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# Orthogonal polynomials in two variables and second-order partial differential equations

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**Abstract**

We study the second-order partial differential equations

$$L[u] = Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y = \lambda_n u,$$

which have orthogonal polynomials in two variables as solutions. By using formal functional calculus on moment functionals, we first give new simpler proofs and improvements of the results by Krall and Sheffer and Littlejohn. We then give a two-variable version of Al-Salam and Chihara's characterization of classical orthogonal polynomials in one variable. We also study in detail the case when $L[\cdot]$ belongs to the basic class, that is, $A_y = C_x = 0$. In particular, we characterize all such differential equations which have a product of two classical orthogonal polynomials in one variable as solutions.

*Keywords:* Orthogonal polynomials in two variables; Second-order partial differential equations

*AMS classification:* 33C50; 35P99

## 1. Introduction

In this work, we are concerned with orthogonal polynomial solutions of second-order partial differential equations of spectral type

$$L[u] := Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y = \lambda_n u, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where $A(x, y), \ldots, E(x, y)$ are polynomials, independent of $n$, with $|A| + |B| + \cdots + |E| \neq 0$ and $\lambda_n$ is the eigenvalue parameter.

In 1967, Krall and Sheffer [3] posed and partially solved the following problem:

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Classify, up to a linear change of variables, all orthogonal polynomial systems that arise as eigenfunctions of Eq. (1.1), assuming that Eq. (1.1) is admissible (see Definition 3.1).

Krall and Sheffer first found necessary and sufficient conditions for weak orthogonal polynomials (see Definition 2.1) to satisfy an admissible Eq. (1.1) in terms of the recurrence relations for moments of the orthogonalizing measure. In doing so, they used a formal generating series

$$G(x,y,s,t) := \sum_{m,n=0}^{\infty} P_{mn}(x,y)s^m t^n$$

of a polynomial sequence $\{P_{mn}(x,y)\}$. See Section 4 (in particular, Theorems 4.3 and 4.4) in [3].

Later, Littlejohn [7] employed a functional approach to investigate the same problem and made an important observation that the recurrence relations found by Krall and Sheffer can be restated in a much simpler closed form (see, [7, p. 117]) using a weak solution of the weight equations for the differential operator $L[\cdot]$.

Instead of moments and a formal generating series $G\ (x,y,s,t)$ used in [3] and a weak solution of the weight equations used in [7], we use directly moment functionals (i.e., linear functionals on the space of polynomials) and their formal calculus, which turn out to be quite successful in the study of orthogonal polynomials in one variable (see [4–6, 9, 10]).

In this way, we can provide much simpler new proofs and some improvements of results in [3, 7]. For example, the technical assumption, such as the unique solvability of the moment equations, needed in [7, Theorem 2.3.1] can now be removed.

In [3, 7], it is always assumed that Eq. (1.1) is admissible. However, since it is unknown whether Eq. (1.1) is admissible when it has orthogonal polynomials as solutions, we do not assume the admissibility of Eq. (1.1), whenever possible.

We then find a characterization of orthogonal polynomials satisfying the differential Eq. (1.1) via the so-called structure relation, which was first proved for classical orthogonal polynomials in one variable by Al-Salam and Chihara [1] (see also [4, 9]).

Lastly in Section 4, we consider the particular case when $L[\cdot]$ belongs to the basic class (cf. [8]), that is, when $A_y = C_x = 0$. We characterize differential Eq. (1.1) which have a product of two classical orthogonal polynomials in one variable as solutions. We also find conditions under which derivatives of any orthogonal polynomial solutions to Eq. (1.1) are also orthogonal polynomials satisfying the same type of equations as (1.1).

We refer to [2, 11] for works closely related to ours. See also [12, 13] and references therein for general theory of multi-variable orthogonal polynomials including Favard's Theorem and Christoffel–Darboux formula.

## 2. Preliminaries

For any integer $n \geqslant 0$, let $\mathscr{P}_n$ be the space of real polynomials in two variables of (total) degree $\leqslant n$ and $\mathscr{P} = \bigcup_{n \geqslant 0} \mathscr{P}_n$. By a polynomial system (PS), we mean a sequence of polynomials $\{\phi_{mn}(x,y)\}_{m,n=0}^{\infty}$ such that $\deg(\phi_{mn}) = m+n$ for $m$ and $n \geqslant 0$ and $\{\phi_{n-j,j}\}_{j=0}^{n}$ are linearly independent modulo $\mathscr{P}_{n-1}$ for $n \geqslant 0$ ($\mathscr{P}_{-1} = \{0\}$).

A PS $\{P_{mn}\}$ is said to be monic if

$$P_{mn}(x,y) = x^m y^n + R_{mn}(x,y), \quad m \text{ and } n \geqslant 0,$$

where $R_{mn}(x,y)$ is a polynomial of degree $\leqslant m+n-1$.

For any moment functional $\sigma$ on $\mathscr{P}$, we let

$$\langle \partial_x\sigma, \phi\rangle = -\langle \sigma, \partial_x\phi\rangle, \qquad \langle \partial_y\sigma, \phi\rangle = -\langle \sigma, \partial_y\phi\rangle, \qquad \langle \psi\sigma, \phi\rangle = \langle \sigma, \psi\phi\rangle$$

for any polynomials $\phi(x,y)$ and $\psi(x,y)$.

**Definition 2.1** (*Krall and Sheffer* [3]). A PS $\{\phi_{mn}\}$ is a weak orthogonal polynomial system (WOPS) if there is a nonzero moment functional $\sigma$ such that

$$\langle \sigma, \phi_{mn}\phi_{kl}\rangle = 0 \quad \text{if } m+n \neq k+l.$$

If, furthermore,

$$\langle \sigma, \phi_{mn}\phi_{kl}\rangle = K_{mn}\delta_{mk}\delta_{nl},$$

where $K_{mn}$ are nonzero (respectively, positive) constants, we call $\{\phi_{mn}\}$ an orthogonal polynomial system (OPS) (respectively, a positive-definite OPS). In this case, we say that $\{\phi_{mn}\}$ is a WOPS or an OPS relative to $\sigma$.

A PS $\{\phi_{mn}\}$ is a WOPS relative to $\sigma$ if and only if $\langle \sigma, \phi_{mn}R\rangle = 0$ for any polynomial $R(x,y)$ of degree $\leqslant m+n-1$.

For any PS $\{\phi_{mn}\}$, there is a unique moment functional $\sigma$, called the canonical moment functional of $\{\phi_{mn}\}$, defined by the conditions

$$\langle \sigma, 1\rangle = 1 \quad \text{and} \quad \langle \sigma, \phi_{mn}\rangle = 0, \quad m+n \geqslant 1.$$

Note that if $\{\phi_{mn}\}$ is a WOPS relative to $\sigma$, then $\sigma$ must be a nonzero constant multiple of the canonical moment functional of $\{\phi_{mn}\}$.

**Definition 2.2.** A moment functional $\sigma$ is quasi-definite (respectively, positive-definite) if there is an OPS (respectively, a positive-definite OPS) relative to $\sigma$.

In treating multi-variable orthogonal polynomials, it is convenient (see [12]) to use the following vector notations:

For a PS $\{\phi_{mn}\}$, we let

$$\Phi_n := [\phi_{n0}, \phi_{n-1,1}, \ldots, \phi_{0n}]^{\mathrm{T}}, \quad n \geqslant 0,$$

and use also $\{\Phi_n\}_{n=0}^{\infty}$ to denote the PS $\{\phi_{mn}\}$. For a matrix $\Psi = [\psi_{ij}(x,y)]_{i=0,j=0}^{m\ \ n}$ of polynomials and a moment functional $\sigma$, we let

$$\langle \sigma, \Psi\rangle = [\langle \sigma, \psi_{ij}\rangle]_{i=0,j=0}^{m\ \ n}.$$

Then, a PS $\{\Phi_n\}_{n=0}^{\infty}$ is an OPS (respectively, a positive-definite OPS) relative to $\sigma$ if and only if $\langle \sigma, \Phi_m \Phi_n^{\mathrm{T}} \rangle = H_n \delta_{mn}$, $m$ and $n \geqslant 0$ and $H_n := \langle \sigma, \Phi_n \Phi_n^{\mathrm{T}} \rangle$, $n \geqslant 0$, is a nonsingular (respectively, positive-definite) diagonal matrix.

The following was proved in [3].

**Proposition 2.1.** *For a moment functional $\sigma \neq 0$, the following statements are equivalent:*

(i) *$\sigma$ is quasi-definite (respectively, positive-definite);*

(ii) *There is a unique monic WOPS $\{\mathbb{P}_n\}_{n=0}^{\infty}$ relative to $\sigma$;*

(iii) *There is a monic WOPS $\{\mathbb{P}_n\}_{n=0}^{\infty}$ such that $H_n := \langle \sigma, \mathbb{P}_n \mathbb{P}_n^{\mathrm{T}} \rangle$, $n \geqslant 0$, is a nonsingular (respectively, positive-definite) symmetric matrix;*

(iv) *$\Delta_n = |D_n| \neq 0$ (respectively, $D_n$ is positive-definite), where*

$$D_n := \begin{bmatrix} \sigma_{00} & \sigma_{10} & \sigma_{01} & \cdots & \sigma_{n0} & \cdots & \sigma_{0n} \\ \sigma_{10} & \sigma_{20} & \sigma_{11} & \cdots & \sigma_{n+1,0} & \cdots & \sigma_{1n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ \sigma_{0n} & \sigma_{1n} & \sigma_{0,n+1} & \cdots & \sigma_{nn} & \cdots & \sigma_{0,2n} \end{bmatrix}, \quad n \geqslant 0,$$

*and $\sigma_{ij} = \langle \sigma, x^i y^j \rangle$, $i$ and $j \geqslant 0$, are the moments of $\sigma$.*

In fact, Krall and Sheffer proved Proposition 2.1 only for the quasi-definite case. But, the proof for the positive-definite case is similar. It is also easy (cf. [12]) to see that $\sigma$ is positive-definite if and only if $\langle \sigma, \phi^2 \rangle > 0$ for any polynomial $\phi(x, y) \neq 0$.

**Lemma 2.2.** *Let $\sigma$ and $\tau$ be moment functionals and $R(x, y)$ a polynomial. Then*

(i) *$\sigma = 0$ if and only if $\partial_x \sigma = 0$ or $\partial_y \sigma = 0$.*

*Assume that $\sigma$ is quasi-definite and let $\{\Phi_n\}_{n=0}^{\infty}$ be an OPS relative to $\sigma$. Then*

(ii) *$R(x, y)\sigma = 0$ if and only if $R(x, y) = 0$;*

(iii) *$\langle \tau, \phi_{mn} \rangle = 0$, $m + n > k$ ($k \geqslant 0$ an integer) if and only if $\tau = \psi(x, y)\sigma$ for some polynomial $\psi(x, y)$ of degree $\leqslant k$.*

**Proof.** (i) and (ii) are obvious.

(iii) $\Leftarrow$: It is trivial from the orthogonality of $\{\Phi_n\}_{n=0}^{\infty}$ relative to $\sigma$.

(iii) $\Rightarrow$: Consider a moment functional $\tilde{\tau} = (\sum_{j=0}^{k} \mathbb{C}_j \Phi_j)\sigma$,
where $\mathbb{C}_j = (C_{j0}, C_{j-1,1}, \ldots, C_{0j})$, $0 \leqslant j \leqslant k$, are arbitrary constant row vectors.
Then

$$\langle \tilde{\tau}, \Phi_n \rangle = \sum_{j=0}^{k} \langle \sigma, \Phi_n \Phi_j^{\mathrm{T}} \rangle \mathbb{C}_j^{\mathrm{T}} = \begin{cases} 0, & n > k, \\ \langle \sigma, \Phi_n \Phi_n^{\mathrm{T}} \rangle \mathbb{C}_n^{\mathrm{T}}, & 0 \leqslant n \leqslant k. \end{cases}$$

Hence, if we take $\mathbb{C}_n = \langle \tau, \Phi_n \rangle^{\mathrm{T}} H_n^{-1}$, $0 \leqslant n \leqslant k$, then $\langle \tau, \Phi_n \rangle = \langle \tilde{\tau}, \Phi_n \rangle$, $n \geqslant 0$, so that $\tau = \tilde{\tau}$. $\square$

## 3. Second-order differential equations

It is easy to see (cf. [3]) that if the differential Eq. (1.1) has a PS $\{\Phi_n\}_{n=0}^{\infty}$ as solutions, then it must be of the form

$$\begin{aligned} L[u] &= Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y \\ &= (ax^2 + d_1x + e_1y + f_1)u_{xx} + (2axy + d_2x + e_2y + f_2)u_{xy} \\ &\quad +(ay^2 + d_3x + e_3y + f_3)u_{yy} + (gx + h_1)u_x + (gy + h_2)u_y \\ &= \lambda_n u, \end{aligned} \tag{3.1}$$

where $\lambda_n := an(n-1) + gn$.

**Definition 3.1** (*Krall and Sheffer* [3]). The differential equation (3.1) or the differential operator $L[\cdot]$ is admissible if $\lambda_m \neq \lambda_n$ for $m \neq n$ or equivalently $an + g \neq 0, n \geqslant 1$.

It is shown in [3] that Eq. (3.1) is admissible if and only if Eq. (3.1) has a unique monic PS as solutions.

Assume that $A = B = C = 0$. Then, Eq. (3.1) becomes

$$L[u] = (gx + h_1)u_x + (gy + h_2)u_y = gnu, \tag{3.2}$$

which can have a PS as solutions only when $g \neq 0$. Hence, by setting $x^* = gx + h_1$ and $y^* = gy + h_2$, Eq. (3.2) becomes

$$L[u] = xu_x + yu_y = nu,$$

which has a unique monic PS $\{x^{n-j}y^j\}_{n=0,j=0}^{\infty \ \ n}$ as solutions. However, the PS $\{x^{n-j}y^j\}_{n=0,j=0}^{\infty \ \ n}$ is a WOPS but cannot be an OPS since its canonical moment functional $\delta(x,y)$ is not quasi-definite (see Proposition 2.1). Therefore, from now on, we always assume that $|A| + |B| + |C| \neq 0$ and $|a| + |g| \neq 0$ in (3.1).

**Lemma 3.1.** *If Eq.* (3.1) *has a PS* $\{\Phi_n\}_{n=0}^{\infty}$ *as solutions, then the canonical moment functional* $\sigma$ *of* $\{\Phi_n\}_{n=0}^{\infty}$ *must satisfy*

$$L^*[\sigma] = 0, \tag{3.3}$$

$$\langle \sigma, D \rangle = \langle \sigma, E \rangle = \langle \sigma, A + xD \rangle = \langle \sigma, C + yE \rangle = \langle \sigma, B + yD \rangle = \langle \sigma, B + xE \rangle = 0, \tag{3.4}$$

*where* $L^*[\cdot]$ *is the formal Lagrange adjoint of* $L[\cdot]$ *given by*

$$L^*[u] := (Au)_{xx} + 2(Bu)_{xy} + (Cu)_{yy} - (Du)_x - (Eu)_y.$$

**Proof.** For any $m$ and $n \geqslant 0$

$$\langle L^*[\sigma], \phi_{mn} \rangle = \langle \sigma, L[\phi_{mn}] \rangle = \lambda_{m+n} \langle \sigma, \phi_{mn} \rangle = 0,$$

since $\lambda_0 = 0$ and $\langle \sigma, \phi_{mn} \rangle = 0$ for $m+n \geqslant 1$. Hence, $L^*[\sigma] = 0$, that is, for any polynomial $Q(x,y)$

$$\langle L^*[\sigma], Q \rangle = 0.$$

In particular, $\langle L^*[\sigma], x \rangle = \langle \sigma, L[x] \rangle = \langle \sigma, D \rangle = 0$ and similarly $\langle \sigma, E \rangle = 0$. We may assume that $\{\Phi_n\}_{n=0}^{\infty}$ is a monic PS. Then

$$D = \lambda_1 \phi_{10} \quad \text{and} \quad E = \lambda_1 \phi_{01},$$

so that $\langle \sigma, yD \rangle = \lambda_1 \langle \sigma, y\phi_{10} \rangle = \lambda_1 \langle \sigma, \phi_{10}\phi_{01} \rangle = \langle \sigma, xE \rangle$.

Now take $Q(x,y) = x^2$, $xy$, and $y^2$ to obtain

$$\langle L^*[\sigma], x^2 \rangle = \langle \sigma, L[x^2] \rangle = 2\langle \sigma, A + xD \rangle = 0,$$

$$\langle L^*[\sigma], xy \rangle = \langle \sigma, L[xy] \rangle = \langle \sigma, 2B + yD + xE \rangle = 0,$$

and

$$\langle L^*[\sigma], y^2 \rangle = \langle \sigma, L[y^2] \rangle = 2\langle \sigma, C + yE \rangle = 0.$$

Hence, $\langle \sigma, A + xD \rangle = \langle \sigma, C + yE \rangle = \langle \sigma, B + yD \rangle = \langle \sigma, B + xE \rangle = 0$. □

**Proposition 3.2.** *Assume that Eq.* (3.1) *has a PS* $\{\Phi_n\}_{n=0}^{\infty}$ *as solutions. If the canonical moment functional* $\sigma$ *of* $\{\Phi_n\}_{n=0}^{\infty}$ *satisfies*

$$\langle \sigma, x\phi_{mn} \rangle = \langle \sigma, y\phi_{mn} \rangle = 0, \quad m+n \geqslant 2, \tag{3.5}$$

*then*

$$M_1[\sigma] := (A\sigma)_x + (B\sigma)_y - D\sigma = 0 \quad \textit{and} \quad M_2[\sigma] := (B\sigma)_x + (C\sigma)_y - E\sigma = 0. \tag{3.6}$$

**Proof.** We may assume that $\{\Phi_n\}_{n=0}^{\infty}$ is a monic PS. Then for any polynomial $\psi(x,y)$

$$\begin{aligned}\lambda_{m+n}\langle \sigma, \phi_{mn}\psi \rangle &= \langle \sigma, L[\phi_{mn}]\psi \rangle = \langle L^*[\psi\sigma], \phi_{mn} \rangle \\ &= \langle [A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy}]\sigma + \psi_x[2(A\sigma)_x + 2(B\sigma)_y - D\sigma] \\ &\quad + \psi_y[2(B\sigma)_x + 2(C\sigma)_y - E\sigma] + \psi L^*[\sigma], \phi_{mn} \rangle \\ &= \langle [A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy}]\sigma + \psi_x[2(A\sigma)_x + 2(B\sigma)_y - D\sigma] \\ &\quad + \psi_y[2(B\sigma)_x + 2(C\sigma)_y - E\sigma], \phi_{mn} \rangle,\end{aligned}$$

since $L^*[\sigma] = 0$ by Lemma 3.1.

We now take $\psi(x,y) = x$ and $y$. Then we have, by (3.5), for $m+n \geqslant 2$ and $m+n = 0$

$$\langle 2(A\sigma)_x + 2(B\sigma)_y - D\sigma, \phi_{mn} \rangle = \lambda_{m+n}\langle \sigma, x\phi_{mn} \rangle = 0, \tag{3.7}$$

$$\langle 2(B\sigma)_x + 2(C\sigma)_y - E\sigma, \phi_{mn} \rangle = \lambda_{m+n}\langle \sigma, y\phi_{mn} \rangle = 0. \tag{3.8}$$

We also have by (3.5)

$$\langle D\sigma, \phi_{mn} \rangle = \langle E\sigma, \phi_{mn} \rangle = 0, \quad m+n \geqslant 2. \tag{3.9}$$

Hence, we have by (3.7)–(3.9)

$$\langle M_1[\sigma], \phi_{mn}\rangle = \langle M_2[\sigma], \phi_{mn}\rangle = 0, \quad m+n \geqslant 2.$$

On the other hand,

$$\langle M_1[\sigma], \phi_{00}\rangle = \langle (A\sigma)_x + (B\sigma)_y - D\sigma, 1\rangle = -\langle \sigma, D\rangle = 0$$

and

$$\langle M_2[\sigma], \phi_{00}\rangle = \langle (B\sigma)_x + (C\sigma)_y - E\sigma, 1\rangle = -\langle \sigma, E\rangle = 0,$$

by (3.4) so that in order to show (3.6) it only remains to show

$$\langle M_j[\sigma], \phi_{01}\rangle = \langle M_j[\sigma], \phi_{10}\rangle = 0, \quad j = 1, 2.$$

$$\langle M_1[\sigma], \phi_{10}\rangle = \langle (A\sigma)_x + (B\sigma)_y - D\sigma, \phi_{10}\rangle = -\langle \sigma, A + D\phi_{10}\rangle = -\langle \sigma, A + xD\rangle = 0,$$

$$\langle M_1[\sigma], \phi_{01}\rangle = \langle (A\sigma)_x + (B\sigma)_y - D\sigma, \phi_{01}\rangle = -\langle \sigma, B + D\phi_{01}\rangle = -\langle \sigma, B + yD\rangle = 0$$

by (3.4). Similarly, $\langle M_2[\sigma], \phi_{10}\rangle = \langle M_2[\sigma], \phi_{01}\rangle = 0$. □

We call $M_1[\sigma] = M_2[\sigma] = 0$ the moment equations for the Eq. (3.1).

**Corollary 3.3.** *If Eq. (3.1) has a WOPS $\{\Phi_n\}_{n=0}^{\infty}$ as solution, then the canonical moment functional $\sigma$ of $\{\Phi_n\}_{n=0}^{\infty}$ satisfies $M_1[\sigma] = M_2[\sigma] = L^*[\sigma] = 0$.*

**Proof.** Condition (3.5) is satisfied for any WOPS $\{\Phi_n\}_{n=0}^{\infty}$ so that $M_1[\sigma] = M_2[\sigma] = 0$ by Proposition 3.2. Finally, $L^*[\sigma] = 0$ follows from Lemma 3.1 or from the relation

$$L^*[\sigma] = (M_1[\sigma])_x + (M_2[\sigma])_y. \quad \square \tag{3.10}$$

In terms of moments of $\sigma$, we can express $M_1[\sigma] = 0$, $M_2[\sigma] = 0$, and $L^*[\sigma] = 0$ as

$$\begin{aligned} C_{mn} := -2\langle M_1[\sigma], x^m y^n\rangle &= 2\{a(m+n) + g\}\sigma_{m+1,n} \\ &+ (2d_1 m + e_2 n + 2h_1)\sigma_{mn} + d_2 n\sigma_{m+1,n-1} + 2f_1 m\sigma_{m-1,n} \\ &+ f_2 n\sigma_{m,n-1} + 2e_1 m\sigma_{m-1,n+1} = 0; \end{aligned} \tag{3.11}$$

$$\begin{aligned} B_{mn} := -2\langle M_2[\sigma], x^m y^n\rangle &= 2\{a(m+n) + g\}\sigma_{m,n+1} \\ &+ e_2 m\sigma_{m-1,n+1} + (d_2 m + 2e_3 n + 2h_2)\sigma_{mn} + f_2 m\sigma_{m-1,n} \\ &+ 2f_3 n\sigma_{m,n-1} + 2d_3 n\sigma_{m+1,n-1} = 0; \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} A_{mn} := \langle L^*[\sigma], x^m y^n\rangle &= \tfrac{1}{2} m C_{m-1,n} + \tfrac{1}{2} n B_{m,n-1} \\ &= \lambda_{m+n}\sigma_{mn} + m[d_1(m-1) + e_2 n + h_1]\sigma_{m-1,n} \end{aligned}$$

$$+n[d_2m+e_3(n-1)+h_2]\sigma_{m,n-1}$$
$$+f_1m(m-1)\sigma_{m-2,n}+f_2mn\sigma_{m-1,n-1}$$
$$+f_3n(n-1)\sigma_{m,n-2}+e_1m(m-1)\sigma_{m-2,n+1}$$
$$+d_3n(n-1)\sigma_{m+1,n-2}=0 \tag{3.13}$$

for $m$ and $n \geqslant 0$, where $\sigma_{mn}=0$ if $m$ or $n<0$.

The relation (3.10) is equivalent to (see [3, Lemma 4.1])

$$2A_{mn}=nB_{m,n-1}+mC_{m-1,n}\,(B_{0,-1}=C_{-1,0}=0),\ m \text{ and } n\geqslant 0.$$

Note that Proposition 3.2 gives a much simpler proof and some improvement of similar results in [3, 7]. To be precise, Krall and Sheffer (see [3, Theorem 4.3]) proved Corollary 3.3 in the form of (3.11)–(3.13) when Eq. (3.1) is admissible, using the formal generating series

$$G(x,y,s,t):=\sum_{m,n=0}^{\infty}\phi_{mn}(x,y)s^mt^n$$

of $\{\Phi_n\}_{n=0}^{\infty}$.

Littlejohn (see [7, p. 117]) found the relations (3.11)–(3.13) using a weak solution $\Lambda(x,y)$ to the weight equations $M_1[\Lambda]=0$ and $M_2[\Lambda]=0$ for $L[\cdot]$ when the Eq. (3.1) is admissible.

**Proposition 3.4.** *If Eq.* (3.1) *is admissible, then $L^*[\sigma]=0$ has a unique solution up to a constant factor.*

**Proof.** Let $\sigma$ be the canonical moment functional of any PS $\{\Phi_n\}_{n=0}^{\infty}$ of solutions to Eq. (3.1). Then, $L^*[\sigma]=0$, that is, $A_{mn}=0$ for $m$ and $n\geqslant 0$ by Lemma 3.1 and (3.13).

On the other hand, since $\lambda_{m+n}\neq 0$ for $m+n\geqslant 1$, Eq. (3.13) is uniquely solvable for $\sigma_{mn}$, $m+n\geqslant 1$, once $\sigma_{00}(\neq 0)$ is fixed. Hence, if $L^*[\tau]=0$ and $\tau_{00}=c$, then $\tau=c\sigma$. □

**Corollary 3.5.** *If Eq.* (3.1) *is admissible and has a WOPS as solutions, then the moment equations $M_1[\sigma]=0$ and $M_2[\sigma]=0$ have a unique solution up to a constant factor.*

**Proof.** It follows immediately from Corollary 3.3 and Proposition 3.4. □

The converses of Proposition 3.2 and Corollary 3.3 hold also at least when Eq. (3.1) is admissible as we shall see now.

**Lemma 3.6.** *Let $L[\cdot]$ be the differential operator as in* (3.1). *Then for any moment functional $\sigma$, the following statements are equivalent*:

(i) $M_1[\sigma]=M_2[\sigma]=0$;

(ii) $\sigma L[\cdot]$ *is formally symmetric on polynomials, that is,*

$$\langle L[P]\sigma,Q\rangle=\langle L[Q]\sigma,P\rangle,\quad P \text{ and } Q\in\mathcal{P}. \tag{3.14}$$

*Furthermore, if $L[P]=\lambda P$ and $L[Q]=\mu Q$, $\lambda \neq \mu$, then for any moment functional $\sigma$ satisfying $M_1[\sigma]=M_2[\sigma]=0$,*

$$\langle \sigma, PQ\rangle = 0.$$

**Proof.** Since $\langle L[P]\sigma, Q\rangle = \langle L^*[Q\sigma], P\rangle$, (3.14) is equivalent to

$$L^*[Q\sigma]=L[Q]\sigma, \quad Q(x)\in \mathscr{P}.$$

Since $L^*[Q\sigma]-L[Q]\sigma = 2\partial_x Q\cdot M_1[\sigma]+2\partial_y Q\cdot M_2[\sigma]+QL^*[\sigma]$, (3.14) is equivalent to

$$M_1[\sigma]=M_2[\sigma]=L^*[\sigma]=0, \text{ that is, } M_1[\sigma]=M_2[\sigma]=0.$$

Now, assume that $L[P]=\lambda P, L[Q]=\mu Q, \lambda\neq\mu$, and $M_1[\sigma]=M_2[\sigma]=0$. Then

$$(\lambda-\mu)\langle\sigma,PQ\rangle = \langle\sigma, L[P]Q\rangle - \langle\sigma, L[Q]P\rangle = 0,$$

by the first part of the lemma. Hence, $\langle\sigma,PQ\rangle=0$ since $\lambda-\mu\neq 0$. □

**Theorem 3.7** (cf. [3, Theorems 4.4 and 4.5]). *Let $\{\Phi_n\}_{n=0}^{\infty}$ be a PS satisfying an admissible Eq. (3.1) and $\sigma$ the canonical moment functional of $\{\Phi_n\}_{n=0}^{\infty}$. Then the following statements are equivalent*:

(i) $\{\Phi_n\}_{n=0}^{\infty}$ *is a WOPS relative to* $\sigma$;
(ii) $M_1[\sigma]=0$;
(iii) $M_2[\sigma]=0$;
(iv) $\langle\sigma, x\phi_{mn}\rangle = \langle\sigma, y\phi_{mn}\rangle = 0,\ m+n\geqslant 2$.

**Proof.** By Lemma 3.1 and (3.10), $L^*[\sigma]=(M_1[\sigma])_x+(M_2[\sigma])_y=0$, so that (ii) and (iii) are equivalent by Lemma 2.2 (i). (i) implies (iv) trivially and (iv) implies (ii) and (iii) by Proposition 3.2.

Finally, assume that (ii) holds. Then, we also have $M_2[\sigma]=0$, so that

$$\langle\sigma, \phi_{mn}\phi_{kl}\rangle = 0 \quad \text{if } m+n\neq k+l,$$

by Lemma 3.6 since $\lambda_{m+n}\neq\lambda_{k+l}$ when Eq. (3.1) is admissible and $m+n\neq k+l$. Hence, $\{\Phi_n\}_{n=0}^{\infty}$ is a WOPS relative to $\sigma$, that is, (i) holds. □

We finally give equivalent conditions for an OPS $\{\Phi_n\}_{n=0}^{\infty}$ to satisfy Eq. (3.1). For a PS $\{\Phi_n\}_{n=0}^{\infty}$, where

$$\phi_{n-j,j}(x,y)=\sum_{k=0}^{n} a_{jk}^{n} x^{n-k}y^{k} \text{ modulo } \mathscr{P}_{n-1}, \quad 0\leqslant j\leqslant n,$$

the matrix $A_n := [a_{jk}^n]_{j,k=0}^{n}$ is nonsingular. We then call $\{\mathbb{P}_n\}_{n=0}^{\infty}$, where $\mathbb{P}_n = A_n^{-1}\Phi_n$, $n\geqslant 0$, the normalization of $\{\Phi_n\}_{n=0}^{\infty}$. Note that if $\{\Phi_n\}_{n=0}^{\infty}$ is a PS (respectively, an OPS) satisfying Eq. (3.1), then $\{\mathbb{P}_n\}_{n=0}^{\infty}$ is a monic PS (respectively, a monic WOPS (but not necessarily an OPS)) satisfying Eq. (3.1).

**Theorem 3.8.** *Let $\{\Phi_n\}_{n=0}^{\infty}$ be an OPS relative to a quasi-definite moment functional $\sigma$ and $\{\mathbb{P}_n\}_{n=0}^{\infty}$ the normalization of $\{\Phi_n\}_{n=0}^{\infty}$. Then, the following statements are equivalent:*
(i) *$\{\Phi_n\}_{n=0}^{\infty}$ satisfy Eq. (3.1);*
(ii) $M_1[\sigma]=0$ and $M_2[\sigma]=0$;
(iii) *$\sigma L[\cdot]$ is formally symmetric on polynomials;*
(iv) *there are $(n+1)\times(k+1)$ matrices $F_k^n$ and $G_k^n$ for $k=n-1$, $n$, $n+1$ such that*

$$A\partial_x\Phi_n + B\partial_y\Phi_n = F_{n+1}^n\Phi_{n+1} + F_n^n\Phi_n + F_{n-1}^n\Phi_{n-1}, \quad n \geqslant 1, \tag{3.15}$$

$$B\partial_x\Phi_n + C\partial_y\Phi_n = G_{n+1}^n\Phi_{n+1} + G_n^n\Phi_n + G_{n-1}^n\Phi_{n-1}, \quad n \geqslant 1, \tag{3.16}$$

*and $\sigma$ satisfies*

$$\langle\sigma, P_{10}P_{01}\rangle\langle\sigma, A\rangle - \langle\sigma, P_{10}^2\rangle\langle\sigma, B\rangle = 0, \tag{3.17}$$

$$\langle\sigma, P_{01}^2\rangle\langle\sigma, B\rangle - \langle\sigma, P_{10}P_{01}\rangle\langle\sigma, C\rangle = 0, \tag{3.18}$$

$$\langle\sigma, P_{01}^2\rangle\langle\sigma, A\rangle - \langle\sigma, P_{10}^2\rangle\langle\sigma, C\rangle = 0. \tag{3.19}$$

**Proof.** (i) implies (ii) by Corollary 3.3.
(ii) and (iii) are equivalent by Lemma 3.6.
(ii)$\Rightarrow$(i): we have

$$L[\phi_{mn}] = \lambda_{m+n}\phi_{mn} + R(x,y) = \sum_{i+j=0}^{m+n} c_{ij}\phi_{ij}, \quad \deg(R) \leqslant m+n-1.$$

By Lemma 3.6, for any $0 \leqslant k+l < m+n$,

$$0 = \langle\sigma, \phi_{mn}L[\phi_{kl}]\rangle = \langle\sigma, L[\phi_{mn}]\phi_{kl}\rangle = \sum_{i+j=0}^{m+n} c_{ij}\langle\sigma, \phi_{ij}\phi_{kl}\rangle = \sum_{i+j=k+l} c_{ij}\langle\sigma, \phi_{ij}\phi_{kl}\rangle.$$

Since $\{\Phi_n\}_{n=0}^{\infty}$ is an OPS relative to $\sigma$, $H_{k+l} = [\langle\sigma, \phi_{ij}\phi_{kl}\rangle]_{i+j=k+l}$ is a nonsingular diagonal matrix so that $c_{ij}=0$ for any $i+j<m+n$. Hence,

$$L[\phi_{mn}] = \lambda_{m+n}\phi_{mn} + R(x,y) = \sum_{i+j=m+n} c_{ij}\phi_{ij},$$

so that $c_{mn}=\lambda_{m+n}$ and all other $c_{ij}=0$ since $\{\phi_{ij}\}_{i+j=m+n}$ are linearly independent modulo $\mathscr{P}_{m+n-1}$.
(i)$\Rightarrow$(iv): Since $\deg(A\partial_x\Phi_n + B\partial_y\Phi_n) \leqslant n+1$,

$$A\partial_x\Phi_n + B\partial_y\Phi_n = \sum_{j=0}^{n+1} F_j^n\Phi_j,$$

where $F_j^n$ are $(n+1)\times(j+1)$ matrices. Then

$$\begin{aligned} F_k^n H_k = F_k^n\langle\sigma, \Phi_k\Phi_k^{\mathrm{T}}\rangle &= \left\langle\sigma, \left(\sum_{j=0}^{n+1} F_j^n\Phi_j\right)\Phi_k^{\mathrm{T}}\right\rangle = \langle\sigma, (A\partial_x\Phi_n + B\partial_y\Phi_n)\Phi_k^{\mathrm{T}}\rangle \\ &= -\langle(A\sigma)_x + (B\sigma)_y, \Phi_n\Phi_k^{\mathrm{T}}\rangle - \langle\sigma, \Phi_n(A\partial_x\Phi_k^{\mathrm{T}} + B\partial_y\Phi_k^{\mathrm{T}})\rangle \\ &= -\langle\sigma, \Phi_n(D + A\partial_x + B\partial_y)\Phi_k^{\mathrm{T}}\rangle, \quad 0 \leqslant k \leqslant n+1, \end{aligned}$$

since $M_1[\sigma]=0$. Hence, $F_k^n=0$, $0\leqslant k\leqslant n-2$, since $\deg\{(D+A\partial_x+B\partial_y)\Phi_k^{\mathrm{T}}\}\leqslant k+1\leqslant n-1$ for $0\leqslant k\leqslant n-2$ and $|H_k|\neq 0$. Therefore, we have (3.15). Similarly, we also have (3.16).

On the other hand,

$$\langle M_1[\sigma],P_{10}\rangle=\langle (A\sigma)_x+(B\sigma)_y-D\sigma,P_{10}\rangle=-\langle\sigma,A\rangle-\lambda_1\langle\sigma,P_{10}^2\rangle=0$$

and

$$\langle M_1[\sigma],P_{01}\rangle=\langle (A\sigma)_x+(B\sigma)_y-D\sigma,P_{01}\rangle=-\langle\sigma,B\rangle-\lambda_1\langle\sigma,P_{10}P_{01}\rangle=0,$$

since $D=\lambda_1P_{10}$ and $E=\lambda_1P_{01}$ so that we have (3.17). Similarly, we also have (3.18) using $M_2$ instead of $M_1$. Finally, (3.19) follows from (cf. (3.4))

$$\langle\sigma,A\rangle=-\langle\sigma,xD\rangle=-\lambda_1\langle\sigma,P_{10}^2\rangle \quad\text{and}\quad \langle\sigma,C\rangle=-\langle\sigma,yE\rangle=-\lambda_1\langle\sigma,P_{01}^2\rangle.$$

(iv)$\Rightarrow$(ii): Since Eq. (3.15) implies

$$\begin{aligned}\langle (A\sigma)_x+(B\sigma)_y,\Phi_n\rangle &= -\langle\sigma,A\partial_x\Phi_n+B\partial_y\Phi_n\rangle\\ &= -\langle\sigma,F_{n+1}^n\Phi_{n+1}+F_n^n\Phi_n+F_{n-1}^n\Phi_{n-1}\rangle=0,\quad n\geqslant 2,\end{aligned}$$

there is a polynomial $D(x,y)$ of degree $\leqslant 1$ such that

$$(A\sigma)_x+(B\sigma)_y-D\sigma=0$$

by Lemma 2.2 (iii). We may write $D$ as

$$D=\alpha P_{10}+\beta P_{01}+\gamma \ (\alpha,\beta,\gamma \text{ are constants}).$$

Then

$$0=\langle (A\sigma)_x+(B\sigma)_y,1\rangle=\langle\sigma,D\rangle=\langle\sigma,\alpha P_{10}+\beta P_{01}+\gamma\rangle=\gamma\langle\sigma,1\rangle,$$

so that $\gamma=0$ and

$$\langle\sigma,DP_{10}\rangle=\alpha\langle\sigma,P_{10}^2\rangle+\beta\langle\sigma,P_{10}P_{01}\rangle=\langle (A\sigma)_x+(B\sigma)_y,P_{10}\rangle=-\langle\sigma,A\rangle,$$

$$\langle\sigma,DP_{01}\rangle=\alpha\langle\sigma,P_{10}P_{01}\rangle+\beta\langle\sigma,P_{01}^2\rangle=\langle (A\sigma)_x+(B\sigma)_y,P_{01}\rangle=-\langle\sigma,B\rangle.$$

Hence,

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=-|H_1|^{-1}\begin{bmatrix}\langle\sigma,P_{01}^2\rangle & -\langle\sigma,P_{10}P_{01}\rangle\\ -\langle\sigma,P_{10}P_{01}\rangle & \langle\sigma,P_{10}^2\rangle\end{bmatrix}\begin{bmatrix}\langle\sigma,A\rangle\\ \langle\sigma,B\rangle\end{bmatrix},$$

so that $\beta=0$ by (3.17) and so $D=\alpha P_{10}$.

Similarly, we can see that there is a polynomial $E=\beta P_{01}$ such that

$$(B\sigma)_x+(C\sigma)_y-E\sigma=0,$$

from (3.16) and (3.18). Then (3.19) implies $\alpha=\beta$. $\square$

Note that we may replace $\{\Phi_n\}_{n=0}^{\infty}$ by $\{\mathbb{P}_n\}_{n=0}^{\infty}$ in Theorem 3.8(i) and (iv).

In particular, the condition (iv) in Theorem 3.8 is the two-variable version of the well-known Al-Salam and Chihara [1] characterization of classical orthogonal polynomial in one variable. As in [1], we may call (3.15) and (3.16) the structure relations for an OPS $\{\Phi_n\}_{n=0}^{\infty}$ satisfying Eq. (3.1).

## 4. Differential equations belonging to the basic class

**Definition 4.1** (*Liskova* [8]). The differential operator $L[\cdot]$ in (3.1) belongs to the basic class if $A_y = C_x = 0$, that is,

$$A(x,y) = A(x) = ax^2 + d_1 x + f_1 \quad \text{and} \quad C(x,y) = C(y) = ay^2 + e_3 y + f_3.$$

If $L[\cdot]$ belongs to the basic class and $L[u] = \lambda u$, then $v = \partial_x^j \partial_y^k u$ ($j$ and $k \geqslant 0$) satisfies

$$\begin{aligned} &Av_{xx} + 2Bv_{xy} + Cv_{yy} + (D + jA_x + 2kB_y)v_x + (E + 2jB_x + kC_y)v_y \\ &\quad = (\lambda - jD_x - kE_y - \tfrac{1}{2}j(j-1)A_{xx} - 2jkB_{xy} - \tfrac{1}{2}k(k-1)C_{yy})v. \end{aligned} \tag{4.1}$$

In the following, we use the standard terminologies for orthogonal polynomials in one variable as in [4, 5].

**Proposition 4.1.** *Let* $\{\mathbb{P}_n\}_{n=0}^{\infty}$ *be the monic PS of solutions to the admissible Eq.* (3.1). *Then, the following statements are equivalent*:

(i) $A_y = 0$ (*respectively*, $C_x = 0$);
(ii) $P_{n0}(x,y) = P_{n0}(x)$ (*respectively*, $P_{0n}(x,y) = P_{0n}(y)$), $\quad n \geqslant 0$;
(iii) $P_{20}(x,y) = P_{20}(x)$ (*respectively*, $P_{02}(x,y) = P_{02}(y)$);
(iv) $\{P_{n0}(x,y)\}_{n=0}^{\infty}$ (*respectively*, $\{P_{0n}(x,y)\}_{n=0}^{\infty}$) *satisfy the equation*

$$Au_{xx} + Du_x = \lambda_n u (\text{ respectively, } Cu_{yy} + Eu_y = \lambda_n u); \tag{4.2}$$

(v) $P_{20}(x,y)$ (*respectively*, $P_{02}(x,y)$) *satisfies the equation* (4.2).

*In this case*, $\{P_{n0}(x)\}_{n=0}^{\infty}$ (*respectively*, $\{P_{0n}(y)\}_{n=0}^{\infty}$) *is a WOPS* (*in one variable*). *If moreover, the canonical moment functional* $\sigma$ *of* $\{\mathbb{P}_n\}_{n=0}^{\infty}$ *is positive-definite, then* $\{P_{n0}(x)\}_{n=0}^{\infty}$ (*respectively*, $\{P_{0n}(y)\}_{n=0}^{\infty}$) *is a positive-definite classical OPS.*

**Proof.** (i) $\Rightarrow$ (ii): Assume $A_y = 0$, that is, $A(x,y) = A(x) = ax^2 + d_1 x + f_1$. We will show that $P_{n0}(x,y) = P_{n0}(x)$ by induction on $n \geqslant 0$. For $n = 0$ and 1, it is trivial. Assume that $P_{k0}(x,y) = P_{k0}(x)$, $0 \leqslant k \leqslant n$, for some integer $n \geqslant 1$. Express $P_{n+1,0}(x,y)$ as

$$P_{n+1,0}(x,y) = x^{n+1} + \sum_{k=0}^{n} \mathbb{C}_k \mathbb{P}_k(x,y),$$

where $\mathbb{C}_k = (C_{k0}, \ldots, C_{0k})$. Then, we have from $L[P_{n+1,0}] = \lambda_{n+1} P_{n+1,0}$

$$(n+1)(d_1 n + h_1)x^n + f_1(n+1)nx^{n-1} + \sum_{k=0}^{n} (\lambda_k - \lambda_{n+1})\mathbb{C}_k \mathbb{P}_k = 0. \tag{4.3}$$

Since $P_{k0}(x,y) = P_{k0}(x)$, $0 \leqslant k \leqslant n$, we may express $(n+1)(d_1 n + h_1)x^n$ as

$$(n+1)(d_1 n + h_1)x^n = \sum_{k=0}^{n} \alpha_k P_{k0}(x).$$

Then (4.3) gives

$$\begin{aligned}&\alpha_n P_{n0}(x) + (\lambda_n - \lambda_{n+1})\mathbb{C}_n\mathbb{P}_n\\ &\quad = [\alpha_n + (\lambda_n - \lambda_{n+1})C_{n0}]P_{n0}(x) + (\lambda_n - \lambda_{n+1})\sum_{k=1}^{n} C_{n-k,k}P_{n-k,k}(x,y)\\ &\quad = 0 \text{ modulo } \mathscr{P}_{n-1}.\end{aligned}$$

Hence, $C_{n-k,k} = 0$, $1 \leqslant k \leqslant n$, since $\{P_{n-k,k}\}_{k=0}^{n}$ are linearly independent modulo $\mathscr{P}_{n-1}$ and $L[\cdot]$ is admissible. Then, Eq. (4.3) gives

$$\begin{aligned}&(n+1)(d_1 n + h_1)x^n + f_1(n+1)nx^{n-1} + (\lambda_n - \lambda_{n+1})C_{n0}P_{n0}(x)\\ &\quad +(\lambda_{n-1} - \lambda_{n+1})C_{n-1,0}P_{n-1,0}(x) + (\lambda_{n-1} - \lambda_{n+1})\sum_{k=1}^{n-1} C_{n-1-k,k}P_{n-1-k,k}(x,y)\\ &\quad = 0 \text{ modulo } \mathscr{P}_{n-2},\end{aligned}$$

of which the first four terms can be expressed as $\sum_{k=0}^{n} \beta_k P_{k,0}$. Hence,

$$\begin{aligned}&\beta_n P_{n0} + \beta_{n-1}P_{n-1,0}(x) + (\lambda_{n-1} - \lambda_{n+1})\sum_{k=1}^{n-1} C_{n-1-k,k}P_{n-1-k,k}(x,y)\\ &\quad = 0 \text{ modulo } \mathscr{P}_{n-2},\end{aligned}$$

so that $C_{n-1-k,k} = 0$, $1 \leqslant k \leqslant n-1$ since $P_{n0}(x)$ and $\{P_{n-1-k,k}\}_{k=0}^{n-1}$ are linearly independent modulo $\mathscr{P}_{n-2}$.

Continuing the same process, we obtain

$$C_{k-j,j} = 0, 1 \leqslant k \leqslant n \quad \text{and} \quad 1 \leqslant j \leqslant k$$

so that $P_{n+1,0}(x,y) = x^{n+1} + \sum_{k=0}^{n} C_{k,0}P_{k,0}(x)$.

(ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v): They are trivial.

(ii) $\Rightarrow$ (iv): If $P_{n0}(x,y) = P_{n0}(x)$, $n \geqslant 0$, then $(P_{n0})_y = 0$, $n \geqslant 0$ so that (iv) follows.

(iii) $\Rightarrow$ (i): If $P_{20}(x,y) = P_{20}(x)$, then $\lambda_2 P_{20}(x) = L[P_{20}] = 2A + D(P_{20})_x$ so that $A(x,y) = \frac{1}{2}[\lambda_2 P_{20} - D(P_{20})_x] = A(x)$.

(v) $\Rightarrow$ (iii): Let $P_{20} = x^2 + \alpha x + \beta y + \gamma$. Assume that $A(P_{20})_{xx} + D(P_{20})_x = \lambda_2 P_{20}$. Then

$$L[P_{20}] - A(P_{20})_{xx} - D(P_{20})_x = E(P_{20})_y = \beta E = 0,$$

so that $\beta = 0$ since $E(y) \neq 0$ when $L[\cdot]$ is admissible. Hence, we have (iii).

Now, assume that $A_y = 0$, i.e., $A(x,y) = A(x)$ and let $\sigma$ be the canonical moment functional of $\{\mathbb{P}_n\}_{n=0}^{\infty}$. Let $\tau$ be the restriction of $\sigma$ on the space of polynomials in $x$ only. Then $P_{n0}(x,y) = P_{n0}(x)$, $n \geqslant 0$ by (ii) so that $\langle \tau, P_{n0}\rangle = \langle \sigma, P_{n0}\rangle = \delta_{n0}$, $n \geqslant 0$, that is, $\tau$ is the canonical moment functional of $\{P_{n0}(x)\}_{n=0}^{\infty}$. Then by (iv)

$$0 = \lambda_n\langle \tau, P_{n0}\rangle = \langle \tau, AP_{n0}'' + DP_{n0}'\rangle = \langle D\tau - (A\tau)', P_{n0}'\rangle, \quad n \geqslant 0$$

so that $D\tau - (A\tau)' = 0$. Hence,

$$(\lambda_m - \lambda_n)\langle \tau, P_{m0}P_{n0}\rangle = \langle \tau, AW_{mn}' + DW_{mn}\rangle = \langle D\tau - (A\tau)', W_{mn}\rangle = 0,$$

for $m$ and $n \geqslant 0$, where $W_{mn}(x) = P'_{m0}(x)P_{n0}(x) - P_{m0}(x)P'_{n0}(x)$. Therefore, $\langle \tau, P_{m0}P_{n0} \rangle = 0$ for $m \neq n$, that is, $\{P_{n0}(x)\}_{n=0}^{\infty}$ is a WOPS relative to $\tau$ since $\lambda_m \neq \lambda_n$ for $m \neq n$.

If $\sigma$ is positive definite, then $\langle \tau, P_{n0}^2 \rangle = \langle \sigma, P_{n0}^2 \rangle > 0$, $n \geqslant 0$ so that $\{P_{n0}(x)\}_{n=0}^{\infty}$ is a positive-definite classical OPS satisfying

$$A(x)P''_{n0}(x) + D(x)P'_{n0}(x) = \lambda_n P_{n0}(x), \quad n \geqslant 0.$$

The proof for the case $C_x = 0$ is the same as for the case $A_y = 0$. □

**Corollary 4.2.** *Let $\{\mathbb{P}_n\}_{n=0}^{\infty}$ be the monic PS of solutions to the admissible Eq.* (3.1). *Then, the following statements are equivalent*:

(i) $A_y = C_x = 0$, *that is, $L[\cdot]$ belongs to the basic class*;
(ii) $P_{n0}(x,y) = P_{n0}(x)$ *and* $P_{0n}(x,y) = P_{0n}(y)$, $n \geqslant 0$;
(iii) $P_{20}(x,y) = P_{20}(x)$ *and* $P_{02}(x,y) = P_{02}(y)$;
(iv) $\{P_{n0}(x,y)\}_{n=0}^{\infty}$ *and* $\{P_{0n}(x,y)\}_{n=0}^{\infty}$ *satisfy*

$$A(P_{n0})_{xx} + D(P_{n0})_x = \lambda_n P_{n0} \quad \text{and} \quad C(P_{0n})_{yy} + E(P_{0n})_y = \lambda_n P_{0n}, \; n \geqslant 0; \tag{4.4}$$

(v) $P_{20}(x,y)$ *and* $P_{02}(x,y)$ *satisfy*

$$A(P_{20})_{xx} + D(P_{20})_x = \lambda_2 P_{20} \quad \text{and} \quad C(P_{02})_{yy} + E(P_{02})_y = \lambda_2 P_{02}.$$

*In this case, $\{P_{n0}(x)\}_{n=0}^{\infty}$ and $\{P_{0n}(y)\}_{n=0}^{\infty}$ are WOPS's (in one variable) and we may express $\{\mathbb{P}_n\}_{n=0}^{\infty}$ as*

$$P_{mn}(x,y) = P_{m0}(x)P_{0n}(y) + \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} C_{ij}^{(m,n)} P_{i0}(x)P_{0j}(y), \quad m \text{ and } n \geqslant 0. \tag{4.5}$$

*If, moreover, the canonical moment functional $\sigma$ of $\{\mathbb{P}_n\}_{n=0}^{\infty}$ is positive definite, then $\{P_{n0}(x)\}_{n=0}^{\infty}$ and $\{P_{0n}(y)\}_{n=0}^{\infty}$ are positive definite classical OPS's.*

**Proof.** All others except (4.5) follow from Proposition 4.1. Since each $P_{mn}(x,y)$ is monic and $\{P_{m0}(x)P_{0n}(y)\}_{m+n=0}^{\infty}$ is a monic PS, we have (4.5). □

**Lemma 4.3.** *Consider a second-order ordinary differential equation*

$$\alpha(x)y''(x) + \beta(x)y'(x) = \mu_n y, \tag{4.6}$$

*where $\alpha(x) = ax^2 + bx + c(\neq 0)$, $\beta(x) = dx + e$, and $\mu_n = an(n-1) + dn$.*

*Then, Eq.* (4.6) *has an OPS* (*respectively, a positive-definite OPS*) *as solutions if and only if*

(i) $s_n := an + d \neq 0$, $n \geqslant 0$;
(ii) $\alpha(-(t_n/s_{2n})) \neq 0$ (*respectively*, $(s_{n-1}/s_{2n-1}s_{2n+1})\alpha(-(t_n/s_{2n})) < 0$), $n \geqslant 0$,
*where $t_n = bn + e$ and $s_{-1} = 1$.*

**Proof.** See [5, Theorem 2.9] and [10, Theorem 2].

**Theorem 4.4.** *Let $\{\mathbb{P}_n\}_{n=0}^{\infty}$ be the monic PS of solutions to the admissible equation* (3.1).

(i) *If $A_y = 0$, then $\{P_{n0}(x)\}_{n=0}^{\infty}$ is a classical OPS (respectively, a positive-definite classical OPS) if and only if*

$$A(-(t_n/s_{2n})) \neq 0 \text{ (respectively } (s_{n-1}/s_{2n-1}s_{2n+1})A(-(t_n/s_{2n})) < 0), \quad n \geqslant 0. \tag{4.7}$$

(ii) *If $C_x = 0$, then $\{P_{0n}(y)\}_{n=0}^{\infty}$ is a classical OPS (respectively, a positive definite classical OPS) if and only if*

$$C(-(u_n/s_{2n})) \neq 0 \text{ (respectively } (s_{n-1}/s_{2n-1}s_{2n+1})C(-(u_n/s_{2n})) < 0), \quad n \geqslant 0. \tag{4.8}$$

*Here, $s_n = an + g$, $t_n = d_1 n + h_1$, $u_n = e_3 n + h_2$ for $n \geqslant 0$ and $s_{-1} = 1$.*

**Proof.** It follows immediately from Proposition 4.1 and Lemma 4.3 since $s_n = an + g \neq 0$, $n \geqslant 0$, when $L[\cdot]$ is admissible. □

When is the monic PS $\{\mathbb{P}_n\}_{n=0}^{\infty}$ of solutions to Eq. (3.1) the product of two monic PSs in one variable ?

For any two moment functionals $\tau$ and $\mu$ on the space of polynomials in one variable, we let $\tau \otimes \mu$ be the moment functional on $\mathscr{P}$ defined by

$$\langle \tau \otimes \mu, \phi(x)\psi(y) \rangle = \langle \tau, \phi \rangle \langle \mu, \psi \rangle$$

and linearity.

**Theorem 4.5.** *Let $\{\mathbb{P}_n\}_{n=0}^{\infty}$ be the monic PS of solutions to Eq.* (3.1) *with $A_y = C_x = B = 0$:*

$$L[u] = A(x)u_{xx} + C(y)u_{yy} + D(x)u_x + E(y)u_y = \lambda_n u \tag{4.9}$$

*and let $\sigma$ be the canonical moment functional of $\{\mathbb{P}_n\}_{n=0}^{\infty}$. Then*

(a) *$P_{n0}(x, y) = P_{n0}(x)$, $P_{0n}(x, y) = P_{0n}(y)$, and $P_{mn}(x, y) = P_{m0}(x)P_{0n}(y)$, $m$ and $n \geqslant 0$;*

(b) *$\{\mathbb{P}_n\}_{n=0}^{\infty}$ is a WOPS;*

(c) *the following statements are equivalent:*

  (i) *$\sigma$ is quasi-definite (respectively, positive definite);*

  (ii) *$\{\mathbb{P}_n\}_{n=0}^{\infty}$ is an OPS (respectively, a positive definite OPS);*

  (iii) *$\{P_{n0}(x)\}_{n=0}^{\infty}$ and $\{P_{0n}(y)\}_{n=0}^{\infty}$ are classical OPSs (respectively, positive definite classical OPSs).*

**Proof.** Since $a = 0$, $g \neq 0$ so that $\lambda_n = gn \neq 0$, $n \geqslant 1$ and $L[\cdot]$ is admissible. Hence, there is a unique monic PS $\{\mathbb{P}_n\}_{n=0}^{\infty}$ of solutions to Eq. (4.9).

(a) By Corollary 4.2, $\{P_{n0}(x, y) = P_{n0}(x)\}_{n=0}^{\infty}$ and $\{P_{0n}(x, y) = P_{0n}(y)\}_{n=0}^{\infty}$ are WOPSs satisfying the Eq. in (4.4). Then, $\{P_{n-k,0}(x)P_{0k}(y)\}_{n=0,k=0}^{\infty\ \ n}$ is also a monic PS satisfying Eq. (4.9), so that

$$P_{n-k,0}(x)P_{0k}(y) = P_{n-k,k}(x, y), \quad 0 \leqslant k \leqslant n,$$

since the admissible Eq. (4.9) has a unique monic PS as solutions.

(b) Let $\tau$ and $\mu$ be the restrictions of $\sigma$ on the space of polynomials of $x$ only and $y$ only, respectively. Then $\{P_{n0}(x)\}_{n=0}^{\infty}$ and $\{P_{0n}(y)\}_{n=0}^{\infty}$ are WOPSs relative to $\tau$ and $\mu$, respectively by Corollary 4.2. We also have $\sigma = \tau \otimes \mu$ since we have

$$\langle \tau \otimes \mu, P_{mn} \rangle = \langle \tau, P_{m0} \rangle \langle \mu, P_{0n} \rangle = \langle \sigma, P_{m0} \rangle \langle \sigma, P_{0n} \rangle = \delta_{m+n,0} = \langle \sigma, P_{mn} \rangle$$

for $m$ and $n \geqslant 0$. Hence,

$$\langle \sigma, P_{mn} x^k y^l \rangle = \langle \tau \otimes \mu, P_{mn} x^k y^l \rangle = \langle \tau, P_{m0} x^k \rangle \langle \mu, P_{0n} y^l \rangle = 0, \quad 0 \leqslant k + l < m + n,$$

so that $\{\mathbb{P}_n\}_{n=0}^{\infty}$ is a WOPS relative to $\sigma$.

(c) We have by (a) and (b)

$$H_n := \langle \sigma, \mathbb{P}_n \mathbb{P}_n^{\mathrm{T}} \rangle = [\langle \tau, P_{n-k,0}^2 \rangle \langle \mu, P_{0k}^2 \rangle \delta_{jk}]_{j,k=0}^n, \quad n \geqslant 0,$$

is a diagonal matrix. Hence, $\sigma$ is quasi-definite (respectively, positive definite) (cf. Proposition 2.1) if and only if

$$\langle \tau, P_{n0}^2 \rangle \quad \text{and} \quad \langle \mu, P_{0n}^2 \rangle, \quad n \geqslant 0,$$

are nonzero (respectively, positive). Therefore, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). □

We can now characterize completely Eq. (3.1) with $B = 0$, which has an OPS as solutions.

**Theorem 4.6.** *We assume $B = 0$ in Eq.* (3.1)*. Then, Eq.* (3.1) *has an OPS* (*respectively, a positive definite OPS*) *as solutions if and only if $L[\cdot]$ belongs to the basic class and*

$$g f_1 - d_1(d_1 n + h_1) \neq 0 \ (\textit{respectively}, < 0), \quad n \geqslant 0; \tag{4.10}$$

$$g f_3 - e_3(e_3 n + h_2) \neq 0 \ (\textit{respectively}, < 0), \quad n \geqslant 0. \tag{4.11}$$

*In this case, Eq.* (3.1) *has a monic OPS $\{\mathbb{P}_n\}_{n=0}^{\infty}$ as solutions, which is a product of two classical OPSs in one variable.*

**Proof.** Since $a = 0$, $g \neq 0$ so that $L[\cdot]$ is admissible and Eq. (3.1) has a monic PS $\{\mathbb{P}_n\}_{n=0}^{\infty}$ as solutions. Note that when $L[\cdot]$ belongs to the basic class, that is, $A(x) = d_1 x + f_1$ and $C(y) = e_3 y + f_3$, the conditions (4.7) and (4.8) are equivalent to the conditions (4.10) and (4.11) since $s_n = g$, $n \geqslant 0$.

Assume that $L[\cdot]$ belongs to the basic class and the Conditions (4.10) and (4.11) hold. Then, by Theorem 4.4, $\{P_{n0}(x)\}_{n=0}^{\infty}$ and $\{P_{0n}(y)\}_{n=0}^{\infty}$ are classical OPSs (respectively, positive definite classical OPSs). Then, by Theorem 4.5, $\{\mathbb{P}_n\}_{n=0}^{\infty}$ is a monic OPS (respectively, a monic positive definite OPS) which is a product of two classical OPSs $\{P_{n0}(x)\}_{n=0}^{\infty}$ and $\{P_{0n}(y)\}_{n=0}^{\infty}$.

Conversely, assume that Eq. (3.1) has an OPS (respectively, a positive definite OPS) $\{\Phi_n\}_{n=0}^{\infty}$ as solutions and let $\sigma$ be the canonical moment functional of $\{\Phi_n\}_{n=0}^{\infty}$. Since $L[\cdot]$ is admissible, we may assume that $D(x) = x$ and $E(y) = y$ by a linear change of variables. Hence, we have from (3.4) and $M_1[\sigma] = M_2[\sigma] = L^*[\sigma] = 0$ (cf. Corollary 3.3)

$$\sigma_{10} = \sigma_{01} = \sigma_{11} = 0; \qquad \sigma_{20} = -f_1, \qquad \sigma_{02} = -f_3$$

and

$$\langle M_1[\sigma], y^2\rangle = -\langle \sigma, y^2 D\rangle = -\sigma_{12} = 0,$$
$$\langle M_2[\sigma], x^2\rangle = -\langle \sigma, x^2 E\rangle = -\sigma_{21} = 0,$$
$$\langle L^*[\sigma], x^2 y\rangle = 2\langle \sigma, yA + xyD\rangle + \langle \sigma, x^2 E\rangle = 0,$$
$$\langle L^*[\sigma], xy^2\rangle = 2\langle \sigma, xC + xyE\rangle + \langle \sigma, y^2 D\rangle = 0,$$

so that

$$\sigma_{12} = \sigma_{21} = 0, \qquad d_3\sigma_{20} = -d_3 f_1 = 0, \qquad e_1\sigma_{02} = -e_1 f_3 = 0.$$

On the other hand, since $\sigma$ is quasi-definite,

$$\Delta_1 = \sigma_{20}\sigma_{02} = f_1 f_3 \neq 0,$$

so that $d_3 = e_1 = 0$, which means that $L[\cdot]$ belongs to the basic class.

Now, let $\{\mathbb{P}_n\}_{n=0}^{\infty}$ be the normalization of $\{\Phi_n\}_{n=0}^{\infty}$. Then $\{\mathbb{P}_n\}_{n=0}^{\infty}$ is a monic WOPS relative to $\sigma$ satisfying Eq. (4.9) and $\sigma$ is the canonical moment functional of $\{\mathbb{P}_n\}_{n=0}^{\infty}$. Since $\sigma$ is quasi-definite, condition (iii) in Theorem 4.5 holds so that we have (4.7) and (4.8) or equivalently (4.10) and (4.11) by Theorem 4.4. □

Theorem 4.6 implies that if Eq. (3.1) with $B=0$ has an OPS $\{\Phi_n\}_{n=0}^{\infty}$ as solutions, then the normalization $\{\mathbb{P}_n\}_{n=0}^{\infty}$ of $\{\Phi_n\}_{n=0}^{\infty}$ must be a product of two classical OPSs, which are either Laguerre or Hermite polynomials since $\deg(A) \leqslant 1$ and $\deg(C) \leqslant 1$.

**Example 4.1.** Consider the following equation:

$$(x+\alpha)u_{xx} + 2(y+1)u_{yy} + xu_x + yu_y = nu. \tag{4.12}$$

Krall and Sheffer [3] showed that Eq. (4.12) has an OPS as solutions. But, they might not recognize that Eq. (4.12) has a product of two monic PS's in one variable as solutions.

Since Eq. (4.12) is admissible, it always has a unique monic PS as solutions, which is a product of two monic PS's in one variable for any choice of $\alpha$. On the other hand, by Theorem 4.6, Eq. (4.12) has a monic OPS (which cannot be positive definite) as solutions if and only if $\alpha \neq 0, 1, 2, \ldots$. To be precise, by setting $x^* = -x - \alpha$ and $y^* = -\frac{1}{2}(y+1)$, Eq. (4.12) becomes

$$xu_{xx} + yu_{yy} + (-\alpha - x)u_x + (-\tfrac{1}{2} - y)u_y = -nu. \tag{4.13}$$

of which the monic PS of solutions is

$$\{\mathbb{P}_n^{(\alpha)}\}_{n=0}^{\infty} = \{L_{n-k}^{(-\alpha-1)}(x) L_k^{(-3/2)}(y)\}_{n=0,k=0}^{\infty\ \ n},$$

where $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ is the monic Laguerre polynomials of the order $\alpha$. Hence, $\{\mathbb{P}_n^{(\alpha)}\}_{n=0}^{\infty}$ is a monic OPS (but not a positive definite OPS) if and only if $\alpha \neq 0, 1, 2, \ldots$.

**Example 4.2.** Consider the following equation:

$$3yu_{xx} + 2u_{xy} - xu_x - yu_y = -nu. \tag{4.14}$$

Krall and Sheffer [3] showed that Eq. (4.14) has an OPS $\{\Phi_n\}_{n=0}^{\infty}$ as solutions. If $\{\mathbb{P}_n\}_{n=0}^{\infty}$ is the normalization of $\{\Phi_n\}_{n=0}^{\infty}$, then we have by Proposition 4.1

$$P_{0n}(x,y)=P_{0n}(y) \quad \text{and} \quad yP_{0n}'(y)=nP_{0n}(y), \quad n\geqslant 0$$

so that $P_{0n}(y)=y^n$, $n\geqslant 0$. Since $\{y^n\}_{n=0}^{\infty}$ is not an OPS, $\{\Phi_n\}_{n=0}^{\infty}$ cannot be a positive definite OPS.

As a converse to Theorem 4.5, we have:

**Theorem 4.7.** *Assume that Eq.* (3.1) *has a monic PS* $\{\mathbb{P}_n\}_{n=0}^{\infty}$ *as solutions, which is a product of two monic PSs in one variable. Then we have*:

(i) $A_y=C_x=0$, *that is,* $L[\cdot]$ *belongs to the basic class;*

(ii) *if* $a=0$, *then* $B=0$ *so that Eq.* (3.1) *must be of the form* (4.9) *and the conclusions* (a), (b), *and* (c) *of Theorem* 4.5 *hold;*

(iii) *if* $a\neq 0$, *then Eq.* (3.1) *must be of the form*

$$a(x-\alpha)^2u_{xx}+2a(x-\alpha)(y-\beta)u_{xy}+a(y-\beta)^2u_{yy} +g(x-\alpha)u_x+g(y-\beta)u_y=\lambda_n u \tag{4.15}$$

*for some constants* $\alpha$ *and* $\beta$.

**Proof.** Assume that $P_{mn}(x,y)=\phi_m(x)\psi_n(y)$, $m$ and $n\geqslant 0$, where $\{\phi_n(x)\}_{n=0}^{\infty}$ and $\{\psi_n(y)\}_{n=0}^{\infty}$ are monic PSs in one variable. Then $P_{n0}(x,y)=\phi_n(x)$ and $P_{0n}(x,y)=\psi_n(y)$, $n\geqslant 0$. Hence, $A_y=C_x=0$ by Proposition 4.1. Here, we note that we do not need the admissibility of $L[\cdot]$ in proving (iii) $\Rightarrow$ (i) in Proposition 4.1. Since $D=\lambda_1P_{10}$, $E=\lambda_1P_{01}$, and $L[P_{11}]=L[P_{10}P_{01}]=2B+DP_{01}+EP_{10}=\lambda_2P_{11}$,

$$B(x,y)=\tfrac{1}{2}(\lambda_2-2\lambda_1)P_{11}=aP_{11}. \tag{4.16}$$

Hence, if $a=0$, then $B=0$ so that (ii) follows from (i) and Theorem 4.5. We now assume $a\neq 0$. Since $P_{mn}(x,y)=P_{m0}(x)P_{0n}(y)$, we have from (4.4)

$$A(P_{mn})_{xx}+C(P_{mn})_{yy}+D(P_{mn})_x+E(P_{mn})_y=(\lambda_m+\lambda_n)P_{mn}. \tag{4.17}$$

Subtracting (4.17) from $L[P_{mn}]=\lambda_{m+n}P_{mn}$ and using (4.16), we have

$$B(P_{mn})_{xy}=aP_{11}(P_{mn})_{xy}=amnP_{mn}, \quad m \text{ and } n\geqslant 0.$$

Since $a\neq 0$, we have

$$P_{10}(x)P_{01}(y)P_{m0}'(x)P_{0n}'(y)=mnP_{m0}(x)P_{0n}(y), \quad m \text{ and } n\geqslant 0$$

so that

$$P_{10}(x)P_{m0}'(x)=mP_{m0}(x), \qquad P_{01}(y)P_{0n}'(y)=nP_{0n}(y), \qquad m \text{ and } n\geqslant 0.$$

We may express $P_{m0}(x)$ as $P_{m0}(x)=\sum_{j=0}^{m}a_j(P_{10}(x))^j$, where $a_m=1$. Then

$$P_{10}(x)P_{m0}'(x)-mP_{m0}(x)=\sum_{j=0}^{m}(j-m)a_j(P_{10}(x))^j=0.$$

Hence $a_j = 0$, $0 \leqslant j \leqslant m-1$ so that $P_{m0}(x) = (P_{10}(x))^m$, $m \geqslant 0$. Similarly, $P_{0n}(y) = (P_{01}(y))^n$, $n \geqslant 0$. Set $P_{10}(x) = x - \alpha$ and $P_{01}(y) = y - \beta$. Then

$$\begin{aligned}
&L[P_{10}] = D(P_{10})_x = D = \lambda_1 P_{10};\\
&L[P_{01}] = E(P_{01})_y = E = \lambda_1 P_{01};\\
&L[P_{20}] = 2A + D(P_{20})_x = 2A + 2DP_{10} = \lambda_2 P_{20} = \lambda_2 (P_{10})^2;\\
&L[P_{11}] = 2B + D(P_{11})_x + E(P_{11})_y = 2B + DP_{01} + EP_{10} = \lambda_2 P_{11} = \lambda_2 P_{10} P_{01};\\
&L[P_{02}] = 2C + E(P_{02})_y = 2C + 2EP_{01} = \lambda_2 P_{02} = \lambda_2 (P_{01})^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&A(x) = \tfrac{1}{2}(\lambda_2 - 2\lambda_1)(P_{10})^2 = a(x-\alpha)^2,\\
&B(x,y) = \tfrac{1}{2}(\lambda_2 - 2\lambda_1)P_{10}P_{01} = a(x-\alpha)(y-\beta),\\
&C(y) = \tfrac{1}{2}(\lambda_2 - 2\lambda_1)(P_{01})^2 = a(y-\beta)^2,\\
&D(x) = \lambda_1 P_{10} = g(x-\alpha),\\
&E(y) = \lambda_1 P_{01} = g(y-\beta),
\end{aligned}$$

so that (4.15) follows. □

We now claim that Eq. (4.15) cannot have an OPS as solutions. By a linear change of variables, we may transform Eq. (4.15) into

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} + g u_x + g u_y = \lambda_n u. \tag{4.18}$$

Assume that Eq. (4.18) has an OPS $\{\Phi_n\}_{n=0}^{\infty}$ relative to $\sigma$ as solutions. Then $L^*[\sigma] = 0$ by Lemma 3.1 so that by (3.13)

$$A_{mn} = \lambda_{m+n}\sigma_{mn} = (m+n)(m+n-1+g)\sigma_{mn} = 0, \quad m \text{ and } n \geqslant 0.$$

Hence, $\sigma_{mn} = 0$ for $m+n \neq 0$ and $m+n > g-1$ so that $\Delta_n = 0$ for $n > \max(0, g-1)$, which contradicts the quasi-definiteness of $\sigma$. Hence, Eq. (4.18) or equivalently Eq. (4.15) cannot have an OPS as solutions.

Therefore, from Theorems 4.5 and 4.7, we obtain: if Eq. (3.1) has a monic OPS $\{\mathbb{P}_n\}_{n=0}^{\infty}$ as solutions, which is a product of two monic PSs in one variable, then Eq. (3.1) must be of the form (4.9) and $\{\mathbb{P}_n\}_{n=0}^{\infty}$ must be a product of two classical OPSs in one variable.

It is well known (see, for example, [4, 10]) that classical OPSs (in one variable) are the only OPSs such that their derivatives are also orthogonal. In particular, if $\{\phi_n(x)\}_{n=0}^{\infty}$ is a classical OPS satisfying Eq. (4.6), then $\{\phi_n'(x)\}_{n=1}^{\infty}$ is also a classical OPS satisfying the same type of the equation as (4.6).

We finally consider two-variable analogue of this property. Assume that Eq. (3.1) is admissible and has an OPS $\{\Phi_n\}_{n=0}^{\infty}$ as solutions. Let $\sigma$ be the canonical moment functional of $\{\Phi_n\}_{n=0}^{\infty}$. Then, we may assume that $D(x)=x$ and $E(y)=y$ so that we have by (3.4):

$$\langle\sigma,D\rangle=\sigma_{10}=0,\qquad \langle\sigma,E\rangle=\sigma_{01}=0,$$
$$\langle\sigma,A+xD\rangle=(a+1)\sigma_{20}+f_1=0,$$
$$\langle\sigma,B+yD\rangle=(a+1)\sigma_{11}+\tfrac{1}{2}f_2=0,$$
$$\langle\sigma,C+yE\rangle=(a+1)\sigma_{02}+f_3=0.$$

Therefore, $\Delta_1=\sigma_{20}\sigma_{02}-(\sigma_{11})^2=(1/(a+1)^2)(f_1f_3-\frac{1}{4}(f_2)^2)\neq 0$. In particular, $B^2-AC\neq 0$ since $\frac{1}{4}(f_2)^2-f_1f_3$ is the constant term of $B^2-AC$.

We now consider Eq. (3.1), which is admissible and belongs to the basic class

$$L[u]=A(x)u_{xx}+2B(x,y)u_{xy}+C(y)u_{yy}+D(x)u_x+E(y)u_y=\lambda_n u. \tag{4.19}$$

Let $\{\mathbb{P}_n\}_{n=0}^{\infty}$ be the unique monic PS of solutions to Eq. (4.19). We know by Corollary 4.2 that $P_{n0}(x,y)=P_{n0}(x)$ and $P_{0n}(x,y)=P_{0n}(y)$, $n\geqslant 0$. Set

$$P^{(x)}_{n-j,j}=\frac{1}{n-j+1}(P_{n-j+1,j})_x,\quad 0\leqslant j\leqslant n,$$
$$P^{(y)}_{n-j,j}=\frac{1}{j+1}(P_{n-j,j+1})_y,\quad 0\leqslant j\leqslant n.$$

Then, $\{\mathbb{P}^{(x)}_n\}_{n=0}^{\infty}=\{P^{(x)}_{n-j,j}\}_{n=0,j=0}^{\infty\ \ n}$ and $\{\mathbb{P}^{(y)}_n\}_{n=0}^{\infty}=\{P^{(y)}_{n-j,j}\}_{n=0,j=0}^{\infty\ \ n}$ are monic PS's satisfying (cf. (4.1))

$$L^{(1)}[u]=Au_{xx}+2Bu_{xy}+Cu_{yy}+(D+A_x)u_x+(E+2B_x)u_y$$
$$=n[a(n+1)+g]u, \tag{4.20}$$

$$L^{(2)}[u]=Au_{xx}+2Bu_{xy}+Cu_{yy}+(D+2B_y)u_x+(E+C_y)u_y$$
$$=n[a(n+1)+g]u, \tag{4.21}$$

respectively. Note that Eqs. (4.20) and (4.21) are also admissible.

**Theorem 4.8.** *Assume that Eq.* (4.19) *is admissible and has an OPS* $\{\Phi_n\}_{n=0}^{\infty}$ *as solutions. Let* $\{\mathbb{P}_n\}_{n=0}^{\infty}$ *be the normalization of* $\{\Phi_n\}_{n=0}^{\infty}$ *and* $\sigma$ *the canonical moment functional of* $\{\Phi_n\}_{n=0}^{\infty}$.

(i) *If* $\{\mathbb{P}^{(x)}_n\}_{n=0}^{\infty}$ *is a WOPS relative to* $f(x,y)\sigma$ *for some polynomial* $f(x,y)(\neq 0)$, *then* $f$ *must satisfy*

$$Af_x+Bf_y-A_xf=0; \tag{4.22}$$
$$Bf_x+Cf_y-2B_xf=0. \tag{4.23}$$

*Moreover, A, B, and C must satisfy the compatibility condition*

$$\frac{\partial}{\partial y}\left[\frac{2BB_x-A_xC}{B^2-AC}\right]=\frac{\partial}{\partial x}\left[\frac{A_xB-2AB_x}{B^2-AC}\right]. \tag{4.24}$$

*Conversely,*

(ii) *if Eqs.* (4.22) *and* (4.23) *have a nonzero polynomial solution* $f$, *then* $\{\mathbb{P}^{(x)}_n\}_{n=0}^{\infty}$ *is a monic WOPS relative to* $f\sigma$.

**Proof.** We first note that by Theorem 3.8, $\sigma$ satisfies

$$M_1[\sigma]=(A\sigma)_x+(B\sigma)_y-D\sigma=0 \quad \text{and} \quad M_2[\sigma]=(B\sigma)_x+(C\sigma)_y-E\sigma=0.$$

(i) Assume that $\{\mathbb{P}_n^{(x)}\}_{n=0}^{\infty}$ is a WOPS relative to $f(x,y)\sigma$ for some polynomial $f\neq 0$. Then by Theorem 3.7,

$$M_1[f\sigma]=(Af\sigma)_x+(Bf\sigma)_y-(D+A_x)f\sigma=0, \tag{4.25}$$

$$M_2[f\sigma]=(Bf\sigma)_x+(Cf\sigma)_y-(E+2B_x)f\sigma=0. \tag{4.26}$$

Since $M_1[f\sigma]=f[(A\sigma)_x+(B\sigma)_y-D\sigma]+(Af_x+Bf_y-A_xf)\sigma=(Af_x+Bf_y-A_xf)\sigma=0$ and $\sigma$ is quasi-definite, we have (4.22). Similarly, we have (4.23) from $M_2[f\sigma]=0$. We may solve Eqs. (4.22) and (4.23) for $f_x$ and $f_y$ as

$$(B^2-AC)f_x=(2BB_x-A_xC)f \quad \text{and} \quad (B^2-AC)f_y=(A_xB-2AB_x)f,$$

from which (4.24) follows since $B^2-AC\neq 0$.

(ii) Assume that $f(\neq 0)$ satisfies Eqs. (4.22) and (4.23). Then $f\sigma$ is a nonzero moment functional and satisfies Eqs. (4.25) and (4.26). Let $\tau$ be the canonical moment functional of $\{\mathbb{P}_n^{(x)}\}_{n=0}^{\infty}$. Then, both $f\sigma$ and $\tau$ satisfy

$$L^{(1)^*}[f\sigma]=L^{(1)^*}[\tau]=0$$

by (3.10) for $f\sigma$ and by Lemma 3.1 for $\tau$. Since $L^{(1)}[\cdot]$ is admissible, we must have

$$f\sigma=c\tau,$$

where $c=\langle f\sigma,1\rangle=\langle\sigma,f\rangle\neq 0$ by Proposition 3.4. Hence, $\tau$ also satisfies (4.25) and (4.26). Therefore, $\{\mathbb{P}_n^{(x)}\}_{n=0}^{\infty}$ is a monic WOPS relative to $f\sigma$ by Theorem 3.7. $\square$

We can also state Theorem 4.8 for $\{\mathbb{P}_n^{(y)}\}_{n=0}^{\infty}$ instead of $\{\mathbb{P}_n^{(x)}\}_{n=0}^{\infty}$, for which Eqs. (4.22)–(4.24) must be replaced by

$$Af_x+Bf_y-2B_yf=0, \tag{4.27}$$

$$Bf_x+Cf_y-C_yf=0, \tag{4.28}$$

and

$$\frac{\partial}{\partial y}\left[\frac{BC_y-2B_yC}{B^2-AC}\right]=\frac{\partial}{\partial x}\left[\frac{2BB_y-AC_y}{B^2-AC}\right].$$

It is easy to see that when the compatibility condition (4.24) is satisfied, Eqs. (4.22) and (4.23) always have a nonzero solution $f(x,y)$, which however may or may not be a polynomial.

When $B=0$, $f=A(x)$ or $f=C(y)$ satisfies (4.22), (4.23) or (4.27), (4.28), respectively. Moreover, it is easy to see that if $\{\mathbb{P}_n\}_{n=0}^{\infty}$ is a monic OPS relative to $\sigma$ satisfying Eq. (4.9) so that $\{\mathbb{P}_n\}_{n=0}^{\infty}$ is a product of two classical OPSs (cf. Theorem 4.5), then $\{\mathbb{P}_n^{(x)}\}_{n=0}^{\infty}$ and $\{\mathbb{P}_n^{(y)}\}_{n=0}^{\infty}$ are also monic OPSs relative to $A(x)\sigma$ and $C(y)\sigma$ satisfying (4.20) and (4.21), respectively.

**Example 4.3.** Consider the following equation:

$$(x^2-1)u_{xx}+2xyu_{xy}+(y^2-1)u_{yy}+g(xu_x+yu_y)=n(n-1+g)u \quad (g>1), \tag{4.29}$$

which is known to have an OPS, called circle polynomials, as solutions (cf. [3]). Let $\{\mathbb{P}_n\}_{n=0}^{\infty}$ be the monic polynomial solutions to Eq. (4.29). Then, $\{\mathbb{P}_n\}_{n=0}^{\infty}$ is orthogonal on $\mathscr{D}=\{(x,y)|x^2+y^2<1\}$ relative to the weight function $w(x,y)=(1-x^2-y^2)^{(g-3)/2}$. In this case, $f(x,y)=1-x^2-y^2$ satisfies both (4.22), (4.23) and (4.27), (4.28) so that by Theorem 4.8, $\{\mathbb{P}_n^{(x)}\}_{n=0}^{\infty}$ and $\{\mathbb{P}_n^{(y)}\}_{n=0}^{\infty}$ are also orthogonal relative to $(1-x^2-y^2)^{(g-1)/2}$ on $\mathscr{D}$.

We finally note that Theorem 4.8 remains to hold for $\{\mathbb{P}_n^{(x)}\}_{n=0}^{\infty}$ or $\{\mathbb{P}_n^{(y)}\}_{n=0}^{\infty}$, respectively, when either $C_x=0$, but $A_y\neq 0$ or $A_y=0$, but $C_x\neq 0$. For example, if we consider Eq. (4.14), where $C=0$, but $A_y=3\neq 0$, then Eqs. (4.22) and (4.23) have $f(x,y)=1$ as a nonzero solution. In fact, if $\{\mathbb{P}_n\}_{n=0}^{\infty}$ is the monic PS of solutions to Eq. (4.14), then $\{\mathbb{P}_n^{(x)}\}_{n=0}^{\infty}$ is also a monic PS satisfying the Eq. (4.20), which in this case, is the same as (4.14) so that $\{\mathbb{P}_n^{(x)}\}_{n=0}^{\infty}=\{\mathbb{P}_n\}_{n=0}^{\infty}$.

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