^t \mathrm{e}^{\xi_{s-}} \mathrm{d}L_s$, where L is an auxiliary Lévy process composed of elements of ξ and η. Note that in [12] the sign of the process ξ is reversed in the definition of the GOU. For our purposes it suits us to have the GOU in the form $V_t := \mathrm{e}^{\xi_t}(z + Z_t)$ and to study the behaviour of V in terms of Z.

Our main results are presented in Section 2. Theorem 1 presents exact necessary and sufficient conditions under which the infinite horizon ruin probability for the GOU is zero. These conditions do not relate to the convergence of Z or stationarity of V or to any moment conditions. Instead they are expressed at a more basic level, directly on the Lévy measure of (ξ, η). Theorem 3 shows that $P(Z_t < 0) > 0$ for all $t > 0$ as long as η is not a subordinator. This result is an important building block in the proof of Theorem 1. Finally in Section 2, Theorem 4 extends a ruin probability formula of Paulsen [5], presenting a slightly different version which deals with the general dependent case, and applies whenever Z_t converges almost surely to a random variable Z_∞ as $t \to \infty$.

Section 3 contains technical results of interest, which characterize what we call the lower bound function of the GOU, and are used to prove the main ruin probability theorem. Section 4 contains proofs of the results stated in Sections 2 and 3.

1.1. Notation

We now set out our theoretical framework and notation. Let $(X_t)_{t\geq 0} := (\xi_t, \eta_t)_{t\geq 0}$ be a bivariate Lévy process with $\xi_0 = \eta_0 = 0$, adapted to a filtered complete probability space

$(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{0 \le t \le \infty}, P)$ satisfying the "usual hypotheses" (see Protter [13] p. 3), where ξ and η are not identically zero. Assume that the σ-algebra $\mathscr{F}$ and the filtration $\mathbb{F}$ are generated by (ξ, η), that is, $\mathscr{F} := \sigma((\xi, \eta)_t : 0 \le t < \infty)$ and $\mathscr{F}_t := \sigma((\xi, \eta)_s : 0 \le s \le t)$. Note that the processes V and Z are defined with respect to $\mathbb{F}$.

The characteristic triplet of (ξ, η) will be written as $\big((\tilde{\gamma}_\xi, \tilde{\gamma}_\eta), \Sigma_{\xi,\eta}, \Pi_{\xi,\eta}\big)$ where $(\tilde{\gamma}_\xi, \tilde{\gamma}_\eta) \in \mathbb{R}^2$, the Gaussian covariance matrix $\Sigma_{\xi,\eta}$ is a non-stochastic 2×2 positive definite matrix, and the Lévy measure $\Pi_{\xi,\eta}$ is a σ-finite measure on $\mathbb{R}^2 \backslash \{0\}$ satisfying the condition $\int_{\mathbb{R}^2} \min\{|z|^2, 1\} \Pi_{\xi,\eta}(\mathrm{d}z) < \infty$, where $|\cdot|$ denotes Euclidean distance. For details on Lévy processes see Bertoin [14] and Sato [15].

The Lévy–Ito decomposition (Sato [15], Ch. 4,) breaks (ξ, η) down into a sum of four mutually independent Lévy processes:

$$(\xi_t, \eta_t) = (\tilde{\gamma}_\xi, \tilde{\gamma}_\eta)t + (B_{\xi,t}, B_{\eta,t}) + \int_{|z|<1} z \left(N_{\xi,\eta,t}(\cdot, \mathrm{d}z) - t\Pi_{\xi,\eta}(\mathrm{d}z)\right) + \int_{|z| \ge 1} z N_{\xi,\eta,t}(\cdot, \mathrm{d}z), \tag{3}$$

where B_ξ and B_η are Brownian motions such that (B_ξ, B_η) has covariance matrix $\Sigma_{\xi,\eta}$, and $N_{\xi,\eta,t}(\omega, \cdot)$ is the random jump measure of (ξ, η) such that $E\left(N_{\xi,\eta,1}(\omega, \Lambda)\right) = \Pi_{\xi,\eta}(\Lambda)$ for Λ a Borel subset of $\mathbb{R}^2 \backslash \{0\}$ whose closure does not contain 0. We can write (see Protter [13], p. 31)

$$(\tilde{\gamma}_\xi, \tilde{\gamma}_\eta) = E\left((\xi_1, \eta_1) - \int_{|z| \ge 1} z N_{\xi,\eta,1}(\cdot, \mathrm{d}z)\right). \tag{4}$$

The characteristic triplets of ξ and η as one-dimensional Lévy processes are denoted as $(\gamma_\xi, \sigma_\xi^2, \Pi_\xi)$ and $(\gamma_\eta, \sigma_\eta^2, \Pi_\eta)$ respectively, where

$$\Pi_\xi(\Gamma) = \Pi_{\xi,\eta}(\Gamma \times \mathbb{R}) \quad \text{and} \quad \Pi_\eta(\Gamma) = \Pi_{\xi,\eta}(\mathbb{R} \times \Gamma) \tag{5}$$

for Γ a Borel subset of $\mathbb{R} \backslash \{0\}$ whose closure does not contain 0,

$$\gamma_\xi = \tilde{\gamma}_\xi + \int_{\{|x|<1\} \cap \{x^2+y^2 \ge 1\}} x \Pi_{\xi,\eta}(\mathrm{d}(x, y)), \tag{6}$$

$$\gamma_\eta = \tilde{\gamma}_\eta + \int_{\{|y|<1\} \cap \{x^2+y^2 \ge 1\}} y \Pi_{\xi,\eta}(\mathrm{d}(x, y)), \tag{7}$$

and σ_ξ^2 and σ_η^2 are the upper left and lower right entries, respectively, in the matrix $\Sigma_{\xi,\eta}$. With the one-dimensional random jump measures of ξ and η denoted by $N_\xi(\omega, \cdot)$ and $N_\eta(\omega, \cdot)$ respectively, we can write the Lévy–Ito decomposition of ξ as

$$\xi_t = \gamma_\xi t + B_{\xi,t} + \int_{|x|<1} x \left(N_{\xi,t}(\cdot, \mathrm{d}x) - t\Pi_\xi(\mathrm{d}x)\right) + \int_{|x| \ge 1} x N_{\xi,t}(\cdot, \mathrm{d}x), \tag{8}$$

where

$$\gamma_\xi = E\left(\xi_1 - \int_{|x| \ge 1} x N_{\xi,1}(\cdot, \mathrm{d}x)\right), \tag{9}$$

and similarly for η. A Lévy process is said to be a subordinator if it takes only non-negative values, which implies that its sample paths are non-decreasing (Bertoin [14], p. 71).

Stochastic integrals are interpreted according to Protter [13]. The integral $\int_a^b$ is interpreted as $\int_{[a,b]}$ and the integral $\int_{a+}^b$ as $\int_{(a,b]}$. The jump of a process Y at t is denoted by $\Delta Y_t := Y_t - Y_{t-}$. The Lévy measure of a Lévy process Y is denoted by Π_Y. If T is a fixed time or a stopping time denote the process Y stopped at T by Y^T and define it by $Y_t^T := Y_{t\wedge T} := Y_{\min\{t,T\}}$. For a function $f(x)$ define $f^+(x) := f(x) \vee 0 := \max\{f(x), 0\}$ and $f^-(x) := \max\{-f(x), 0\}$. The symbol 1_Λ will denote the characteristic function of a set Λ. The symbol $=_D$ will denote equality in distribution of two random variables. The initials "iff" will denote the phrase "if and only if". The symbol "a.s." will denote equality, or convergence, almost surely. Let T_z denote the first time V drops below zero, so

$$T_z := \inf\{t > 0 : V_t < 0 | V_0 = z\}$$

and $T_z := \infty$ whenever $V_t > 0\ \forall t > 0$ and $V_0 = z$. For $z \geq 0$, define the infinite horizon ruin probability function to be

$$\psi(z) := P\left(\inf_{t\geq 0} V_t < 0 \,\middle|\, V_0 = z\right) = P(T_z < \infty).$$

2. Ruin probability results

Our results are given in terms of regions of support of the Lévy measure $\Pi_{\xi,\eta}$. We define some notation, beginning with the following quadrants of the plane. Let $A_1 := \left\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\right\}$, and similarly, let A_2, A_3 and A_4 be the quadrants in which $\{x \geq 0, y \leq 0\}$, $\{x \leq 0, y \leq 0\}$ and $\{x \leq 0, y \geq 0\}$ respectively. For each $i = 1, 2, 3, 4$ and $u \in \mathbb{R}$ define

$$A_i^u := \left\{(x, y) \in A_i : y - u(e^{-x} - 1) < 0\right\}.$$

These sets are defined such that if $(\Delta\xi_t, \Delta\eta_t) \in A_i^u$ and $V_{t-} = u$, then $\Delta V_t < 0$, as we see from the equation

$$\begin{aligned}
\Delta V_t &= V_t - V_{t-} \\
&= \mathrm{e}^{\xi_t}\left(z + \int_0^{t-} \mathrm{e}^{-\xi_{s-}}\mathrm{d}\eta_s + \mathrm{e}^{-\xi_{t-}}\Delta\eta_t\right) - \mathrm{e}^{\xi_{t-}}\left(z + \int_0^{t-} \mathrm{e}^{-\xi_{s-}}\mathrm{d}\eta_s\right) \\
&= (\mathrm{e}^{\xi_t} - \mathrm{e}^{\xi_{t-}})\left(z + \int_0^{t-} \mathrm{e}^{-\xi_{s-}}\mathrm{d}\eta_s\right) + \mathrm{e}^{\xi_t}\mathrm{e}^{-\xi_{t-}}\Delta\eta_t \\
&= (\mathrm{e}^{\Delta\xi_t} - 1)V_{t-} + \mathrm{e}^{\Delta\xi_t}\Delta\eta_t.
\end{aligned} \tag{10}$$

If $u \leq 0$ then $A_2^u = A_2$ and $A_4^u = \emptyset$. As u decreases to $-\infty$, the sets A_1^u expand, whilst the A_3^u shrink. Define

$$\theta_1 := \begin{cases} \sup\left\{u \leq 0 : \Pi_{\xi,\eta}(A_1^u) > 0\right\} \\ -\infty \quad \text{if } \Pi_{\xi,\eta}(A_1) = 0, \end{cases} \qquad \theta_3 := \begin{cases} \inf\left\{u \leq 0 : \Pi_{\xi,\eta}(A_3^u) > 0\right\} \\ 0 \quad \text{if } \Pi_{\xi,\eta}(A_3) = 0. \end{cases}$$

If $u \geq 0$ then $A_3^u = A_3$ and $A_1^u = \emptyset$. As u increases to ∞, the sets A_2^u shrink, whilst the A_4^u expand. Define

$$\theta_2 := \begin{cases} \sup\left\{u \geq 0 : \Pi_{\xi,\eta}(A_2^u) > 0\right\} \\ 0 \quad \text{if } \Pi_{\xi,\eta}(A_2) = 0, \end{cases} \qquad \theta_4 := \begin{cases} \inf\left\{u \geq 0 : \Pi_{\xi,\eta}(A_4^u) > 0\right\} \\ \infty \quad \text{if } \Pi_{\xi,\eta}(A_4) = 0. \end{cases}$$

For each $i = 1, 2, 3, 4$, note that $\Pi_{\xi,\eta}(A_i^{\theta_i}) = 0$, since in the definitions of A_i^u we are requiring that $y - u(\mathrm{e}^{-x} - 1)$ be strictly less than zero.

Theorem 1 (*Exact Conditions for No Ruin for the GOU*). *The ruin probability function satisfies $\psi(0) = 0$ if and only if η is a subordinator. If η is not a subordinator then there exists $c > 0$ such that the ruin probability function satisfies $\psi(c) = 0$ if and only if the Lévy measure satisfies $\Pi_{\xi,\eta}(A_3) = 0$, $\theta_2 \leq \theta_4$, and:*

- *when $\sigma_\xi^2 \neq 0$ the Gaussian covariance matrix is of form $\Sigma_{\xi,\eta} = \begin{bmatrix} 1 & -u \\ -u & u^2 \end{bmatrix} \sigma_\xi^2$ for some $u \in [\theta_2, \theta_4]$ satisfying*

$$g(u) := \tilde{\gamma}_\eta + u\tilde{\gamma}_\xi - \frac{1}{2}u\sigma_\xi^2 - \int_{\{x^2+y^2<1\}} (ux + y)\Pi_{\xi,\eta}(\mathrm{d}(x, y)) \geq 0; \tag{11}$$

- *when $\sigma_\xi^2 = 0$ the Gaussian covariance matrix is of form $\Sigma_{\xi,\eta} = 0$ and there exists $u \in [\theta_2, \theta_4]$ satisfying $g(u) \geq 0$.*

If $\sigma_\xi^2 \neq 0$ and the conditions of the theorem hold, then $\psi(z) = 0$ for all $z \geq c := -\frac{\sigma_{\xi,\eta}}{\sigma_\xi^2}$, whilst $\psi(z) > 0$ for all $z < c$.

If $\sigma_\xi^2 = 0$ and the conditions of the theorem hold, then $\psi(z) = 0$ for all $z \geq c :=$ $\inf\{u \in [\theta_2, \theta_4] : g(u) \geq 0\}$, whilst $\psi(z) > 0$ for all $z < c$.

We now discuss some examples and special cases which illustrate and amplify the results in Theorem 1.

Remark 2. (1) Suppose that (ξ, η) is continuous. We can then write $(\xi_t, \eta_t) = (\gamma_\xi t, \gamma_\eta t) + (B_{\xi,t}, B_{\eta,t})$. Theorem 1 states that $\psi(z) = 0$ for all $z \geq u$ and $\psi(z) > 0$ for all $z < u$, if and only if there exists $u \geq 0$ such that $B_\eta = -uB_\xi$, and $(\gamma_\xi - \frac{1}{2}\sigma_\xi^2)u + \gamma_\eta \geq 0$. For example we could have

$$(\xi_t, \eta_t) := (B_t + ct, -B_t + (1/2 - c)t), \tag{12}$$

where $c \in \mathbb{R}$ and $\sigma_\xi^2 = 1$. Then Theorem 1 implies that $\psi(z) = 0$ for all $z \geq -\frac{\sigma_{\xi,\eta}}{\sigma_\xi^2} = 1$ whilst $\psi(z) > 0$ for all $z < 1$. For this simple case, we can check the result directly. Using Ito's formula we obtain

$$Z_t = -\int_0^t \mathrm{e}^{-(B_s+cs)}\mathrm{d}B_s + (1/2 - c)\int_0^t \mathrm{e}^{-(B_s+cs)}\mathrm{d}s = \mathrm{e}^{-(B_t+ct)} - 1,$$

and hence a lower bound for Z is -1.

(2) Suppose that (ξ, η) is a finite variation Lévy process. Then $\Sigma_{\xi,\eta} = 0$ and $\int_{|z|<1} |z|\Pi_{\xi,\eta}(\mathrm{d}z) < \infty$. We can define the drift vector as

$$(d_\xi, d_\eta) := (\tilde{\gamma}_\xi, \tilde{\gamma}_\eta) - \int_{|z|<1} z\Pi_{\xi,\eta}(\mathrm{d}z) \tag{13}$$

and write

$$(\xi_t, \eta_t) = (d_\xi, d_\eta)t + \int_{\mathbb{R}^2} zN_{\xi,\eta,t}(\cdot, \mathrm{d}z).$$

In this situation, Theorem 1 simplifies to the following statement: $\psi(0) = 0$ iff η is a subordinator. If η is not a subordinator then $\psi(c) = 0$ for some $c > 0$ iff $\Pi_{\xi,\eta}(A_3) = 0$, $\theta_2 \leq \theta_4$, and at least one of the following is true:

- $d_\xi = 0$, and $d_\eta \geq 0$; or
- $d_\xi > 0$ and $-\frac{d_\eta}{d_\xi} \leq \theta_4$; or
- $d_\eta > 0$, and $d_\xi < 0$, such that $-\frac{d_\eta}{d_\xi} \geq \theta_2$.

If the second property holds, then $\psi(z) = 0$ for all $z \geq c := \max\{\theta_2, -\frac{d_\eta}{d_\xi}\}$ and $\psi(z) > 0$ for all $z < c$. If the other properties hold, then $\psi(z) = 0$ for all $z \geq c := \theta_2$ and $\psi(z) > 0$ for all $z < c$.

These results are easily obtained by using (13) to transform condition (11) into the equation $g(u) = d_\eta + u d_\xi \geq 0$. For a simple example, let N_t be a Poisson process with parameter λ, let $c > 0$ and let

$$(\xi_t, \eta_t) := (-ct + N_t, 2ct - N_t). \tag{14}$$

Then we are in the third case above, and $\psi(z) = 0$ for all $z \geq \theta_2 = \frac{\mathrm{e}}{\mathrm{e}-1}$, and $\psi(z) > 0$ for all $z < \frac{\mathrm{e}}{\mathrm{e}-1}$. For this simple case, we can verify the results by direct but tedious calculations which we omit here.

(3) Suppose that ξ and η are independent. This implies that ξ and η jump separately, which means that all jumps occur at the axes of the sets A_i. Further, there is zero covariance between the Brownian components of ξ and η, namely $\sigma_{\xi,\eta} = 0$. With a little work, Theorem 1 simplifies to the following statement: $\psi(0) = 0$ iff η is a subordinator. If η is not a subordinator then $\psi(z) = 0$ for $z > 0$ iff ξ and η are each of finite variation and have no negative jumps, and $g(z) = d_\eta + z d_\xi \geq 0$. Note that for this situation to occur, it must be the case that $d_\eta < 0$ (since η is not a subordinator), which implies that $d_\xi > 0$. Hence $E(\xi_1) > 0$.

(4) Paulsen [5] states conditions for certain ruin when ξ and η are independent. For the cases $E(\xi_1) < 0$ and $E(\xi_1) = 0$, and under certain moment conditions, he shows that $\psi(z) = 1$ for all $z \geq 0$. Theorem 1 shows that the situation changes when dependence is allowed. The continuous process defined in (12), and the jump process defined in (14), illustrate this difference. Each process trivially satisfies Paulsen's moment conditions and can satisfy $E(\xi_1) < 0$, or $E(\xi_1) = 0$, depending on the choices of c and λ; however it is not the case that $\psi(z) = 1$ for all $z \geq 0$. Note that Paulsen does not comment on the possibility of zero ruin for the independent case. The above statement (3) completely explains this situation.

(5) We make some comments on subordinators and explain why Theorem 1 has to have a separate statement for the simple case in which η is a subordinator. By Sato [15], p. 137, η is a subordinator if and only if the following three conditions hold:
- $\sigma_\eta^2 = 0$, so η has no Brownian component;
- $\Pi_\eta((-\infty, 0)) = 0$, so η has no negative jumps;
- $d_\eta \geq 0$, where

$$d_\eta := \gamma_\eta - \int_{(0,1)} y \Pi_\eta(\mathrm{d}y) = E\left(\eta_1 - \int_{(0,\infty)} y N_{\eta,1}(\cdot, \mathrm{d}y)\right).$$

Note that, by definition, $d_\eta \in [-\infty, \infty)$, and $d_\eta = -\infty$ iff $\int_{(0,1)} y \Pi_\eta(\mathrm{d}y) = \infty$.

Suppose that η is a subordinator. Since $\sigma_\eta^2 = 0$ the covariance matrix is of form $\Sigma_{\xi,\eta} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \sigma_\xi^2$. Using the fact that η has no negative jumps, and Eq. (7), we obtain

$$d_\eta = \gamma_\eta - \int_{\mathbb{R}\times\{|y|<1\}} y \Pi_{\xi,\eta}(\mathrm{d}(x,y)) = \tilde{\gamma}_\eta - \int_{\{x^2+y^2<1\}} y \Pi_{\xi,\eta}(\mathrm{d}(x,y))$$
$$= g(0).$$

Thus, the fact that $d_\eta \geq 0$ implies that (11) is satisfied for $u = 0$. Also, since η has no negative jumps, $\theta_2 = 0$, and hence the condition $\theta_2 \leq \theta_4$ is satisfied. However there is one condition that might not be satisfied. Even though η has no negative jumps, we cannot say $\Pi_{\xi,\eta}(A_3) = 0$, since it may be the case that $\Pi_{\xi,\eta}((-\infty, 0) \times \{0\}) > 0$. Namely, ξ may make a negative jump at the same time as η has no jump.

(6) If $\Pi_{\xi,\eta}(A_3) = 0$, and $\theta_2 \leq \theta_4$, then the function $g(u)$ from (11) exists for any $u \in [\theta_2, \theta_4]$, and $g(u) \in [-\infty, \infty)$. Under such conditions, the domain of integration for the integral component of g can be decreased using the fact that

$$\Pi_{\xi,\eta}\left(\{y - u(\mathrm{e}^{-x} - 1) \leq 0\}\right) = 0. \tag{15}$$

Further, if $g(u)$ is finite for some $u \in [\theta_2, \theta_4]$, then

$$\int_{\{y-u(\mathrm{e}^{-x}-1)\in(0,1)\}} \left(y - u(\mathrm{e}^{-x} - 1)\right) \Pi_{\xi,\eta}(\mathrm{d}(x, y)) < \infty. \tag{16}$$

On first viewing, (16) may seem counterintuitive, as it places a constraint on the size of the positive jumps of V. However, if (16) does not hold, and all the other conditions, excluding (11), are satisfied, then the Lévy properties of (ξ, η) imply that V_t can drift negatively when $V_{t-} = u$. These statements, and the Eqs. (15) and (16), are discussed further in Remark 10 following Theorem 9.

Theorem 3. *The Lévy process η is not a subordinator if and only if $P(Z_T < 0) > 0$ for any fixed time $T > 0$.*

One direction of this result is trivial and has been noted above, namely, if η is a subordinator then $P(Z_T < 0) = 0$ for any $T > 0$. The other direction seems quite intuitive and in fact is implicitly assumed by Paulsen [5] for the case when ξ and η are independent. However even for the independent case the proof is non-trivial. We prove it for the general case using a change of measure argument and some analytic lemmas. As well as being of independent interest, this result is essential in proving Theorem 1.

The final theorem in this section provides a formula for the ruin probability for the case where Z converges. Recall that T_z denotes the first time V drops below zero when $V_0 = z$, or equivalently, the first time Z drops below $-z$.

Theorem 4. *Suppose Z_t converges a.s. to a finite random variable Z_∞ as $t \to \infty$, and let $G(z) := P(Z_\infty \leq z)$. Then*

$$\psi(z) = \frac{G(-z)}{E\left(G(-V_{T_z})\middle| T_z < \infty\right)}.$$

Remark 5. (1) For the case where ξ and η are independent, Paulsen [5] shows that, under a number of side conditions which ensure that Z_t converges a.s. to a finite random variable Z_∞ with distribution function $H(z) := P(Z_\infty < z)$ as $t \to \infty$,

$$\psi(z) = \frac{H(-z)}{E\left(H(-V_{T_z})\middle| T_z < \infty\right)}.$$

This formula is a modification of a result given by Harrison [16] for the special case in which ξ is deterministic drift and η is a Lévy process with finite variance. Theorem 4 extends the formula to the general dependent case. Our proof is similar to those of Paulsen and Harrison; however we write it out in full because some details are different.

(2) Erickson and Maller [10] prove that Z_t converges a.s. to a finite random variable Z_∞ as $t \to \infty$ if and only if

$$\lim_{t\to\infty} \xi_t = +\infty \text{ a.s.} \quad \text{and} \quad \int_{\mathbb{R}\setminus[-e,e]} \left(\frac{\ln|y|}{A_\xi(\ln|y|)}\right) \Pi_\eta(\mathrm{d}y) < \infty,$$

where, for $x \geq 1$,

$$A_\xi(x) := 1 + \int_1^x \Pi_\xi((z,\infty))\mathrm{d}z.$$

Lindner and Maller [12] prove that if V is not a constant process, then V is strictly stationary if and only if $\int_0^\infty \mathrm{e}^{\xi_{s-}}\mathrm{d}L_s$ converges a.s. to a finite random variable as $t \to \infty$, where L is the Lévy process

$$L_t := \eta_t + \sum_{0<s\leq t} \left(\mathrm{e}^{\Delta\xi_s} - 1\right)\Delta\eta_s + t\mathrm{Cov}(B_{\xi,1}, B_{\eta,1}), \quad t \geq 0.$$

In neither of these cases do the conditions of Theorem 1 simplify. Each of the processes defined in (12) and (14) can belong to either of these cases, or neither, depending on the choice of constant c and parameter λ.

(3) Bertoin et al. [11] prove that if Z_t converges a.s. to a finite random variable Z_∞ as $t \to \infty$, then Z_∞ has an atom iff Z_∞ is a constant value k iff $P\left(Z_t = k(1-\mathrm{e}^{-\xi_t})\ \forall t > 0\right) = 1$ iff $\mathrm{e}^{-\xi} = \epsilon(-\eta/k)$, where $\epsilon(\cdot)$ denotes the stochastic exponential. In this case it is trivial that $\psi(z) = 0$ for all $z \geq -k$. Theorem 1 produces the same result; however this will not become immediately clear until Remark 8(2) following Theorem 7.

3. Technical results of interest

This section contains technical results needed in the proofs of Theorems 1 and 3, which also have some independent interest. The proofs of these results are given in Section 4. Recall that the stochastic, or Doléans–Dade, exponential of a semimartingale X_t is denoted by $\epsilon(X)_t$. The first proposition introduces a process W which will play an important role throughout the rest of the paper. This proposition is adapted from Proposition 8.22 of [17] and is presented without proof.

Proposition 6. *Given a bivariate Lévy process (ξ,η) there exists a Lévy process W such that $\mathrm{e}^{-\xi_t} = \epsilon(W)_t$ and (ξ,η,W) is a trivariate Lévy process. If ξ has characteristic triplet $(\gamma_\xi, \sigma_\xi, \Pi_\xi)$ then*

$$W_t = -\xi_t + \frac{\sigma_\xi^2 t}{2} + \sum_{0<s\leq t}\left(\mathrm{e}^{-\Delta\xi_s} + \Delta\xi_s - 1\right) \tag{17}$$

and the characteristic triplet of W is given by $\sigma_W^2 = \sigma_\xi^2$ and

$$\Pi_W(\Lambda) = \Pi_\xi\left(\{x : \mathrm{e}^{-x} - 1 \in \Lambda\}\right) \tag{18}$$

and

$$\gamma_W = -\gamma_\xi \frac{1}{2}\sigma_\xi^2 + \int_{\mathbb{R}} \left(x 1_{(-1,1)}(x) + (\mathrm{e}^{-x}-1)1_{(-\ln 2,\infty)}(x)\right) \Pi_\xi(\mathrm{d}x), \tag{19}$$

where the integral converges.

We define the lower bound function δ for V in (1) as

$$\delta(z) = \inf\left\{u \in \mathbb{R} : P\left(\inf_{t\geq 0} V_t \leq u \middle| V_0 = z\right) > 0\right\}.$$

The following theorem exactly characterizes the lower bound function.

Theorem 7. *The lower bound function satisfies the following properties:*

(1) *For all* $z \in \mathbb{R}$, $\delta(z) \leq z$.
(2) *If* $z_1 < z_2$ *then* $\delta(z_1) \leq \delta(z_2)$.
(3) *For all* $z \in \mathbb{R}$, $\delta(z) = z$ *if and only if* $\eta - zW$ *is a subordinator.*
(4) *For all* $z \in \mathbb{R}$, $\delta(z) = \delta(\delta(z))$, *and*

$$\delta(z) = \sup\left\{u : u \leq z, \eta - uW \text{ is a subordinator}\right\}.$$

Remark 8. (1) If η is a subordinator then $\delta(0) = 0$, so V cannot drop below zero when $V_0 = z \geq 0$.
(2) As noted in Remark 5(3), if Z_t converges a.s. to a finite random variable Z_∞ as $t \to \infty$, then Z_∞ has an atom iff $\mathrm{e}^{-\xi} = \epsilon(-\eta/k)$. If this holds then $\delta(-k) = -k$, since $\eta + k(-\eta/k) = 0$ and hence is a subordinator. Thus $\psi(z) = 0$ for all $z \geq -k$, as mentioned in Remark 5(3).

Theorem 9. *Let* $u \in \mathbb{R}\backslash\{0\}$ *and let* (ξ, η, W) *be the trivariate Lévy process from Proposition 6. The Lévy process* $\eta - uW$ *is a subordinator if and only if the following three conditions are satisfied: the Gaussian covariance matrix is of the form*

$$\Sigma_{\xi,\eta} = \begin{bmatrix} 1 & -u \\ -u & u^2 \end{bmatrix} \sigma_\xi^2, \tag{20}$$

at least one of the following is true:

- $\Pi_{\xi,\eta}(A_3) = 0$ *and* $\theta_2 \leq \theta_4$ *and* $u \in [\theta_2, \theta_4]$;
- $\Pi_{\xi,\eta}(A_2) = 0$ *and* $\theta_1 \leq \theta_3$ *and* $u \in [\theta_1, \theta_3]$;
- $\Pi_{\xi,\eta}(A_3) = \Pi_{\xi,\eta}(A_2) = 0$ *and* $u \in [\theta_1, \theta_4]$;

and in addition, u satisfies (11).

Remark 10. In Remark 2(5) we stated three necessary and sufficient conditions for a Lévy process to be a subordinator. These three conditions correspond respectively with the three conditions in Theorem 9, as we shall see in the proof. In particular, if one of the dot point conditions holds, and $u \in [\theta_i, \theta_j]$ for its respective i, j, then $\Pi_{\eta-uW}((-\infty, 0]) = 0$, which we will show to be equivalent to (15), and the function g from (11) satisfies $g(u) = d_{\eta-uW} \in [-\infty, \infty)$. Further, if $g(u)$ is finite for some $u \in [\theta_i, \theta_j]$ then $\int_{(0,1)} z\Pi_{\eta-uW}(\mathrm{d}z) < \infty$, which we will show to be equivalent to (16). Note that if $\eta - uW$ has no Brownian component, no negative jumps, but $\int_{(0,1)} z\Pi_{\eta-uW}(\mathrm{d}z) = \infty$, then, somewhat surprisingly, $\eta - uW$ is fluctuating and hence not a subordinator, regardless of the value of the shift constant $\gamma_{\eta-uW}$. This behaviour occurs since $d_{\eta-uW} = -\infty$, and is explained in Sato [15], p. 138.

4. Proofs

We begin by proving Theorem 3. For this proof, some lemmas are required. In these we assume that $X = (\xi, \eta)$ has bounded jumps so that X has finite absolute moments of all orders. Then, to prove Theorem 3 we reduce to this case.

Lemma 11. *Suppose $X = (\xi, \eta)$ has bounded jumps and $E(\eta_1) = 0$. If we let $T > 0$ be a fixed time then Z^T is a mean-zero martingale with respect to $\mathbb{F}$.*

Proof. Since η is a Lévy process the assumption $E(\eta_1) = 0$ implies that η is a càdlàg martingale. Since ξ is càdlàg , $\mathrm{e}^{-\xi}$ is a locally bounded process and hence Z is a local martingale for $\mathbb{F}$ by Protter [13], p. 171. If we show that $E\left(\sup_{s\le t} |Z_s^T|\right) < \infty$ for every $t \ge 0$ then Protter [13], p. 38, implies that Z^T is a martingale. This is equivalent to showing $E\left(\sup_{t\le T} |Z_t|\right) < \infty$. Since Z is a local martingale and $Z_0 = 0$, the Burkholder–Davis–Gundy inequalities in Lipster and Shiryaev [18], p. 70 and p. 75, ensure the existence of $b > 0$ such that

$$\begin{aligned}
E\left(\sup_{0\le t\le T}\left|\int_0^t \mathrm{e}^{-\xi_{s-}}\mathrm{d}\eta_s\right|\right) &\le bE\left(\left[\int_0^\bullet \mathrm{e}^{-\xi_{s-}}\mathrm{d}\eta_s, \int_0^\bullet \mathrm{e}^{-\xi_{s-}}\mathrm{d}\eta_s\right]_T^{1/2}\right)\\
&= bE\left(\left(\int_0^T \mathrm{e}^{-2\xi_{s-}}\mathrm{d}[\eta,\eta]_s\right)^{1/2}\right)\\
&\le bE\left(\left(\int_0^T \sup_{0\le t\le T} \mathrm{e}^{-2\xi_t}\mathrm{d}[\eta,\eta]_s\right)^{1/2}\right)\\
&= bE\left(\sup_{0\le t\le T} \mathrm{e}^{-\xi_t}[\eta,\eta]_T^{1/2}\right)\\
&\le b\left(E\left(\sup_{0\le t\le T} \mathrm{e}^{-2\xi_t}\right)\right)^{1/2} \left(E\left([\eta,\eta]_T\right)\right)^{1/2},
\end{aligned}$$

where the second inequality follows from the fact that $[\eta,\eta]_s$ is increasing and the final inequality follows by the Cauchy–Schwarz inequality. (The notation $[\cdot,\cdot]$ denotes the quadratic variation process.) Now, by Protter [13], p. 70,

$$E\left([\eta,\eta]_T\right) = \sigma_\eta^2 T + E\left(\sum_{0\le s\le T} (\Delta\eta)^2\right) = \sigma_\eta^2 T + T\int x^2 \Pi_\eta(\mathrm{d}x),$$

which is finite since η has bounded jumps. Thus it suffices to prove that $E\left(\sup_{0\le t\le T} \mathrm{e}^{-2\xi_t}\right) < \infty$. Setting $Y_t = \mathrm{e}^{-\xi_t}/E(\mathrm{e}^{-\xi_t})$, a non-negative martingale, it follows by Doob's maximal inequality, as expressed in Shiryaev [19], p. 765, that

$$E\left(\sup_{0\le t\le T} \frac{\mathrm{e}^{-2\xi_t}}{\left(E(\mathrm{e}^{-\xi_t})\right)^2}\right) \le 4\frac{E\left(\mathrm{e}^{-2\xi_T}\right)}{\left(E(\mathrm{e}^{-\xi_T})\right)^2},$$

which is finite since ξ has bounded jumps and hence has finite exponential moments of all orders (Sato [15], p. 161). It is shown in Sato [15], p. 165, that $\left(E(\mathrm{e}^{-\xi_t})\right)^2 = \left(E(\mathrm{e}^{-\xi_1})\right)^{2t}$. Letting $c := \left(E(\mathrm{e}^{-\xi_1})\right)^2 \in (0,\infty)$, the above inequality implies that

$$E\left(\sup_{0\le t\le T} \mathrm{e}^{-2\xi_t}\right) \le \max\{1, c^T\} E\left(\sup_{0\le t\le T} \frac{\mathrm{e}^{-2\xi_t}}{c^t}\right) < \infty. \quad \square$$

We now present two lemmas dealing with absolute continuity of measures. These lemmas will be used to construct a new process W such that W^T is a mean-zero martingale which is mutually absolutely continuous with Z^T. Then $P(Z_T < 0) > 0$ if and only if $P(W_T < 0) > 0$, and the latter statement will follow immediately from the fact that W^T is a mean-zero martingale.

Lemma 12. *Let $X := (\xi, \eta)$ and $Y := (\tau, \nu)$ be bivariate Lévy processes adapted to $(\Omega, \mathscr{F}, \mathbb{F}, P)$, and let $Z_t := \int_0^t \mathrm{e}^{-\xi_{s-}} \mathrm{d}\eta_s$ and $W_t := \int_0^t \mathrm{e}^{-\tau_{s-}} \mathrm{d}\nu_s$. If the induced probability measures of X^T and Y^T are mutually absolutely continuous, then the induced probability measures of Z^T and W^T are mutually absolutely continuous.*

Proof. Let $D([0,T] \to \mathbb{R}^2)$ denote the set of càdlàg functions from $[0,T]$ to $\mathbb{R}^2$ and $\mathscr{B}^{2[0,T]}$ denote the σ-algebra generated in this set by the Borel cylinder sets (see Kallenberg [20]). Then the induced probability measures of X^T and Y^T can be written as P_{X^T} and P_{Y^T} on the measure space $\left(D([0,T] \to \mathbb{R}^2), \mathscr{B}^{2[0,T]}\right)$. Let $C := (C', C'')$ be the coordinate mapping of $\left(D([0,T] \to \mathbb{R}^2), \mathscr{B}^{2[0,T]}\right)$ to itself. Define Z' on the probability space $\left(D([0,T] \to \mathbb{R}^2), \mathscr{B}^{2[0,T]}, P_{X^T}\right)$ by $Z'_t := \int_0^t \mathrm{e}^{-C'_{s-}} \mathrm{d}C''_s$. Define W' on $\left(D([0,T] \to \mathbb{R}^2), \mathscr{B}^{2[0,T]}, P_{Y^T}\right)$ by $W'_t := \int_0^t \mathrm{e}^{-C'_{s-}} \mathrm{d}C''_s$. Note that Z' and W' are different processes since they are being evaluated under different measures. Now $Z = X \circ Z'$ and $W = Y \circ W'$. Hence $P(Z^T \in \Lambda) = P_{X^T}(Z' \in \Lambda)$ and $P(W^T \in \Lambda) = P_{Y^T}(W' \in \Lambda)$. Since P_{X^T} and P_{Y^T} are mutually absolutely continuous, Protter [13], p. 60, implies that Z' and W' are P_{X^T}-indistinguishable, and P_{Y^T}-indistinguishable. So $P_{X^T}(Z' \in \Lambda) = P_{X^T}(W' \in \Lambda)$. Since P_{X^T} and P_{Y^T} are mutually absolutely continuous, $P_{X^T}(W' \in \Lambda) = 0$ iff $P_{Y^T}(W' \in \Lambda) = 0$ which proves $P(Z^T \in \Lambda) = 0$ iff $P(W^T \in \Lambda) = 0$, as required. $\square$

Lemma 13. *If $X := (\xi, \eta)$ has bounded jumps, $E(\eta_1) \ge 0$, η is not a subordinator, and η is not pure deterministic drift, then there exists a bivariate Lévy process $Y := (\tau, \nu)$ with bounded jumps, adapted to $(\Omega, \mathscr{F}, \mathbb{F}, P)$, such that X^T and Y^T are mutually absolutely continuous for all $T > 0$, and $E(\nu_1) = 0$.*

Proof. As mentioned in Remark 2(5), the Lévy process η is a subordinator if and only if the following three conditions hold: $\sigma_\eta^2 = 0$, $\Pi_\eta((-\infty, 0)) = 0$, and $d_\eta \ge 0$ where $d_\eta := \gamma_\eta - \int_{(0,1)} y \Pi_\eta(\mathrm{d}y)$. Thus it suffices to prove the lemma for the following three cases.

Case 1: Suppose $\sigma_\eta \ne 0$. Given dependent Brownian motions B_ξ and B_η there exists a Brownian motion B' independent of B_η, and constants a_1 and a_2 such that $(B_\xi, B_\eta) = (a_1 B' + a_2 B_\eta, B_\eta)$. Using the Lévy–Ito decomposition, X can be written as the sum of two independent processes as follows:

$$X_t = (\xi_t, \eta_t) = (\xi'_t + B_{\xi,t}, \eta'_t + B_{\eta,t}) =_D (\xi'_t + a_1 B'_t, \eta'_t) + (a_2 B_{\eta,t}, B_{\eta,t}),$$

where (ξ', η') is a pure jump Lévy process with drift, independent of (B_ξ, B_η). Let $c := E(\eta_1)$ and define the Lévy process Y by

$$Y_t := (\xi'_t + a_1 B'_t, \eta'_t) + \left(a_2(B_{\eta,t} - ct), B_{\eta,t} - ct\right).$$

It is a simple consequence of Girsanov's theorem for Brownian motion, e.g. Klebaner [21], p. 241, that the induced measures of the processes $B_{\eta,t}$ and $B_{\eta,t} - ct$ on $\left(D([0,T] \to \mathbb{R}), \mathscr{B}^{[0,T]}\right)$

are mutually absolutely continuous. It is trivial to show that this implies that the induced probability measures of $(a_2 B_{\eta,t}, B_{\eta,t})^{\mathrm{T}}$ and $(a_2(B_{\eta,t} - ct), B_{\eta,t} - ct)^{\mathrm{T}}$ are mutually absolutely continuous. Using independence, this implies that the induced probability measures of X^T and Y^T are mutually absolutely continuous. Note that if we write Y as $Y = (\tau, \nu)$ then $\nu_t = \eta_t - ct$ so $E(\nu_1) = 0$ as required.

Case 2: Suppose $\sigma_\eta = 0$ and $\Pi_\eta((-\infty, 0)) > 0$. We can assume that X has jumps contained in Λ, a square in $\mathbb{R}^2$, i.e. for all $t > 0$

$$(\Delta\xi_t, \Delta\eta_t) \in \Lambda := \{(x, y) \in \mathbb{R}^2 : -a \le x \le a, -a \le y \le a\}.$$

For any $0 < b < a$ define the set $\Gamma \subset \Lambda$ by

$$\Gamma := \{(x, y) \in \mathbb{R}^2 : -a \le x \le a, -a \le y \le -b\}.$$

A Lévy measure is σ-finite and $\Pi_\eta((-\infty, 0)) > 0$ so there must exist a $b > 0$ small enough such that $\Pi_X(\Gamma) > 0$. By Protter [13], p. 27, we can write $X = \tilde{X} + \hat{X}$ where $\tilde{X}_t := (\tilde{\xi}_t, \tilde{\eta}_t)$ is a Lévy process with jumps contained in $\Lambda \backslash \Gamma$ and $\hat{X}_t := (\hat{\xi}_t, \hat{\eta}_t)$ is a compound Poisson process independent of $\tilde{X}$, with jumps in Γ and parameter $\lambda := \Pi_X(\Gamma) < \infty$. So we can write $\hat{X}_t = \sum_{i=1}^{N_t} C_i$ where N is a Poisson process with parameter λ and $(C_i)_{i\ge1} := (C_i', C_i'')_{i\ge1}$ is an independent identically distributed sequence of two-dimensional random vectors, independent of N, with $C_i \in \Gamma$. Let M be a Poisson process independent of N, C_i and $\tilde{X}$, with parameter $r\lambda$ for some $r \ge 1$. Define the Lévy process Y by $Y_t := \tilde{X}_t + \sum_{i=1}^{M_t} C_i$. We show that the induced probability measures of X^T and Y^T on $\left(D([0, T] \to \mathbb{R}), \mathscr{B}^{[0,T]}\right)$ are mutually absolutely continuous. Since $\tilde{X}$ is independent of both compound Poisson processes, this is equivalent to showing that the induced probability measures of $\sum_{i=1}^{N_t} C_i$ and $\sum_{i=1}^{M_t} C_i$ are mutually absolutely continuous. Let $A \in \mathscr{B}^{[0,T]}$ and note that

$$P\left(\left(\sum_{i=1}^{N_t} C_i\right)_{0\le t\le T} \in A\right) = \sum_{n=0}^{\infty} P\left(\left(\sum_{i=1}^{N_t} C_i\right)_{0\le t\le T} \in A \,\middle|\, N_T = n\right) P\left(N_T = n\right). \quad (21)$$

Since N is a Poisson process, $P(N_t = n) > 0$ for all $n \in \mathbb{N}$. Thus the left hand side of (21) is zero if and only if $P\left(\left(\sum_{i=1}^{N_t} C_i\right)_{0\le t\le T} \in A | N_T = n\right) = 0$ for all $n \in \mathbb{N}$.

For any Poisson processes, regardless of the parameter, Kallenberg [20], p. 179, shows that once we condition on the event that n jumps have occurred in time $(0, T]$, then the jump times are uniformly distributed over $(0, T]$. This implies that

$$P\left(\left(\sum_{i=1}^{N_t} C_i\right)_{0\le t\le T} \in A \,\middle|\, N_T = n\right) = P\left(\left(\sum_{i=1}^{M_t} C_i\right)_{0\le t\le T} \in A \,\middle|\, M_T = n\right).$$

Thus $P\left(\left(\sum_{i=1}^{N_t} C_i\right)_{0\le t\le T} \in A\right) = 0$ if and only if $P\left(\left(\sum_{i=1}^{M_t} C_i\right)_{0\le t\le T} \in A\right) = 0$, which proves that the two measures are mutually absolutely continuous, as required.

Recall that $Y_t =: (\tau_t, \nu_t) = \tilde{X}_t + \sum_{i=1}^{M_t} C_i$ where $\tilde{X} := (\tilde{\xi}, \tilde{\eta})$ and $C_i := (C_i', C_i'') \in \Gamma$. Thus $\nu_t = \tilde{\eta}_t + \sum_{i=1}^{M_t} C_i''$ which implies that $tE(\nu_1) = tE(\tilde{\eta}_1) + r\lambda t E(C_i'')$ where $E(\tilde{\eta}_1) > E(\eta_1) \ge 0$. Choosing $r = E(\tilde{\eta}_1)/|\lambda E(C_i'')|$ gives $E(\nu_1) = 0$ as required.

Case 3: Suppose $\sigma_\eta = 0$, $\Pi_\eta((-\infty, 0)) = 0$, and $d_\eta < 0$, where we allow the possibility that $d_\eta = -\infty$. If $\Pi_\eta((0, \infty)) = 0$ then $\eta_t = d_\eta t$ is deterministic, and this possibility has been excluded. So $\Pi_\eta((0,\infty)) > 0$, and we can assume that X has jumps contained in Λ where we define the set $\Lambda := \{(x,y) \in \mathbb{R}^2 : -a \le x \le a, 0 < y \le a\}$. For any $0 < b < a$ define the set $\Gamma^{(b)} \subset \Lambda$ by $\Gamma^{(b)} := \{(x,y) \in \mathbb{R}^2 : -a \le x \le a, b \le y \le a\}$. We can write $X = \tilde{X}^{(b)} + \hat{X}^{(b)}$ where $\tilde{X}^{(b)} := (\tilde{\xi}_t^{(b)}, \tilde{\eta}_t^{(b)})$ is a Lévy process with jumps contained in $\Lambda \backslash \Gamma^{(b)}$ and $\hat{X}^{(b)} := (\hat{\xi}_t^{(b)}, \hat{\eta}_t^{(b)})$ is a compound Poisson process independent of $\tilde{X}^{(b)}$, with jumps in $\Gamma^{(b)}$ and parameter $\lambda^{(b)} := \Pi_X(\Gamma^{(b)}) < \infty$.

If $d_\eta \in (-\infty, 0)$ then we can write $E\left(\tilde{\eta}_t^{(b)}\right) = d_\eta t + t\int_{(0,b)} x\Pi_\eta(\mathrm{d}x)$. Since $\lim_{b\downarrow 0}\int_{(0,b)} x\Pi_\eta(\mathrm{d}x) = 0$, there exists $b > 0$ such that $E\left(\tilde{\eta}_t^{(b)}\right) < 0$. If $d_\eta = -\infty$ then $\int_{(0,1)} x\Pi_\eta(\mathrm{d}x) = \infty$. Note that $E(\eta_1) = E\left(\tilde{\eta}_1^{(b)}\right) + E\left(\hat{\eta}_1^{(b)}\right) \in (0,\infty)$ since jumps are bounded, whilst

$$\lim_{b\downarrow 0} E\left(\hat{\eta}_t^{(b)}\right) = \lim_{b\downarrow 0}\int_{(b,a)} x\Pi_\eta(\mathrm{d}x) = \infty.$$

Hence there again exists $b > 0$ such that $E\left(\tilde{\eta}_t^{(b)}\right) < 0$. From now on we assume that $b > 0$ is small enough that $E\left(\tilde{\eta}_t^{(b)}\right) < 0$. Since a Lévy measure is σ-finite and $\Pi_\eta((0,\infty)) > 0$ we can also assume $\Pi_X(\Gamma^{(b)}) > 0$. Thus we drop the $^{(b)}$ from our labelling. We can write $\hat{X}_t = \sum_{i=1}^{N_t} C_i$ where N is a Poisson process with parameter λ and $(C_i)_{i\ge 1} := (C_i', C_i'')_{i\ge 1}$ is an independent identically distributed sequence of two-dimensional random vectors, independent of N, with $C_i \in \Gamma$. Let M be a Poisson process independent of N, C_i and $\tilde{X}$, with parameter $r\lambda$ for some $r > 0$. Define the Lévy process Y by $Y_t := \tilde{X}_t + \sum_{i=1}^{M_t} C_i$. Then the induced probability measures of X^T and Y^T are mutually absolutely continuous by the same proof as was used for Case 2. If $Y =: (\tau, \nu)$ then $\nu_t = \tilde{\eta}_t + \sum_{i=1}^{M_t} C_i''$ with $C_i'' \in [b,a]$. Since $E(\tilde{\eta}_1) < 0$ for our choice of $0 < b < a$, choosing $r = |E(\tilde{\eta}_1)|/\lambda E(C_i'')$ gives the result. □

Proof (Theorem 3). Take a general (ξ,η), let $a > 0$ and define

$$\Lambda := \{(x,y) \in \mathbb{R}^2 : -a \le x \le a, -a \le y \le a\}.$$

We can write $X = \tilde{X} + \hat{X}$ where $\tilde{X}_t := (\tilde{\xi}_t, \tilde{\eta}_t)$ is a Lévy process with jumps contained in Λ and $\hat{X}_t := (\hat{\xi}_t, \hat{\eta}_t)$ is a compound Poisson process, independent of $\tilde{X}$, with jumps in $\mathbb{R}^2\backslash\Lambda$, and parameter $\lambda := \Pi_X(\mathbb{R}^2\backslash\Lambda) < \infty$. Note that

$$\hat{X}_t := \sum_{0\le s\le t} \Delta X_s 1_{\mathbb{R}^2\backslash\Lambda}(\Delta X_s)$$

and by Poisson properties, $P(\hat{X}_t = 0) > 0$ for any $t \ge 0$. Suppose that $P\left(\int_0^T \mathrm{e}^{-\tilde{\xi}_{s-}}\mathrm{d}\tilde{\eta}_s < 0\right) > 0$. Then $P(Z_T < 0) > 0$, because

$$\begin{aligned} P\left(\int_0^T \mathrm{e}^{-\xi_{s-}}\mathrm{d}\eta_s < 0\right) &\ge P\left(\int_0^T \mathrm{e}^{-\xi_{s-}}\mathrm{d}\eta_s < 0 \,\middle|\, \hat{X}_T = 0\right) P\left(\hat{X}_T = 0\right) \\ &= P\left(\int_0^T \mathrm{e}^{-\tilde{\xi}_{s-}}\mathrm{d}\tilde{\eta}_s < 0 \,\middle|\, \hat{X}_T = 0\right) P\left(\hat{X}_T = 0\right) \\ &= P\left(\int_0^T \mathrm{e}^{-\tilde{\xi}_{s-}}\mathrm{d}\tilde{\eta}_s < 0\right) P\left(\hat{X}_T = 0\right) > 0. \end{aligned}$$

Further, note that η is not a subordinator iff we can choose $a > 0$ such that $\tilde{\eta}$ is not a subordinator. If $\sigma_\eta^2 > 0$ or $d_\eta < 0$ then any $a > 0$ suffices. If $\Pi_\eta((-\infty, 0)) > 0$ then we can choose $a > 0$ large enough that $\Pi_\eta((-a, 0)) > 0$. The converse is obvious. Thus the theorem is proved if we can prove it for the case in which the jumps are bounded. From now on assume that the jumps of $X = (\xi, \eta)$ are contained in the set Λ defined above. Note that this implies that $E(\eta_1)$ is finite.

If η is pure deterministic drift, then $\eta_t = d_\eta t$ where $d_\eta < 0$, since η is not a subordinator. In this case the theorem is trivial, since Z is strictly decreasing. Thus, assume that η is not deterministic drift. We first prove the theorem for the case where $-c := E(\eta_1) < 0$. Note that

$$\begin{aligned} P(Z_T < 0) &= P\left(\int_0^T \mathrm{e}^{-\xi_{s-}}\mathrm{d}(\eta_s + cs) - \int_0^T \mathrm{e}^{-\xi_{s-}}\mathrm{d}(cs) < 0\right) \\ &\geq P\left(\int_0^T \mathrm{e}^{-\xi_{s-}}\mathrm{d}(\eta_s + cs) < 0\right) > 0. \end{aligned}$$

The final inequality follows by Lemma 11, which implies that $\int_0^T \mathrm{e}^{-\xi_{s-}}\mathrm{d}(\eta_s + cs)$ is a martingale, so $E\left(\int_0^T \mathrm{e}^{-\xi_{s-}}\mathrm{d}(\eta_s + cs)\right) = 0$. Note that $\int_0^T \mathrm{e}^{-\xi_{s-}}\mathrm{d}(\eta_s + cs)$ is not identically zero due to our assumption that η is not deterministic drift.

Now we assume that $c := E(\eta_1) \geq 0$. Lemma 13 ensures that there exists $Y := (\tau, \nu)$ with bounded jumps, adapted to $(\Omega, \mathscr{F}, \mathbb{F}, P)$, such that X^T and Y^T are mutually absolutely continuous for all $T > 0$, and $E(\nu_1) = 0$. If we let $W_t := \int_0^t \mathrm{e}^{-\tau_{s-}}\mathrm{d}\nu_s$ then Lemma 11 ensures that W^T is a mean-zero martingale. We prove that W_T is not identically zero. Firstly if ν is deterministic drift then W is either strictly increasing, or strictly decreasing; hence W_T is not identically zero. If ν is not deterministic drift then the quadratic variation $[\nu, \nu]$ is an increasing process. Hence

$$\left[\int_0^\bullet \mathrm{e}^{-\tau_{s-}}\mathrm{d}\nu_s, \int_0^\bullet \mathrm{e}^{-\tau_{s-}}\mathrm{d}\nu_s\right]_T = \left(\int_0^T \mathrm{e}^{-2\tau_{s-}}\mathrm{d}[\nu, \nu]_s\right) > 0.$$

If W_T is identically zero then W_t must be identically zero for all $t \leq T$, since W^T is a martingale. Thus $[W, W]_T = 0$, which gives a contradiction. Now, since W is not identically zero, and $E(W_T) = 0$, we conclude that $P(W_T < 0) > 0$. However, Lemma 12 ensures that the induced probability measures of Z^T and W^T are mutually absolutely continuous. Hence $P(Z_T < 0) > 0$. □

Proof (Theorem 7). Property 1 is immediate from the definition while Property 2 follows from the fact that V_t is increasing in z for all $t \geq 0$. Let W be the process such $\mathrm{e}^{-\xi_t} = \epsilon(W)_t$. Then for any $u \in \mathbb{R}$,

$$\begin{aligned} V_t &= \mathrm{e}^{\xi_t}\left(z + \int_0^t \mathrm{e}^{-\xi_{s-}}\mathrm{d}\eta_s\right) \\ &= \mathrm{e}^{\xi_t}\left(z + \int_0^t \mathrm{e}^{-\xi_{s-}}\mathrm{d}(\eta_s - uW_s) + u\int_0^t \mathrm{e}^{-\xi_{s-}}\mathrm{d}W_s\right) \\ &= \mathrm{e}^{\xi_t}\left(z + \int_0^t \mathrm{e}^{-\xi_{s-}}\mathrm{d}(\eta_s - uW_s) + u(\mathrm{e}^{-\xi_t} - 1)\right) \\ &= u + \mathrm{e}^{\xi_t}\left(z - u + \int_0^t \mathrm{e}^{-\xi_{s-}}\mathrm{d}(\eta_s - uW_s)\right). \end{aligned}$$

Now if $\eta - zW$ is a subordinator then $\int_0^t e^{-\xi_{s-}} d(\eta_s - zW_s) \geq 0$ so $\delta(z) = z$. By Theorem 3 if $\eta - zW$ is not a subordinator then for some t and some $\epsilon > 0$,

$$P\left(\int_0^t e^{-\xi_{s-}} d(\eta_s - zW_s) < -\epsilon\right) > 0$$

and so, with $V_0 = z + \epsilon$ and $u = z$,

$$P\left(\inf_{t\geq 0} V_t < z \,\middle|\, V_0 = z + \epsilon\right) = P\left(\inf_{t\geq 0}\left\{z + e^{\xi_t}\left(\epsilon + \int_0^t e^{-\xi_{s-}} d(\eta_s - zW_s)\right)\right\} < z\right)$$

which is strictly positive. This implies that $\delta(z) \leq \delta(z+\epsilon) < z$ and establishes Property 3. Now Property 3 implies Property 4 if $\eta - \delta(z)W$ is a subordinator. So suppose that $\eta - \delta(z)W$ is not a subordinator. Then from the argument above we know that for some $\epsilon > 0$, $\delta(\delta(z) + \epsilon) < \delta(z)$. Let $T_u = \inf\{t > 0 : V_t \leq u\}$. By the definition of δ we have that $P(T_{\delta(u)+\epsilon} < \infty) > 0$. By the strong Markov property of V_t, if $u < z$,

$$\begin{aligned}
&P\left(\inf_{t\geq 0} V_t < \delta(u) \,\middle|\, V_0 = z\right)\\
&\quad = P\left(\inf_{t\geq 0} V_{t+T_{\delta(u)+\epsilon}} < \delta(u) \,\middle|\, V_0 = z\right)\\
&\quad = P\left(\inf_{t\geq 0} V_{t+T_{\delta(u)+\epsilon}} < \delta(u) \,\middle|\, T_{\delta(u)+\epsilon} < \infty, V_0 = z\right) P\left(T_{\delta(u)+\epsilon} < \infty\right)\\
&\quad \geq P\left(\inf_{t\geq 0} V_t < \delta(u) \,\middle|\, V_0 = \delta(u) + \epsilon\right) P\left(T_{\delta(u)+\epsilon} < \infty\right) > 0.
\end{aligned}$$

This contradiction proves Property 4. □

Proof (Theorem 9). The Lévy process $S^{(u)} := \eta - uW$ is a subordinator if and only if the following three conditions hold: $\sigma^2_{S^{(u)}} = 0$, $\Pi_{S^{(u)}}((-\infty, 0)) = 0$, and $d_{S^{(u)}} \geq 0$ where $d_{S^{(u)}} := E\left(S_1^{(u)} - \int_{(0,\infty)} z N_{S^{(u)},1}(\cdot, dz)\right)$. Note that $\sigma^2_{S^{(u)}} = 0$ is equivalent to $B_\eta - uB_W = 0$, which is equivalent to $B_\eta = -uB_\xi$ by (17), which establishes (20). We show that $S^{(u)}$ has no negative jumps for $u \neq 0$ if and only at least one of the dot point conditions of the theorem holds. Using (18) we see that $\Delta S_t^{(u)} = \Delta\eta_t - u\left(e^{-\Delta\xi_t} - 1.\right)$. If $u > 0$ then $\Delta S_t^{(u)} < 0$ requires that $(\Delta\xi_t, \Delta\eta_t)$ be contained within A_2^u, A_3, or A_4^u. Every $(\Delta\xi_t, \Delta\eta_t) \in A_3$ produces a $\Delta S_t^{(u)} < 0$. Recall that the value θ_2 is the supremum of all the values of $u \geq 0$ at which there can be a negative jump $\Delta S_t^{(u)}$ with $(\Delta\xi, \Delta\eta) \in A_2$. Note that at $u = \theta_2$ such a jump is not possible. The obvious symmetric statement holds for θ_4. Hence, if $u > 0$ then $S^{(u)}$ has no negative jumps if and only if $\Pi_{\xi,\eta}(A_3) = 0$, $\theta_2 \leq \theta_4$ and $u \in [\theta_2, \theta_4]$.

If $u < 0$ then $\Delta S_t^{(u)} < 0$ requires that $(\Delta\xi_t, \Delta\eta_t)$ be contained within A_1^u, A_2, or A_3^u. Every $(\Delta\xi_t, \Delta\eta_t) \in A_2$ produces a $\Delta S_t^{(u)} < 0$. Recall that the value θ_1 is the supremum of all the values of $u \leq 0$ at which there can be a negative jump $\Delta S_t^{(u)}$ with $(\Delta\xi, \Delta\eta) \in A_1$, and at $u = \theta_1$ such a jump is not possible. The obvious symmetric statement holds for θ_3. Hence, if $u < 0$ then $S^{(u)}$ can have no negative jumps if and only if $\Pi_{\xi,\eta}(A_2) = 0$, $\theta_1 \leq \theta_3$ and $u \in [\theta_1, \theta_3]$. Finally, if $\Pi_{\xi,\eta}(A_3) = \Pi_{\xi,\eta}(A_2) = 0$ then $\theta_3 = \theta_2 = 0$ and so both of the above are satisfied when $u \in [\theta_1, \theta_4]$. Now suppose that at least one of the dot point conditions holds. We let $u \in [\theta_i, \theta_j]$

for suitable i, j, and prove that $g(u) = d_{S^{(u)}}$. The following holds:

$$\begin{aligned}
d_{S^{(u)}} &= \gamma_\eta - u\gamma_W + E\left(\int_{|y|\geq 1} yN_{\eta,1}(\cdot, \mathrm{d}y) - u\int_{|x|\geq 1} xN_{W,1}(\cdot, \mathrm{d}x)\right. \\
&\quad \left. - \int_{(0,\infty)} zN_{\eta_1 - uW_1}(\cdot, \mathrm{d}z)\right) \\
&= \gamma_\eta - u\gamma_W + E\left(\int_{|y|\geq 1} yN_{\eta,1}(\cdot, \mathrm{d}y) - u\int_{(-\infty,-\ln 2)} \left(\mathrm{e}^{-x} - 1\right) N_{\xi,1}(\cdot, \mathrm{d}x)\right. \\
&\quad \left. - \int_{\{y-u(\mathrm{e}^{-x}-1)>0\}} \left(y - u(\mathrm{e}^{-x} - 1)\right) N_{\xi,\eta,1}\left(\cdot, \mathrm{d}(x, y)\right)\right) \\
&= \gamma_\eta + u\gamma_\xi - \frac{1}{2}u\sigma_\xi^2 + E\left(\int_{\mathbb{R}^2} \left(y1_{|y|\geq 1} - ux1_{|x|<1} - u(\mathrm{e}^{-x} - 1)\right.\right. \\
&\quad \left.\left. - \left(y - u(\mathrm{e}^{-x} - 1)\right) 1_{\{y-u(\mathrm{e}^{-x}-1)>0\}}\right) N_{\xi,\eta,1}(\cdot, \mathrm{d}(x, y))\right) \\
&= \gamma_\eta + u\gamma_\xi - \frac{1}{2}u\sigma_\xi^2 - E\left(\int_{\mathbb{R}^2} (ux1_{|x|<1} + y1_{|y|<1})N_{\xi,\eta,1}(\cdot, \mathrm{d}(x, y))\right) \\
&= \tilde{\gamma}_\eta + u\tilde{\gamma}_\xi - \frac{1}{2}u\sigma_\xi^2 - E\left(\int_{\{x^2+y^2<1\}} (ux + y)N_{\xi,\eta,1}(\cdot, \mathrm{d}(x, y))\right).
\end{aligned}$$

The first equality follows because the expected values of each of the Brownian motion components of η and W are zero, as is the expected value of the compensated small jump processes of η and W. The second and third equalities follow by (18) and (19) respectively. The fourth equality follows since u is contained in suitable $[\theta_i, \theta_j]$ which implies that $S^{(u)}$ has no negative jumps, and correspondingly $N_{\xi,\eta,1}\left(\{y - u(\mathrm{e}^{-x} - 1) \leq 0\}\right) = 0$. The final equality follows by (6) and (7). Thus we are done if we can exchange integration and expectation in the above expression. Now if $f(x, y)$ is a non-negative measurable function and Λ is a Borel set in $\mathbb{R}^2$ then the monotone convergence theorem implies that

$$E\left(\int_\Lambda f(x, y)N_{\xi,\eta,1}(\cdot, \mathrm{d}(x, y))\right) = \int_\Lambda f(x, y)\Pi_{\xi,\eta}(\mathrm{d}(x, y)).$$

For general $f(x, y)$, if $\int_\Lambda f^+(x, y)\Pi_{\xi,\eta}(\mathrm{d}(x, y))$ or $\int_\Lambda f^-(x, y)\Pi_{\xi,\eta}(\mathrm{d}(x, y))$ is finite, then the following is a well-defined member of the extended real numbers:

$$\begin{aligned}
&E\left(\int_\Lambda f(x, y)N_{\xi,\eta,1}(\cdot, \mathrm{d}(x, y))\right) \\
&\quad = \int_\Lambda f^+(x, y)\Pi_{\xi,\eta}(\mathrm{d}(x, y)) - \int_\Lambda f^-(x, y)\Pi_{\xi,\eta}(\mathrm{d}(x, y)) \\
&\quad = \int_\Lambda f(x, y)\Pi_{\xi,\eta}(\mathrm{d}(x, y)).
\end{aligned}$$

However, using the fact that $0 < \mathrm{e}^{-x} - 1 + x < x^2$ whenever $|x| < 1$, we have

$$\begin{aligned}
&\int_{\{x^2+y^2<1\}} (ux + y)^- \Pi_{\xi,\eta}(\mathrm{d}(x, y)) \\
&\quad = \int_{\{x^2+y^2<1\}} -(ux + y)1_{\{ux+y\leq 0\}}\Pi_{\xi,\eta}(\mathrm{d}(x, y))
\end{aligned}$$

$$\leq \int_{\{x^2+y^2<1\}} \left(y - u(\mathrm{e}^{-x} - 1) - (ux + y)\right) 1_{\{ux+y\leq 0\}} \Pi_{\xi,\eta}(\mathrm{d}(x,y))$$
$$= \int_{\{x^2+y^2<1\}} -u(\mathrm{e}^{-x} - 1 + x) 1_{\{ux+y\leq 0\}} \Pi_{\xi,\eta}(\mathrm{d}(x,y))$$
$$\leq \int_{\{x^2+y^2<1\}} |u| x^2 1_{\{ux+y\leq 0\}} \Pi_{\xi,\eta}(\mathrm{d}(x,y))$$
$$\leq |u| \int_{\mathbb{R}} \min\left\{1, x^2\right\} \Pi_\xi(\mathrm{d}x),$$

which is finite since Π_ξ is a Lévy measure. □

Proof (Theorem 1). By Theorem 7, $\psi(0) = 0$ iff $\delta(0) = 0$ iff η is a subordinator. Suppose η is not a subordinator and let $c > 0$. Clearly $\psi(c) = 0$ if and only if $\delta(c) \geq 0$. By Theorem 7, this is equivalent to the condition that there exists $0 < u \leq c$ such that $\delta(u) = u$. Combining this fact with Theorem 9 proves Theorem 1. □

Proof (Theorem 4). Define

$$U_t := \mathrm{e}^{\xi_t}(Z_\infty - Z_t) = \mathrm{e}^{\xi_t} \int_{t+}^{\infty} \mathrm{e}^{-\xi_{s-}} \mathrm{d}\eta_s.$$

Since we are integrating over (t, ∞) there are no predictability problems moving e^{ξ_t} under the integral sign. Thus $U_t = \int_{t+}^{\infty} \mathrm{e}^{-(\xi_{s-}-\xi_t)} \mathrm{d}\eta_s$, from which it follows, from Lévy properties, that U_t is independent of $\mathscr{F}_t$ and that U_{T_z} conditioned on $T_z < \infty$ is independent of $\mathscr{F}_{T_z}$. Since (ξ, η) is a Lévy process, for any $u > 0$ and $t > 0$

$$(\hat{\xi}_{u-}, \hat{\eta}_u) := (\xi_{(t+u)-} - \xi_t, \eta_{t+u} - \eta_t) =_D (\xi_{u-}, \eta_u). \tag{22}$$

Thus

$$U_t = \int_{s\in(t,\infty)} \mathrm{e}^{-(\xi_{s-}-\xi_t)} \mathrm{d}\eta_s = \int_{u\in(0,\infty)} \mathrm{e}^{-(\xi_{(t+u)-}-\xi_t)} \mathrm{d}\eta_{t+u}$$
$$= \int_{u\in(0,\infty)} \mathrm{e}^{-(\xi_{(t+u)-}-\xi_t)} \mathrm{d}(\eta_{t+u} - \eta_t) = \int_{u\in(0,\infty)} \mathrm{e}^{-\hat{\xi}_{u-}} \mathrm{d}\hat{\eta}_u$$
$$=_D \int_{u\in(0,\infty)} \mathrm{e}^{-\xi_{u-}} \mathrm{d}\eta_u \quad \text{(by (22))} = Z_\infty \quad (\text{since } \Delta\eta_0 = 0).$$

In particular, for any Borel set A,

$$P\left(U_{T_z} \in A \,\middle|\, T_z < \infty\right) = P(Z_\infty \in A). \tag{23}$$

Next note that if $\omega \in \{T_z < \infty\}$ then by definition of U,

$$z + Z_\infty = z + Z_{T_z} + \mathrm{e}^{-\xi_{T_z}} U_{T_z} = \mathrm{e}^{-\xi_{T_z}} \left(\mathrm{e}^{\xi_{T_z}}(z + Z_{T_z}) + U_{T_z}\right)$$
$$= \mathrm{e}^{-\xi_{T_z}} (V_{T_z} + U_{T_z}).$$

This implies that

$$P(T_z < \infty, z + Z_\infty < 0) = P(T_z < \infty, V_{T_z} + U_{T_z} < 0). \tag{24}$$

Finally note that $(Z_\infty < -z) \subset (T_z < \infty)$ since the convergence from Z_t to Z_∞ is a.s. convergence. Thus

$$\begin{aligned} P\,(z + Z_\infty < 0) &= P\,(T_z < \infty, z + Z_\infty < 0) \\ &= P\left(T_z < \infty, V_{T_z} + U_{T_z} < 0\right) \quad \text{(by (24))} \\ &= E\left(P(T_z < \infty, V_{T_z} + U_{T_z} < 0 \middle| \mathscr{F}_{T_z})\right) \\ &= \int_{T_z < \infty} P\left(V_{T_z} + U_{T_z} < 0 \middle| \mathscr{F}_{T_z}\right)(\omega) P(\mathrm{d}\omega). \end{aligned}$$

But if $T_z(\omega) < \infty$ then

$$\begin{aligned} P\left(V_{T_z} + U_{T_z} < 0 \middle| \mathscr{F}_{T_z}\right)(\omega) &= P\left(V_{T_z}(\omega) + U_{T_z} < 0 \middle| \mathscr{F}_{T_z}\right)(\omega) \\ &= P\left(U_{T_z} < -V_{T_z}(\omega) \middle| T_z < \infty\right) \\ &= P\left(Z_\infty < -V_{T_z}(\omega)\right) \quad \text{(by (23))}. \end{aligned}$$

The second to last equality follows since U_{T_z} conditioned on $T_z < \infty$ is independent of $\mathscr{F}_{T_z}$. Thus we obtain the required formula from

$$\begin{aligned} G(-z) &= \int_{T_z < \infty} G(-V_{T_z})(\omega) P(\mathrm{d}\omega) \\ &= E\left(G(-V_{T_z}) 1_{T_z < \infty}\right) \\ &= E\left(G(-V_{T_z}) 1_{T_z < \infty} \middle| T_z < \infty\right) P(T_z < \infty) \\ &\quad + E\left(G(-V_{T_z}) 1_{T_z < \infty} \middle| T_z = \infty\right) P(T_z = \infty) \\ &= E\left(G(-V_{T_z}) \middle| T_z < \infty\right) P(T_z < \infty). \quad \square \end{aligned}$$

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