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ALMOST ARCWISE CONNECTED CONTINUA WITHOUT DENSE ARC COMPONENTS

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A continuum M is *almost arcwise connected* if each pair of nonempty open subsets of M can be joined by an arc in M. An almost arcwise connected plane continuum without a dense arc component can be defined by identifying pairs of endpoints of three copies of the Knaster indecomposable continuum that has two endpoints. In [7] K.R. Kellum gave this example and asked if every almost arcwise connected continuum without a dense arc component has uncountably many arc components. We answer Kellum's question by defining an almost arcwise connected plane continuum with only three arc components none of which are dense. A continuum M is *almost Peano* if for each finite collection $\mathscr{C}$ of nonempty open subsets of M there is a Peano continuum in M that intersects each element of $\mathscr{C}$. We define a hereditarily unicoherent almost Peano plane continuum that does not have a dense arc component. We prove that every almost arcwise connected planar λ-dendroid has exactly one dense arc component. It follows that every hereditarily unicoherent almost arcwise connected plane continuum without a dense arc component has uncountably many arc components. Using an example of J. Krasinkiewicz and P. Minc [8], we define an almost Peano λ-dendroid that does not have a dense arc component. Using a theorem of J.B. Fugate and L. Mohler [3], we prove that every almost arcwise connected λ-dendroid without a dense arc component has uncountably many arc components. In Euclidean 3-space we define an almost Peano continuum with only countably many arc components no one of which is dense. It is not known if the plane contains a continuum with these properties.

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almost Peano continua
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almost continuous function
fixed-point property
λ-dendroid

Introduction

Let f be a function of a continuum X into a continuum Y. In [13] J. Stallings defined f to be *almost continuous* if each open set in $X \times Y$ that contains f contains a continuous function of X into Y. Kellum [7] proved that if X is Peano and f is an almost continuous surjection, then Y is almost Peano. Kellum [7] also proved that

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every almost Peano continuum is an almost continuous image of every Peano continuum.

Every almost continuous retract of a 2-cell has the fixed-point property [13, Theorem 3 and Proposition 1] (also see [4]). K. Borsuk [13, p. 263] observed that this theorem provides an interesting approach to the problem of whether every nonseparating plane continuum has the fixed-point property. For more information about almost continuous functions see [4–7] and [13].

Definitions. A *continuum* is a nondegenerate compact connected metric space. A *Peano continuum* is a locally connected continuum. A continuum is *hereditarily unicoherent* if each pair of its intersecting subcontinua has a connected intersection. A continuum is *decomposable* if it is the union of two of its proper subcontinua; otherwise, it is *indecomposable*. It is *hereditarily decomposable* if each of its subcontinua is decomposable. A continuum is a λ-*dendroid* if it is hereditarily unicoherent and hereditarily decomposable.

In this paper E^n is Euclidean n-space. The closure of a given set S is denoted by $\mathrm{Cl}\, S$.

Example 1. There exists an almost arcwise connected continuum Z_1 with only three arc components none of which are dense in Z_1.[1]

To define Z_1, let C be the Cantor ternary set in the unit interval $[0, 1]$. Let

$$P = \{(x, y) \in E^2 : x \in C, \tfrac{2}{3} \leq y \leq \tfrac{5}{6}\}.$$

For each positive integer n, let

$$Q_n = \left\{(x, y) \in E^2 : x \in C, x = \frac{i}{3^n}\ (0 \leq i \leq 3^n), \frac{1}{2n+1} \leq y \leq \frac{1}{2n}\right\}.$$

For each n, let U_n be the union of the components of $C \times [0, 1]$ that intersect Q_n. Let $R_1 = Q_1$ and for $n = 2, 3, \ldots$, let $R_n = Q_n \setminus U_{n-1}$. Let $x_1, x_2, \ldots$ be a sequence of inaccessible points of C that is dense in C. For each n, let S_n be the interval

$$\{x_n\} \times \left[1 - \frac{1}{25n}, 1 - \frac{1}{50n}\right].$$

In E^3, modify $C \times [0, 1]$ so that each component of $P \cup \bigcup_{n=1}^{\infty} R_n \cup S_n$ is the limit bar of a $\sin 1/x$ curve as indicated in Fig. 1. Call the resulting space V.

Let $\mathcal{T}$ be the set of all components T of P such that the projection of $C \times [0, 1]$ onto C sends T to a point of $x_1, x_2, \ldots$ or to an accessible point of C. Let $T_1, T_2, \ldots$ be the elements of $\mathcal{T}$. Let $D_1, D_2, \ldots$ be a sequence of disjoint rectangular disks with the following properties. For each n,

(1) the base interval of D_n is T_n,

(2) the height of D_n is less than $1/n$, and

(3) $D_n \cap V$ is a $\sin 1/x$ curve.

[1] In a recent conversation, Kellum observed that every arcwise connected continuum without a dense arc component has at least three arc components.

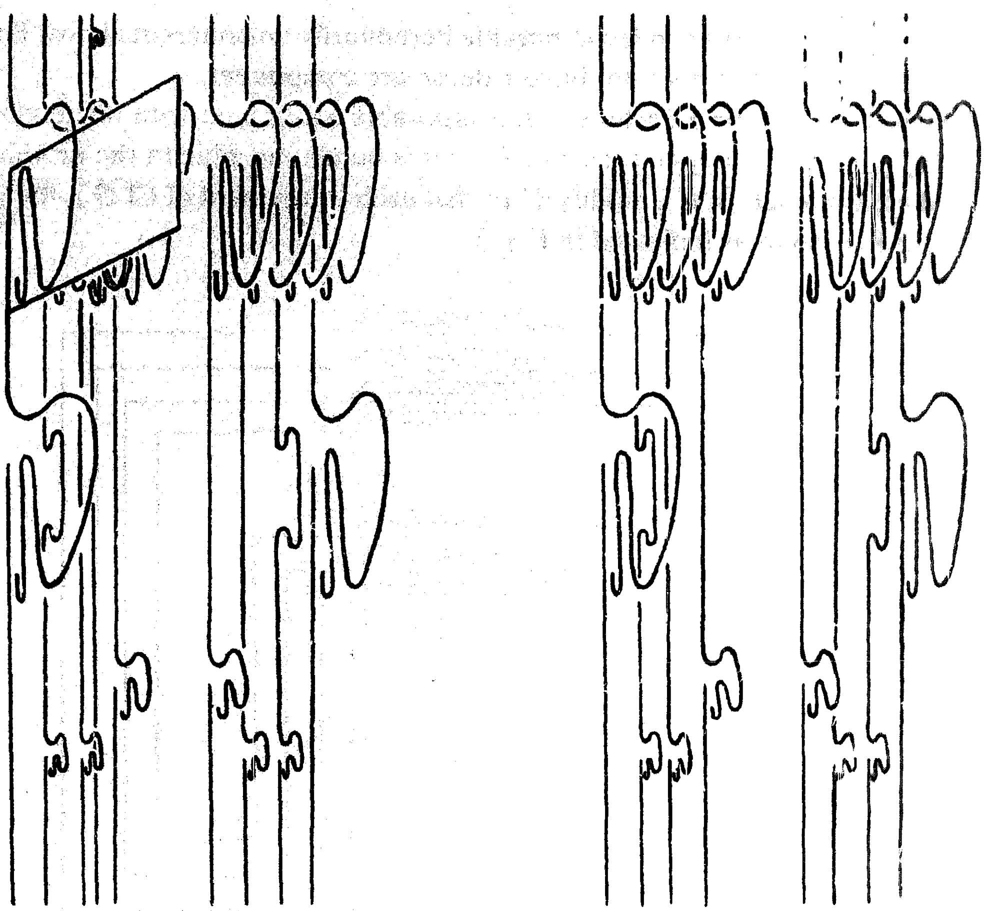

Fig. 1.

Let $W = V \cup \bigcup_{n=1}^{\infty} D_n$. For each n, let d_n be the projection of D_n straight down onto T_n. Let d be the map of W into W such that d is the identity on $W \setminus \bigcup_{n=1}^{\infty} D_n$ and for each n, the restriction of d to D_n is d_n. Note that d defines a monotone upper semi-continuous decomposition of W.

Let X be the decomposition space associated with d. Note that the components of P that miss $\bigcup_{n=1}^{\infty} D_n$ are still limit bars of $\sin 1/x$ curves. Each component of X is a chainable continuum with two endpoints. Using methods similar to L.G. Oversteegen's [11, Theorem 2.2], embed X in E^2 between two parallel lines K and L so that each component of X meets $K \cup L$ only at its endpoints and has one endpoint in K and the other endpoint in L.

Let Y be the plane continuum $X/\{C \times \{0\}, C \times \{1\}\}$ [9, p. 533]. Let A and B be the arc components of Y that contain $C \times \{0\}$ and $C \times \{1\}$, respectively. Note that $Y = A \cup B$. Both A and B are dense in Y.

Let h_1 and h_2 be homeomorphisms of Y such that Y, $h_1[Y]$, and $h_2[Y]$ are disjoint. Define Z_1 to be the plane continuum obtained from $Y \cup h_1[Y] \cup h_2[Y]$ by identifying $C \times \{0\}$ with $h_1(C \times \{1\})$, $h_1(C \times \{0\})$ with $h_2(C \times \{1\})$, and $h_2(C \times \{0\})$ with $C \times \{1\}$. The three arc components of Z_1 are $A \cup h_1[B]$, $h_1[A] \cup h_2[B]$, and $h_2[A] \cup B$. Although Z_1 is almost arcwise connected, no arc component is dense in Z_1. Note that Z_1 is not hereditarily unicoherent and not almost Peano.

Example 2. There exists an indecomposable hereditarily unicoherent almost Peano plane continuum Z_2 that does not have a dense arc component.

To define Z_2, let U be Knaster's indecomposable continuum with one endpoint [9, p. 204]. Let G be an open subset of U that is homeomorphic to the product of $(0, 1)$ and a Cantor set. In E^3, modify U so that each component of Cl G is the limit bar of a sin $1/x$ curve as indicated in Fig. 2.

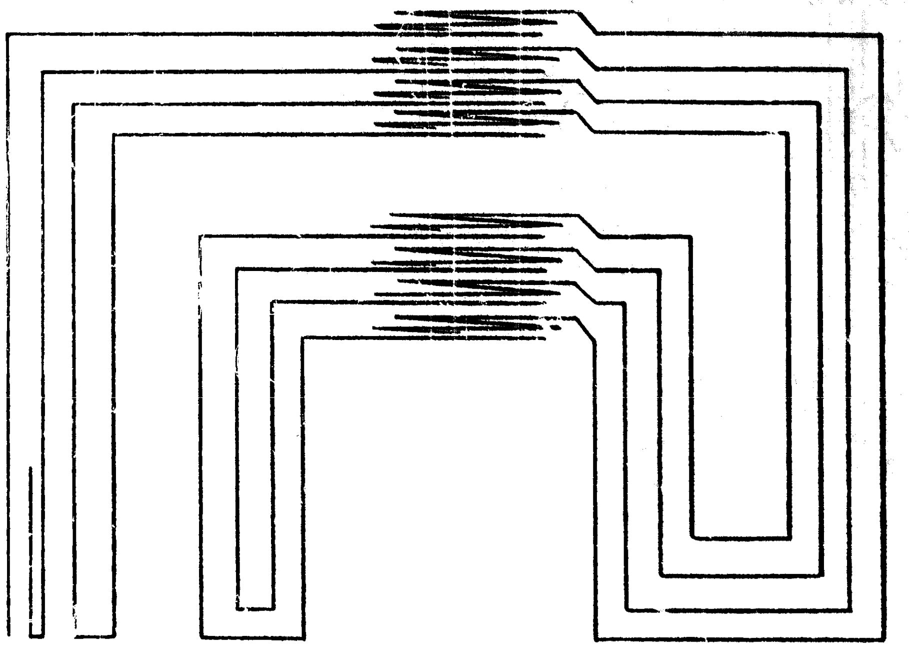

Fig. 2.

Note that the resulting continuum V is indecomposable. Let $C_1, C_2, \ldots$ be a sequence of distinct composants of V.

For each positive integer n, let $\mathscr{D}_n$ be a finite collection of rectangular disks with the following properties. For each element D of $\mathscr{D}_n$,

(1) the base interval of D is a component of $C_n \cap \mathrm{Cl}\, G$,

(2) the height of D is less than $1/n$,

(3) $D \cap V$ is a sin $1/x$ curve, and

(4) D misses each element of $\bigcup_{i=1}^{\infty} \mathscr{D}_i \setminus \{D\}$.

Furthermore, an arc component of the union of C_n and the elements of $\mathscr{D}_n$ is $1/n$-dense in V.

Let W be the union of V and the elements of $\bigcup_{n=1}^{\infty} \mathscr{D}_n$. For each n, let d_n be the function that projects each element of $\mathscr{D}_n$ straight down into C_n. Let d be the map of W into W that agrees with each d_n on each element of $\mathscr{D}_n$ and is the identity everywhere else. The map d defines a monotone upper semi-continuous decomposition of W.

Define Z_2 to be the decomposition space associated with d. Since Z_2 is a chainable continuum, it can be embedded in the plane [1, Theorem 4]. Since Z_2 is indecomposable, it has uncountably many arc components. Each arc component is nowhere dense. However Z_2 is almost Peano.

Definition. A continuum M is a *n-od* if there exist a collection $\mathscr{B}$ of n distinct continua and a closed connected set C properly contained in each element of $\mathscr{B}$ such that M is the union of $\mathscr{B}$ and C is the intersection of each pair of elements of $\mathscr{B}$.

Lemma. *If an almost arcwise connected λ-dendroid M is a 4-od in E^2, then M has a dense arc component.*

Proof. Assume C and $\mathscr{B}=\{B_1, B_2, B_3, B_4\}$ have the properties stated in the definition (above). Let D_1, D_2, D_3, and D_4 be disjoint disks such that for each n, D_n is in $E^2\setminus\bigcup_{i\neq n} B_i$ and B_n intersects the interior of D_n. Let H denote the continuum $M\cup\bigcup_{n=1}^4 D_n$.

It follows from a theorem of H. Cook [2] that $E^2\setminus M$ is connected. Each D_n intersects $E^2\setminus M$. Hence there exist an arc segment A_1 in $E^2\setminus H$ and an element K of $\{D_2, D_3, D_4\}$ such that (1) one endpoint of A_1 is in $D_1\setminus M$ and the other endpoint is in $K\setminus M$ and (2) the unbounded component of $E^2\setminus(A_1\cup D_1\cup K\cup M)$ intersects both elements of $\{D_2, D_3, D_4\}\setminus\{K\}$. Assume without loss of generality that $D_2=K$.

There exist an arc segment A_2 in $E^2\setminus(H\cup A_1)$ and an element L of $\{D_3, D_4\}$ such that (1) $D_2\setminus\mathrm{Cl}\,A_1$ and $L\setminus M$ each contain an endpoint of A_2 and (2) the unbounded component of $E^2\setminus(A_2\cup D_2\cup L\cup M)$ intersects both elements of $\{D_1, D_3, D_4\}\setminus\{L\}$. Assume without loss of generality that $D_3=L$.

Let A_3 be an arc segment in $E^2\setminus(H\cup A_1\cup A_2)$ with endpoints in $D_3\setminus\mathrm{Cl}\,A_2$ and $D_4\setminus M$. Let A_4 be an arc segment in $E^2\setminus(H\cup A_1\cup A_2\cup A_3)$ with endpoints in $D_4\setminus\mathrm{Cl}\,A_3$ and $D_1\setminus\mathrm{Cl}\,A_1$.

Let U be the component of $E^2\setminus\bigcup_{n=1}^4 (A_n\cup D_n)$ that contains C. The boundary of U is a simple closed curve S. Furthermore, the arcs $D_1\cap S$ and $D_3\cap S$ are separated in S by $(D_2\cup D_4)\cap S$. Since M is hereditarily unicoherent and almost arcwise connected, there exist in $M\cap U$ an arc segment I from D_1 to D_3 and an arc segment J from D_2 to D_4. Note that $I\cap J$ is a nonempty connected set in C [10, Theorem 28, p. 156].

For each open subset V of M that misses $I\cup J$ there exist an open set W in V and an open set X in $M\cap\bigcup_{n=1}^4 D_n$ such that every arc in M from W to X intersects $I\cup J$. Hence the arc component of M that contains $I\cup J$ is dense in M. Thus the lemma is true.

Theorem 1. *Suppose M is an almost arcwise connected λ-dendroid in E^2. Then M has one dense arc component and every other arc component of M is nowhere dense.*

Proof. Krasinkiewicz and Minc [8] proved that every planar λ-dendroid has at most one dense arc component. They [8] also proved that if a planar λ-dendroid has a dense arc component then every other arc component is nowhere dense. Hence it suffices to show that M has a dense arc component. We consider two cases.

Case 1. Suppose M is irreducible between two points. Then there is a monotone map f of M onto $[0, 1]$ such that the continuum $\mathrm{Cl}\, f^{-1}[(\frac{1}{4}, \frac{3}{4})]$ is irreducible between $f^{-1}[[0, \frac{1}{4}]]$ and $f^{-1}[[\frac{3}{4}, 1]]$ [14, Theorem 8, p. 14]. Since M is almost arcwise connected, there is an arc A in M from $f^{-1}[[0, \frac{1}{4}]]$ to $f^{-1}[[\frac{3}{4}, 1]]$. It follows that $\mathrm{Cl}\, f^{-1}[(\frac{1}{4}, \frac{3}{4})]$ is a subarc of A that contains a nonempty open subset of M. Hence the arc component of M that contains A is dense in M.

Case 2. Suppose M is not irreducible between two points. Then according to a theorem of R.H. Sorgenfrey [12, Theorem 3.2], M is a triod (3-od). Assume C and $\mathscr{B} = \{B_1, B_2, B_3\}$ have the properties stated in the definition (above).

Let H be $\mathrm{Cl}(B_1 \backslash C)$. If H is not a continuum, then M is a 4-od, and, by the lemma (above), M has a dense arc component. Hence we assume that H is a continuum. Let K and L be proper subcontinua of H such that $H = K \cup L$. Since $C \cap H$ is nowhere dense in H, both $K \backslash L$ and $L \backslash K$ intersect $H \backslash C$.

If C intersects $K \cap L$, then the existence of the continua $K \cup C$, $L \cup C$, $B_2 \cup (K \cap L)$, and $B_3 \cup (K \cap L)$ implies that M is a 4-od, and, by the lemma (above), M has a dense arc component. Hence we assume that C misses $K \cap L$. It follows that the connected set $C \cap H$ is in either K or L.

Assume without loss of generality that K contains $C \cap H$ and $L \cap C = \emptyset$. Since M is almost arcwise connected, there is an arc P in K from L to C. Since M is hereditarily unicoherent, P intersects every arc in M that joins an open set in L with an open set in $B_2 \cup B_3$. Hence the arc component Q of M that contains P is dense in $L \cup B_2 \cup B_3$.

By a similar argument, there exists an arc component R of M that contains an arc in B_2 and is dense in $B_1 \cup B_3$. Since R intersects P, it follows that $Q = R$. Hence Q is dense in M. This completes the proof of Theorem 1.

Theorem 2. *If M is a hereditarily unicoherent almost arcwise connected plane continuum without a dense arc component, then M has uncountably many arc components.*

Proof. By Theorem 1, M has an indecomposable subcontinuum I. Since M is hereditarily unicoherent, no arc component of M intersects more than one composant of I. Hence M has uncountably many arc components.

Example 3. There exists an almost Peano λ-dendroid Z_3 that does not have a dense arc component.

In [8], Krasinkiewicz and Minc defined a λ-dendroid X that has uncountably many dense arc components. Define Z_3 by modifying X in the same way that Knaster's continuum was modified to define Z_2 (Example 2). Note that Z_3 has uncountably many arc components.

Theorem 3. *Suppose M is an almost arcwise connected λ-dendroid with only countably many arc components. Then M has one dense arc component and every other arc component of M is nowhere dense.*

Proof. Fugate and Mohler [3, Corollary 1.10] proved that every λ-dendroid with only countably many arc components has at most one dense arc component. If a λ-dendroid with only countably many arc components has a dense arc component, then every other arc component is nowhere dense [3, Corollary 1.9]. Thus it suffices to show that M has a dense arc component.

Assume that no arc component of M is dense. By the Baire category theorem, there exists an arc component R of M such that $\text{Cl}\, R$ contains a nonempty open subset A of M. Let B be a nonempty open subset of $M \setminus \text{Cl}\, R$. There exists an arc component S of M such that $\text{Cl}\, S$ contains a nonempty open subset C of B.

By [3, Corollary 1.5], R and S are G_δ subsets of M. Since $R \cap S = \emptyset$ and R is dense in A, it follows that S is nowhere dense in A. Let D be a nonempty open set in $A \setminus \text{Cl}\, S$. Since M is almost arcwise connected, there is an arc P in M that intersects C and D.

If Q is an arc in M that intersects C and D, then Q intersects P; for otherwise, the intersection of $\text{Cl}\, R$ and $P \cup Q \cup \text{Cl}\, S$ is not connected, and this contradicts the assumption that M is hereditarily unicoherent.

Hence the arc component T of M that contains P is dense in $C \cup D$. Since $T \cap C \neq \emptyset$, $T \neq R$. Therefore $R \cap T = \emptyset$. But R and T are G_δ subsets of M [3, Corollary 1.5] and both $\text{Cl}\, R$ and $\text{Cl}\, T$ contain the open set D. It follows from this contradiction that M has a dense arc component. Hence the proof is complete.

Corollary. *Every almost arcwise connected λ-dendroid without a dense arc component has uncountably many arc components.*

The continuum Z_1 (Example 1) is not almost Peano. This is consistent with the fact that every almost Peano continuum without a dense arc component has infinitely many arc components. The almost Peano continua Z_2 (Example 2) and Z_3 (Example 3) each have uncountably many arc components. One might wonder if every almost Peano continuum without a dense arc component has uncountably many arc components.

Example 4. There exists an almost Peano continuum Z_4 in E^3 with only countably many arc components no one of which is dense in Z_4.

To define Z_4, let C be the Cantor ternary set in $[0, 1]$. For each positive integer n, let

$$P_n = \left\{(x, y) \in E^2 : x \in C, x = \frac{i}{3^n}\ (0 \leq i \leq 3^n), y = 1 - \frac{1}{3^n}\right\}$$

and

$$Q_n = \left\{(x, y) \in E^2 : x \in C, x = \frac{i}{3^n} (0 \leq i \leq 3^n), \frac{1}{2n+1} \leq y \leq \frac{1}{2n}\right\}.$$

For each n, let U_n be the union of the components of $C \times [0, 1]$ that intersect Q_n. Let $R_1 = Q_1$ and $S_1 = P_1$. For $n = 2, 3, \ldots$, let $R_n = Q_n \setminus U_{n-1}$ and $S_n = P_n \setminus U_{n-1}$.

Modify $C \times [0, 1]$ (as in Example 1) so that each component of $\bigcup_{n=1}^{\infty} R_n$ is the limit bar of a sin $1/x$ curve. Call the resulting space X.

Let $\mathscr{D} = \{C \times \{0\}, C \times \{1\}\} \cup \{S_1, S_2, \ldots\}$. In the continuum $X/\mathscr{D}$ replace the point $C \times \{1\}$ with a limit bar L (as indicated in Fig. 3). Do this in such a way that the sequence $S_1, S_2, \ldots$ converges to a point p of L. Call the resulting continuum Y. Note that Y has only countably many arc components and only one arc component is dense in Y. Each S_n misses the dense arc component of Y.

Let h be a homeomorphism of Y such that $Y \cap h[Y] = \emptyset$. Define Z_4 to be the continuum obtained from $Y \cup h[Y]$ by identifying p with $h(p)$, and for each n, identifying S_n with $h(S_n)$.

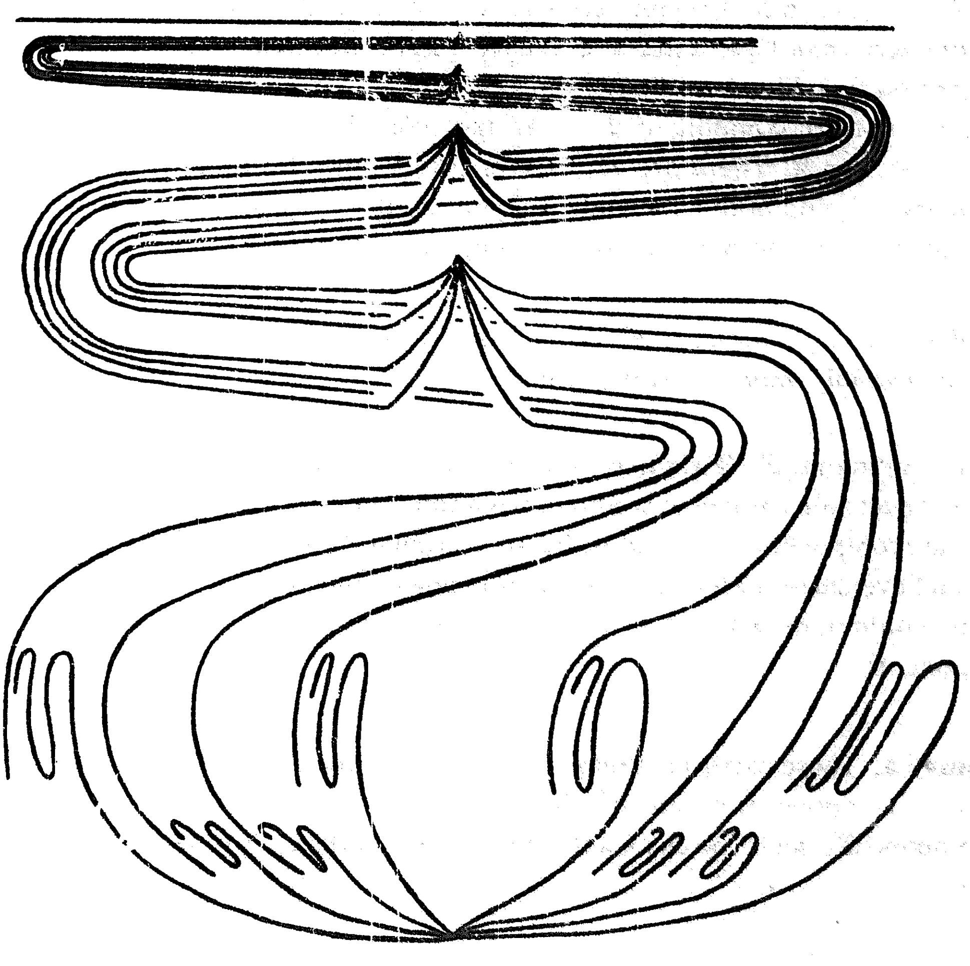

Fig. 3.

Question. Does every almost Peano plane continuum without a dense arc component have uncountably many arc components?

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