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Further Considerations on the Number of Limit Cycles of Vector Fields of the Form $X(v) = Av + f(v)\,Bv$

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In Gasull, Llibre, and Sotomayor. (*J. Differential Equations*, in press) we studied the number of limit cycles of planar vector fields as in the title. The case where the origin is a node with different eigenvalues, which then resisted our analysis, is solved in this paper. © 1987 Academic Press, Inc.

1. INTRODUCTION

In [3] we studied vector fields of the form

$$X(v) = Av + f(v)\,Bv, \tag{1}$$

where A and B are 2×2 matrices, $\det A \neq 0$ and $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a smooth real function such that its expression in polar coordinates is $f(r\cos\theta, r\sin\theta) = r^D \bar{f}(\theta)$ with $D \geqslant 1$. Roughly, we shall say that f is a *homogeneous function of degree* $\mathscr{D}$. There is one hypothesis for the matrices A and B. This hypothesis states that $(JB)_s$ and $(B^tJA)_s$ are definite and have the same sign (for a 2×2 matrix C let C^t denote the transpose of C, $C_s = (C + C^t)/2$ and $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$). When the matrices A and B satisfy this property we shall say that system (1) satisfies *hypothesis* H_1.

We shall say that f is *indefinite* if f takes both positive and negative values.

For vector fields (1) we described in [3] their phase portrait, determin-

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ing the number of their limit cycles, except when f is indefinite and the origin is a node with two different eigenvalues.

In this paper, using techniques learned from [1], we will improve Theorem D of [3], as follows:

THEOREM D′. *Assume that the vector field* (1) *satisfies hypothesis* H_1, *that* f *is an indefinite homogeneous function, and the origin is a node. Then the origin is the unique rest point, it is an attractor, infinity is a repellor and* (1) *has either two limit cycles* (*the inner one being unstable and the* outer *one being stable*), *one limit cycle* (*which is semistable*), *or no limit cycles.*

2. Preliminary Results

Let $\langle u, v\rangle$ denote the usual inner product of $u, v \in \mathbb{R}^2$.

In polar coordinates (r, θ), defined by $x = r^{1/D}\cos\theta$, $y = r^{1/D}\sin\theta$, the expression of the vector field (1) when f is homogeneous of degree D is given by $V(\partial/\partial r) + U(\partial/\partial\theta)$ where

$$V = rDQ' + r^2\bar{f}DQ, \qquad U = R' + r\bar{f}R, \tag{2}$$

and $Q' = Q'(v) = \langle (A)_s v, v\rangle$, $Q = Q(v) = \langle (B)_s v, v\rangle$, $R' = R'(v) = -\langle (JA)_s v, v\rangle$, $R = R(v) = -\langle (JB)_s v, v\rangle$, where $v = (\cos\theta, \sin\theta)$.

The variables ρ, θ were already used in [1, 4, 5] to obtain a simple expression for (2).

LEMMA 1. *In the variables* $\rho = r/(R' + r\bar{f}R)$, $\theta = \theta$ *the expression of the differential system associated to the vector field* (2) *is given by*

$$d\rho/d\theta = A(\theta)\,\rho^3 + B(\theta)\,\rho^2 + C(\theta)\,\rho, \tag{3}$$

where $A(\theta) = D(Q'R - R'Q)\,R\bar{f}^2/R'$, $B(\theta) = D\bar{f}Q - (\bar{f}R)_\theta + \bar{f}R(R'_\theta - 2DQ')/R'$, $C(\theta) = (DQ' - R'_\theta)/R'$.

Remark 2. Note that if the origin of (1) is a focus or a linear center, by Lemmas 2.4(ii) and 2.5 of [3], R' does not vanish. Then, by Lemma 4.4 of [3] the change of variables $\rho = r/(R' + r\bar{f}R)$, $\theta = \theta$ is a diffeomorphism in the region of the plane (r, θ) in which limit cycles can exist. Furthermore, the functions A, B, and C are continuous. If the origin of (1) is a node then R' vanishes. So the functions A, B, and C are not continuous and the differential equation (3) has some points at which it is not well defined.

LEMMA 3. *Assume that the vector field* (1) *satisfies hypothesis* H_1, *and* f *is an indefinite homogeneous function. Then* X *has at most two limit cycles.*

Proof. By Lemma 4.4 of [3] all limit cycles of (1) are integral curves $r(\theta)$ of the vector field (2) such that $r(0)=r(2\pi)$.

Assume that vector field (2) has three limit cycles, $r_1(\theta)>r_2(\theta)>r_3(\theta)>0$. In the variables given by Lemma 1, they write $\rho_1(\theta)$, $\rho_2(\theta)$, and $\rho_3(\theta)$. Set $\rho_4(\theta)\equiv 0$. These four curves satisfy system (3). By making calculations inspired in [7, p. 103], we obtain

$$\frac{\dot\rho_1-\dot\rho_2}{\rho_1-\rho_2}-\frac{\dot\rho_1-\dot\rho_3}{\rho_1-\rho_3}-\frac{\dot\rho_4-\dot\rho_2}{\rho_4-\rho_2}+\frac{\dot\rho_4-\dot\rho_3}{\rho_4-\rho_3}=A(\theta)(\rho_1-\rho_4)(\rho_2-\rho_3),$$

and so

$$\ln[((\rho_1-\rho_2)(\rho_4-\rho_3))/((\rho_1-\rho_3)(\rho_4-\rho_2))]]_0^{2\pi} = \int_0^{2\pi} A(\theta)(\rho_1-\rho_4)(\rho_2-\rho_3)\,d\theta. \tag{4}$$

From $\rho_i-\rho_j=R'(r_i-r_j)/((R'+r_i\bar f R)(R'+r_j\bar f R))$, where $r_4\equiv 0$ we obtain that

$$\ln[((\rho_1-\rho_2)(\rho_4-\rho_3))/((\rho_1-\rho_3)(\rho_4-\rho_2))]]_0^{2\pi} = \ln[(r_3(r_1-r_2))/(r_2(r_1-r_3))]]_0^{2\pi}=0, \tag{5}$$

since the functions r_1, r_2, and r_3 are 2π-periodic. Nevertheless $\int_0^{2\pi} A(\theta)(\rho_1-\rho_4)(\rho_2-\rho_3)\,d\theta=\int_0^{2\pi} H(\theta)\,d\theta$, where

$$H(\theta)=[D\bar f^2(Q'R-R'Q)Rr_1(r_2-r_3)]/[(R'+r_1\bar f R)(R'+r_2\bar f R)(R'+r_3\bar f R)].$$

By hypothesis H_1 and Lemmas 2.6 and 4.4 of [3], $H(\theta)$ does not change sign and is a continuous function. This contradiction with (4) and (5) implies that system (1) has at most two limit cycles. ▌

The idea for the above proof was inspired in [1].

3. Proof of Theorem D'

By Lemma 2.3 of [3] the origin is a stable node and by Theorem 3.3 of [3] infinity is a repellor. Then by Lemma 3, system (1) can have either

(i) two limit cycles, the inner one being unstable and the outer one being stable,

(ii) one semistable limit cycle,

(iii) two semistable limit cycles, or

(iv) no limit cycles.

Examples of (i), (ii), and (iv) were given in Propositions 6.1, 6.2, 6.3 and Remark 6.4 of [3]. So we must show that (iii) is not possible. Assume that we have a system of type (1), under the hypotheses of Theorem D′, and with two semistable limit cycles. By Proposition 6.2 of [3] we only must consider the cases in which either A has two different eigenvalues, or A has a unique eigenvalue but A does not diagonalize. By Lemma 2.4 of [3] we can assume that either $A = A_3 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ or $A = A_2 = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, respectively, where $a < b < 0$. Take the family of differential equations

$$\dot{x} = ax + (1 - a\varepsilon)\, y + f(x, y)(B_1 - \varepsilon B_2) = X_\varepsilon(x, y),$$
$$\dot{y} = ay + f(x, y)\, B_2 = Y(x, y),$$

where $B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ which is such that $\dot{x} = X_0(x, y)$, $\dot{y} = Y(x, y)$ is the system with $A = A_2$ and with two semistable limit cycles. If $\theta(x, y, \varepsilon)$ denotes the angle of the vector field (X_ε, Y) with the x axis in the phase plane, given by $\theta(x, y, \varepsilon) = \tan^{-1}(Y/X_\varepsilon)$, we have

$$\begin{aligned} \partial\theta/\partial\varepsilon &= [X_\varepsilon(\partial Y/\partial\varepsilon) - Y(\partial X_\varepsilon/\partial\varepsilon)]/(X_\varepsilon^2 + Y^2) \\ &= (ay + f(x, y)\, B_2)^2/(X_\varepsilon^2 + Y^2) \geqslant 0. \end{aligned}$$

By using the results on rotated vector fields given in [2, 6] we obtain, taking $|\varepsilon|$ sufficiently small and with the suitable sign, one example with at least four limit cycles and satisfying all the hypotheses of Lemma 3, which is impossible.

The same works for the case $A = A_3$, taking for instance the family $\dot{x} = ax + f(x, y)\, B_1$, $\dot{y} = by + \varepsilon ax + f(x, y)(B_2 + \varepsilon B_1)$. ∎

4. Final Remarks

Note that in Theorem D′ we do not assert that the limit cycles are hyperbolic (when there are two). We do not know if this happens in the general case. When $A = A_2$ and f is homogeneous in the usual sense (i.e., $f(kx, ky) = k^D f(x, y)$) we proved it in Proposition 6.2(ii) of [3]).

The reader is referred to [3] for general references to the works of C. Chicone (preprint, Univ. of Missouri, 1984), W. Coppel (preprint, Austr. Nac. Univ. 1985), and D. Koditschek and K. Narendra (*J. Differential Equations* **54** (1984)), concerning vector fields (1) and related topics.

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