Silverman's Game on Discrete Sets

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ABSTRACT

In a symmetric Silverman game each of the two players chooses a number from a set $S \subset (0,\infty)$. The player with the larger number wins 1, unless the larger is at least T times as large as the other, in which case he loses ν. Such games are investigated for discrete S, for $T>1$ and $\nu>0$. Except for ν too near zero, where there is a proliferation of cases, explicit solutions are obtained. These are of finite type and, except at certain boundary cases, unique.

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1. INTRODUCTION

The symmetric Silverman game (S,T,ν) is defined as follows. Let S be a set of positive real numbers, and let $T>1$, $\nu>0$. Each of two players independently selects an element of S. The player with the larger number wins 1 from his opponent, unless his number is at least T times as large as the other, in which case he must pay the opponent ν. Equal numbers draw. The parameter T is called the threshold, and ν is called the penalty. A version of this game on a special discrete set S (see the Appendix) is described in [3, p. 212]. David Silverman [10] suggested analyzing the game on general sets S. The case where S is an open interval was examined by

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Evans [1], who showed that optimal strategies exist (and gave them) only for certain isolated values of ν, and then only when the interval is sufficiently large. An analogous family of games has been examined by Heuer [5].

The nonsymmetric version, where player I chooses from S_1 and player II from S_2, has been investigated by Heuer [4] and Heuer and Rieder [7]. Solutions are obtained for all disjoint discrete sets S_1 and S_2 for all T when $\nu \geqslant 1$, with partial results in other cases. In [6], Heuer shows that for $\nu \geqslant 1$, Silverman games may be reduced to games on initial segments of the strategy sets, and therefore to finite games when the strategy sets are discrete.

In the present paper we take S to be a discrete set: $S = \{c_1, c_2, c_3, \dots\}$, where $0 < c_1 < c_2 < \cdots$, and S is either finite or unbounded. For positive integers n, let $\nu_n = 2\cos[\pi/(2n+3)] - 1$, and note that $0 < \nu_{n-1} < \nu_n < 1$ and $\lim_{n\to\infty} \nu_n = 1$. The pair (S,T) determines a certain positive integer n, defined in Section 4, which we call the degree of (S,T). In Sections 2–5 we obtain the unique optimal strategy for $\nu > \nu_n$; see Theorem 3, our main result. In particular, this gives the unique optimal strategy when $\nu \geqslant 1$.

In Sections 6–8, we obtain the unique optimal strategy for $\nu_{n-1} < \nu < \nu_n$ when $n > 1$; see Theorems 5–9. These results are considerably more complicated than Theorem 3, because while for $\nu > \nu_n$ there is only a single case, for $\nu_{n-1} < \nu < \nu_n$ there are four different cases, each with its own type of solution. Under certain conditions we show these solutions remain valid in an interval extending below ν_{n-1}, sometimes as far as ν_{n-2}. Since $\nu_0 = 0$, this provides complete solutions for all $\nu > 0$ in some special cases where $n \leqslant 2$. As ν decreases further, the cases seem to proliferate, and there appears to be little hope of describing detailed solutions for all $\nu > 0$ in general. However, for some interesting discrete sets S we can give explicit solutions for all $\nu > 0$. This is done in the Appendix for $S = \{T^{k/2} : k = 0, 1, 2, 3, \dots\}$.

2. THE OPTIMAL PROPORTION VECTOR V_n

Define the sequence of polynomials $F_n(x)$ with integer coefficients by

$$F_{-1}(x) = F_0(x) = 1,$$
$$F_n(x) = (x+1)F_{n-1}(x) - F_{n-2}(x) \qquad \text{for} \quad n \geqslant 1. \tag{2.1}$$

Thus $F_1(x) = x$, $F_2(x) = x^2 + x - 1$, $F_3(x) = x^3 + 2x^2 - x - 1$, etc. (These polynomials are related to the Brewer polynomials $V_k(x,1)$ [2, p. 318] by

$V_k(x+1,1) = F_k(x) + F_{k-1}(x)$.) By standard difference equation methods (e.g. [8, p. 121]) one obtains

$$F_n(x) = (x+3)^{-1}(y^{n+1} + y^{-n-1} + y^n + y^{-n}),$$

where

$$y = \frac{x+1}{2} + \frac{(x^2+2x-3)^{1/2}}{2}.$$

We now show that for each n,

$$F_n(x) > 0 \qquad \text{when} \quad x > \nu_{n-1} = 2\cos\frac{\pi}{2n+1} - 1. \tag{2.2}$$

If $x \geqslant 1$, then $y \geqslant 1$ and $F_n(x) > 0$. If $-3 < x < 1$, then y is nonreal, and $F_n(x) = 0$ if and only if $y^{2n+1} = -1$, i.e., $(x+1)/2 = \operatorname{Re} y \in \{\cos[h\pi/(2n+1)]: h = 1,3,\ldots,2n-1\}$. Thus for $x \geqslant 0$, $F_n(x) = 0$ if and only if $x \in \{2\cos[h\pi/(2n+1)] - 1: h = 1,3,\ldots,2n-1\}$ and $h/(2n+1) \leqslant \frac{1}{3}$. The largest real zero of $F_n(x)$ is ν_{n-1}, and (2.2) follows.

For $n \geqslant 1$ and $\nu > 0$, define the $2n+1$ by 1 column vector $V_n = V_n(\nu)$ by

$$V_n^T = \begin{cases} (F_{n-1}, F_{n-3}, \ldots, F_0; F_1, F_3, \ldots, F_{n-2}; F_n; \\ F_{n-2}, \ldots, F_3, F_1; F_0, \ldots, F_{n-3}, F_{n-1}) & \text{if } n \text{ is odd}, \\ (F_{n-1}, F_{n-3}, \ldots, F_1; F_0, F_2, \ldots, F_{n-2}; \\ F_n; F_{n-2}, \ldots, F_2, F_0; F_1, \ldots, F_{n-3}, F_{n-1}) & \text{if } n \text{ is even}, \end{cases}$$

where $F_i = F_i(\nu)$. For example, $V_1^T = (F_0, F_1, F_0) = (1, \nu, 1)$, $V_2^T = (F_1, F_0, F_2, F_0, F_1) = (\nu, 1, \nu^2 + \nu - 1, 1, \nu)$, and $V_5^T = (F_4, F_2, F_0, F_1, F_3, F_5, F_3, F_1, F_0, F_2, F_4)$. Note that V_n is symmetric about its middle entry F_n, and if $\nu > \nu_{n-1}$, all entries of V_n are positive.

3. THE PAYOFF MATRIX M_n

For $n \geqslant 1$, let M_n be the $2n+1$ by $2n+1$ skew-symmetric (Toeplitz) matrix for which each entry on the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below is $-\nu$. For example,

$$M_3 = \begin{pmatrix} 0 & -1 & -1 & -1 & \nu & \nu & \nu \\ 1 & 0 & -1 & -1 & -1 & \nu & \nu \\ 1 & 1 & 0 & -1 & -1 & -1 & \nu \\ 1 & 1 & 1 & 0 & -1 & -1 & -1 \\ -\nu & 1 & 1 & 1 & 0 & -1 & -1 \\ -\nu & -\nu & 1 & 1 & 1 & 0 & -1 \\ -\nu & -\nu & -\nu & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Let $M_n(i)$ denote the ith row vector of M_n.

LEMMA 1. *For all real ν, the null space of M_n over the reals is the set of real multiples of V_n. Thus, M_n has rank $2n$.*

Proof. We first show that V_n is in the null space of M_n by showing that

$$M_n(i)V_n = 0 \qquad \text{for} \quad 1 \leqslant i \leqslant 2n+1. \tag{3.1}$$

Since $M_n(n+1) = (1,\ldots,1,0,-1,\ldots,-1)$, where 1 and -1 each occur n times, (3.1) holds for $i = n+1$. For $1 \leqslant i \leqslant n$, the vector $M_n(i+1) - M_n(i)$ has exactly three nonzero entries, and its product with V_n is either $F_j(\nu) + F_{j+2}(\nu) - (\nu+1)F_{j+1}(\nu)$ for some nonnegative j depending on i, or $F_0(\nu) + F_1(\nu) - (\nu+1)F_0(\nu)$. In either case, $[M_n(i+1) - M_n(i)]V_n = 0$ by (2.1), so we have the cases $1 \leqslant i \leqslant n+1$ of (3.1). For $1 \leqslant i \leqslant n$, reversing the order of the entries in $M_n(i)$ yields $-M_n(2n+2-i)$, and the remaining cases of (3.1) follow.

It remains to prove that M_n has nullity 1. This is easily checked for $n=1$. Let $n>1$, and assume as an induction hypothesis that M_{n-1} has nullity 1. Assume for the purpose of contradiction that for some $\nu \geqslant 0$ the nullity of M_n exceeds 1. Then since M_n has even rank [9, Theorem 21.1, p. 151], there are at least three linearly independent vectors in the null space of M_n, so there is a nonzero vector U_n in this null space whose middle and last entries are both 0. Let U_n' be the $2n-1$ by 1 vector obtained from U_n by deleting the middle and last entries. Let M_n' be the $2n-1$ by $2n-1$ matrix obtained from M_n by deleting the middle and last rows and columns. Since $M_nU_n = 0$, we have $M_n'U_n' = 0$. It is easy to see that $M_n' = M_{n-1}$, and thus U_n' is in the null space of M_{n-1}. Since $M_{n-1}V_{n-1} = 0$, the induction hypothesis implies that U_n' is a real multiple of V_{n-1}, and without loss of generality we assume that $U_n' = V_{n-1}$. Thus $U_n^T = (—,0,—,0)$, where the first blank is filled with the first n entries of V_{n-1} and the second with the remaining $n-1$

entries of V_{n-1}. Since $M_n U_n = 0$, we have $M_n(n+1)U_n = 0$, which implies that

$$F_{n-1}(\nu) = 0. \tag{3.2}$$

In particular, $\nu < 1$. From $M_n(2n+1)U_n = 0$ it follows that $(\nu - 1)\sum_{r=0}^{n-2} F_r(\nu) = 0$, and therefore

$$\sum_{r=0}^{n-2} F_r(\nu) = 0. \tag{3.3}$$

From (2.1) we have $\sum_{r=1}^{n-1} F_r(\nu) = (\nu+1)\sum_{r=0}^{n-2} F_r(\nu) - \sum_{r=-1}^{n-3} F_r(\nu)$, which in view of (3.2) and (3.3) reduces to $-F_0(\nu) = -F_{-1}(\nu) + F_{n-2}(\nu)$. Therefore $F_{n-2}(\nu) = 0$, which contradicts (3.2). ■

4. THE OPTIMAL SET W

For $x > c_1$, let $\langle x \rangle$ denote the largest element of S which is less than x. Let m be the integer such that $c_m = \langle Tc_1 \rangle$, and define $d_j = \langle Tc_{j+1} \rangle$ for $0 \leqslant j \leqslant m-1$. Let $I = \{j : 1 \leqslant j \leqslant m-1 \text{ and } d_{j-1} < d_j\}$, $E = \{c_j : j \in I\} \cup \{c_m\}$, and $F = \{d_j : j \in I\}$. Without loss of generality, I is nonempty; otherwise the optimal strategy is simply to select c_m. The integer $n = |I|$ is determined by S and T, and will be called deg(S,T), the degree of (S,T). Let $e_1 < e_2 < \cdots < e_{n+1} = c_m$ be the elements of E, and $f_1 < f_2 < \cdots < f_n$ the elements of F. Thus, if $i_1 < \cdots < i_n$ are the elements of I, then $e_1 = c_{i_1}, \ldots, e_n = c_{i_n}$ and $f_1 = d_{i_1}, \ldots, f_n = d_{i_n}$. Let $e_0 = 0$, $f_0 = c_m$. Note that $f_0 < f_1$. One sees then that for $i = 0, 1, \ldots, n$,

$$f_i = \langle Te_{i+1} \rangle, \tag{4.1}$$

and more generally, that

$$f_i = \langle Tc_r \rangle \qquad \text{for} \quad e_i < c_r \leqslant e_{i+1}. \tag{4.2}$$

Let $W = E \cup F$. Then W has $k = 2n+1$ elements, which we denote by $w_1 < w_2 < \cdots < w_k$. Also write $w_0 = c_0 = 0$. Observe that W is determined by S and T, independent of ν. We shall see that W is the optimal, or essential, subset of S for this T in the sense that optimal play in the

Silverman game (S,T,ν) with $\nu > \nu_n$ assigns positive probabilities to precisely the elements of W.

LEMMA 2. *The payoff matrix for the row player in the Silverman game* (W,T,ν) *is* M_n.

Proof. The element in the ith row and jth column of the payoff matrix is

$$\begin{array}{lll} \nu & \text{if} & w_j \geqslant Tw_i, \\ -1 & \text{if} & w_i < w_j < Tw_i, \\ 0 & \text{if} & i = j, \\ 1 & \text{if} & w_j < w_i < Tw_j, \\ -\nu & \text{if} & w_i \geqslant Tw_j. \end{array}$$

It is straightforward then to verify, using (4.1) and the definitions preceding it, that the payoff matrix is precisely M_n. ■

5. THE OPTIMAL STRATEGY FOR $\nu > \nu_n$

Write the vector V_n of Section 2 as $V_n^T = (v_1, \ldots, v_k)$, and let τ be the mixed strategy which assigns probability $v_i/(v_1 + \cdots + v_k)$ to w_i, $1 \leqslant i \leqslant k$. [These components v_i are positive for $\nu > \nu_{n-1}$ by (2.2).] We are now in a position to prove the main theorem for $\nu > \nu_n$.

THEOREM 3. *Let* $\nu > \nu_n = 2\cos[\pi/(2n+3)] - 1$. *Then* τ *is the unique optimal strategy for the Silverman game* (S,T,ν).

Proof. For $b \in S$, denote by $E(b,\tau)$ the expected payoff to player I (the row player) using the pure strategy b against player II's strategy τ. By symmetry of the game, the game value, if it exists, must be 0, so to prove the optimality of τ it suffices to show that for every b in S, $E(b,\tau) \leqslant 0$. If $b \in W$, it follows from Lemmas 1 and 2 that $E(b,\tau) = 0$.

Suppose $w_i < b < w_{i+1}$ for some i, $0 \leqslant i \leqslant n$. For some $r \geqslant 0$, we have $c_r = e_i = w_i < b$, so $c_{r+1} \leqslant b$, since $b \in S$. Then from (4.2), $w_{n+1+i} = f_i =$

$\langle Te_{i+1}\rangle = \langle Tc_{r+1}\rangle < Tb$. If we insert a b-row into the payoff matrix, it looks like this:

	$\cdots$	w_{i+1}	$\cdots$	w_{n+i+1}	$\cdots$
b	$\cdots$	-1	$\cdots$	-1	$\cdots$
w_{i+1}	$\cdots$	0	$\cdots$	-1	$\cdots$

The entries in the b-row and the w_{i+1}-row agree except in the w_{i+1}-column. Thus $E(b,\tau) < E(w_{i+1},\tau) = 0$.

Suppose next that $w_i < b < w_{i+1}$ for some i in the range $n+1 \leqslant i \leqslant 2n$. From (4.1) we have $w_{2n+1} = f_n < Te_{n+1} < Tb$. Also, if $i = n+j$ we have $b > w_{n+j} = f_{j-1} = \langle Te_j\rangle$, so $b \geqslant Te_j = Tw_j$. Thus the b-row and the w_{n+j+1}-row in the payoff matrix are

	$\cdots$	w_j	w_{j+1}	$\cdots$	w_{n+j}	w_{n+j+1}	$\cdots$
b	$\cdots$	$-\nu$	1	$\cdots$	1	-1	$\cdots$
w_{n+j+1}	$\cdots$	$-\nu$	1	$\cdots$	1	0	$\cdots$

The entries in the b-row and the w_{n+j+1}-row agree except in the w_{n+j+1}-column. Again $E(b,\tau) < E(w_{n+j+1},\tau) = 0$.

Finally, if $b > w_{2n+1}$, the above argument showing that $b \geqslant Tw_j$ now shows that $b \geqslant Tw_{n+1}$, so that each of the first $n+1$ entries in the b-row is $-\nu$, and each of the remaining entries in this row is $-\nu$ or 1. Letting Y denote the b row vector, we have $YV_n \leqslant -\nu\sum_{j=0}^{n} F_j(\nu) + \sum_{j=0}^{n-1} F_j(\nu)$, and an easy induction on n shows that the right member of this inequality is $-F_{n+1}(\nu)$. Since $F_{n+1}(\nu) > 0$ for $\nu > \nu_n$, we have $E(b,\tau) < 0$, and τ is optimal.

To prove uniqueness, first note that any optimal strategy σ will select only elements from W, since, as we've just seen, $E(b,\tau) < 0$ when $b \notin W$. If $E(b,\sigma) < 0$ for some $b \in W$, then $E(\tau,\sigma) < 0$, which contradicts the optimality of τ. Thus $E(b,\sigma) = 0$ for all $b \in W$. Uniqueness now follows from Lemma 1. ■

Note that only in the case $b > w_{2n+1}$ of the proof of Theorem 3 did we use the assumption that $\nu > \nu_n$. If S has no elements $> w_{2n+1}$ [in fact, if S has no elements in the interval (w_{2n+1}, Tw_{n+2}); see Theorem 8], the theorem is valid for $\nu > \nu_{n-1}$. In the sequel, we consider games with ν in the interval (ν_{n-2}, ν_n).

6. ESSENTIAL PURE STRATEGIES FOR $\nu < \nu_n$

The essential sets in Theorems 5–7 below will be obtained by augmenting the set W with two additional elements. Several new definitions are required. With $n = \deg(S,T)$ and W as in Section 4, let $g_i = \langle Tf_i \rangle$, $i = 1,2$. Here g_2 is defined only if $n > 1$. Let $U = \{c \in S : e_{n+1} < c \leqslant g_1/T\}$, and if $U \neq \varnothing$ let u be the largest element of U. If $U \neq \varnothing$, then $u \leqslant g_1/T < f_1$, so $e_{n+1} < u < f_1$, and also $f_n < Te_{n+1} < Tu \leqslant g_1$, so $f_n < g_1$. Whether U is empty or not, $f_n \leqslant g_1 \leqslant g_2$.

For $k = -1,0,1,\ldots$, let $G_k(x) = (x^2+2x)F_k(x)$. Then $G_0(x) = x^2+2x$, $G_1(x) = xG_0(x)$, and for $k > 0$ $G_{k+1}(x) = (x+1)G_k(x) - G_{k-1}(x)$.

LEMMA 4. *Let $H_n(x) = (x^2+2x-1)(x+1)F_{n-1}(x) + F_n(x)$, $n \geqslant 1$. Then there exists a positive zero μ_{n-1} of $H_n(x)$ such that $H_n(x) \geqslant 0$ for $x \geqslant \mu_{n-1}$, where $\mu_0 \doteq 0.3247$, and for $n > 1$, $\nu_{n-2} < \mu_{n-1} < \nu_{n-1}$.*

Proof. The function $H_1(x) = (x^2+2x-1)(x+1) + x = x^3+3x^2+2x-1$ is increasing for $x > 0$ and is zero at μ_0, so it is positive for $x > \mu_0$. For $n > 1$, $(x^2+2x-1)(x+1)F_{n-1}(x)$ is 0 at ν_{n-2} and at $\sqrt{2}-1$, both of which are less than ν_{n-1}, and is positive for $x > \max\{\nu_{n-2}, \sqrt{2}-1\}$. Since $F_n(x) > 0$ for $x > \nu_{n-1}$, we have $H_n(x) > 0$ for $x \geqslant \nu_{n-1}$, and by continuity, $H_n(x) > 0$ for all x in some interval (μ_{n-1}, ∞), where $\mu_{n-1} < \nu_{n-1}$. Since $H(\nu_{n-2}) = F_n(\nu_{n-2}) < 0$, $\nu_{n-2} < \mu_{n-1}$. ∎

Following is a short table to illustrate the lemma:

n	$H_n(x)$	μ_{n-1}	ν_{n-1}
1	x^3+3x^2+2x-1	0.3247	0
2	$x^4+3x^3+2x^2-1$	0.5129	0.6180
3	$x^5+4x^4+4x^3-x^2-3x$	0.7106	0.8019
4	$x^6+5x^5+7x^4-6x^2-3x+1$	0.8303	0.8794

7. THE CASE $U \neq \varnothing$

Assume $U \neq \varnothing$, and let $W_1 = W \cup \{u, g_1\} = \{e_1, e_2, \ldots, e_{n+1}, u, f_1, \ldots, f_n, g_1\}$. With $F_k = F_k(\nu)$ and $G_k = G_k(\nu)$, let $Q_n = Q_n(\nu)$ be the column vector defined by

$$Q_n^T = (F_{n-2}; G_{n-2}, G_{n-4}, \ldots, G_0; G_1, G_3, \ldots, G_{n-1}; -F_{n+1}; G_{n-1}, \ldots, F_{n-2})$$

for n even, and

$$(F_{n-2};G_{n-2},G_{n-4},\ldots,G_1;G_0,G_2,\ldots,G_{n-1};-F_{n+1};G_{n-1},\ldots,F_{n-2})$$

for n odd. (In each case the vector has $2n+3$ components and is symmetric about the middle component, $-F_{n+1}$.) Write $Q_n=(q_1,q_2,\cdots,q_{n+1},q_{n+2},q_{n+1},\ldots,q_2,q_1)$, and let B_n be the sum of the components of Q_n. With the help of the paragraph following (2.2), one sees that the components of Q_n are positive for all (positive) ν in the range $\nu_{n-2}<\nu<\nu_n$. Let τ_1 denote the strategy which assigns probability q_i/B_n to the ith element of W_1, $1\leqslant i\leqslant 2n+3$.

THEOREM 5. *Suppose that* $U\neq\varnothing$. *If* $n=1$ *and* $\mu_0<\nu<\nu_1$, *or* $n>1$ *and* $\nu_{n-1}<\nu<\nu_n$, *then* τ_1 *is the unique optimal strategy for the game* (S,T,ν). *The strategy* τ_1 *is no longer optimal for* $\nu<\mu_0$ *when* $n=1$, *if* $S\cap(g_1,Tg_1)\neq\varnothing$, *or for* $\nu<\nu_{n-1}$ *when* $n>1$, *if* $S\cap(f_n,Tu)\neq\varnothing$. *However*:

(a) *Suppose that* $n=1$ *and* $S\cap(g_1,Tg_1)=\varnothing$. *Then* τ_1 *is optimal for* $0<\nu\leqslant\nu_1$.

(b) *Suppose that* $n=2$ *and* $S\cap(f_n,Tu)=\varnothing$. *Then* τ_1 *is optimal for* $\mu_1\leqslant\nu\leqslant\nu_2$. *Moreover,* τ_1 *is optimal for* $0.2720\doteq\alpha\leqslant\nu\leqslant\nu_2$ *if* $S\cap(g_1,Tf_2)=\varnothing$, *and even for* $0<\nu\leqslant\nu_2$ *if* $S\cap(g_1,Tg_1)=\varnothing$, *where* α *is the positive zero of* $(x+1)^2(x^2+2x)-1$.

(c) *Suppose that* $n>2$ *and* $S\cap(f_n,Tu)=\varnothing$. *Then* τ_1 *is optimal for* $\mu_{n-1}\leqslant\nu\leqslant\nu_n$, *and even for* $\nu_{n-2}\leqslant\nu\leqslant\nu_n$ *if* $S\cap(g_1,Tf_2)=\varnothing$.

Proof. We first show that τ_1 is optimal for the subgame on W_1. From the definitions of u, f_i, and g_i, the payoff matrix $\overline{M}_n$ of this subgame, shown in Table 1, is the $2n+3$ by $2n+3$ skew-symmetric matrix which has middle row

$$(-\nu\quad 1\quad\cdots\quad 1\quad 0\quad -1\quad\cdots\quad -1\quad \nu)$$

(with 1 and -1 each occurring n times) and last row

$$(-\nu\quad\cdots\quad -\nu\quad 1\quad\cdots\quad 1\quad 0)$$

(with 1 occurring n times), and which becomes M_n when the middle and last rows and columns are deleted.

for n even, and

$$(F_{n-2};G_{n-2},G_{n-4},\ldots,G_1;G_0,G_2,\ldots,G_{n-1};-F_{n+1};G_{n-1},\ldots,F_{n-2})$$

for n odd. (In each case the vector has $2n+3$ components and is symmetric about the middle component, $-F_{n+1}$.) Write $Q_n=(q_1,q_2,\cdots,q_{n+1},q_{n+2},q_{n+1},\ldots,q_2,q_1)$, and let B_n be the sum of the components of Q_n. With the help of the paragraph following (2.2), one sees that the components of Q_n are positive for all (positive) ν in the range $\nu_{n-2}<\nu<\nu_n$. Let τ_1 denote the strategy which assigns probability q_i/B_n to the ith element of W_1, $1\leqslant i\leqslant 2n+3$.

THEOREM 5. *Suppose that $U\neq\varnothing$. If $n=1$ and $\mu_0<\nu<\nu_1$, or $n>1$ and $\nu_{n-1}<\nu<\nu_n$, then τ_1 is the unique optimal strategy for the game (S,T,ν). The strategy τ_1 is no longer optimal for $\nu<\mu_0$ when $n=1$, if $S\cap(g_1,Tg_1)\neq\varnothing$, or for $\nu<\nu_{n-1}$ when $n>1$, if $S\cap(f_n,Tu)\neq\varnothing$. However:*

(a) *Suppose that $n=1$ and $S\cap(g_1,Tg_1)=\varnothing$. Then τ_1 is optimal for $0<\nu\leqslant\nu_1$.*

(b) *Suppose that $n=2$ and $S\cap(f_n,Tu)=\varnothing$. Then τ_1 is optimal for $\mu_1\leqslant\nu\leqslant\nu_2$. Moreover, τ_1 is optimal for $0.2720\doteq\alpha\leqslant\nu\leqslant\nu_2$ if $S\cap(g_1,Tf_2)=\varnothing$, and even for $0<\nu\leqslant\nu_2$ if $S\cap(g_1,Tg_1)=\varnothing$, where α is the positive zero of $(x+1)^2(x^2+2x)-1$.*

(c) *Suppose that $n>2$ and $S\cap(f_n,Tu)=\varnothing$. Then τ_1 is optimal for $\mu_{n-1}\leqslant\nu\leqslant\nu_n$, and even for $\nu_{n-2}\leqslant\nu\leqslant\nu_n$ if $S\cap(g_1,Tf_2)=\varnothing$.*

Proof. We first show that τ_1 is optimal for the subgame on W_1. From the definitions of u, f_i, and g_i, the payoff matrix $\overline{M}_n$ of this subgame, shown in Table 1, is the $2n+3$ by $2n+3$ skew-symmetric matrix which has middle row

$$(-\nu \quad 1 \quad \cdots \quad 1 \quad 0 \quad -1 \quad \cdots \quad -1 \quad \nu)$$

(with 1 and -1 each occurring n times) and last row

$$(-\nu \quad \cdots \quad -\nu \quad 1 \quad \cdots \quad 1 \quad 0)$$

(with 1 occurring n times), and which becomes M_n when the middle and last rows and columns are deleted.

the w-row of the payoff matrix differ only in the w-column, and $E(b,\tau_1) < E(w,\tau_1) = 0$.

If $f_n < b < g_1$ the f_n and b rows are

	e_1	$\cdots$	e_n	e_{n+1}	u	f_1	$\cdots$	f_{n-1}	f_n	g_1
f_n	$-\nu$	$\cdots$	$-\nu$	1	1	1	$\cdots$	1	0	-1
b	$-\nu$	$\cdots$	$-\nu$	$-\nu$	a	1	$\cdots$	1	1	-1

where $a = -\nu$ or 1 according as $b \geqslant Tu$ or $b < Tu$. First suppose that $a = 1$, so that $f_n < b < Tu$. The $E(b,\tau_1)B_n = -(\nu+1)q_{n+1} + q_2 = -(\nu+1)G_{n-1} + G_{n-2} = -G_n$, which is $\leqslant 0$ for $\nu_{n-1} \leqslant \nu \leqslant \nu_n$ but > 0 for (positive) ν immediately below ν_{n-1}. Next suppose that $a = -\nu$, so that $Tu \leqslant b < g_1$. Then $E(b,\tau_1)B_n = -(\nu+1)(q_{n+1} + q_{n+2}) + q_2 = -G_n - (\nu+1)(-F_{n+1}) = -(\nu^2+2\nu)F_n + (\nu+1)F_{n+1} = -(\nu+1)^2 F_n + (\nu+1)F_{n+1} + F_n = -(\nu+1)F_{n-1} + F_n = -F_{n-2}$, which is < 0 for all $\nu > \nu_{n-2}$ (indeed, for $\nu > \nu_{n-3}$).

Finally, suppose that $b > g_1$. First consider the case of $n = 1$. Then the g_1 and b rows are as follows:

	e_1	e_2	u	f_1	g_1
g_1	$-\nu$	$-\nu$	$-\nu$	1	0
b	$-\nu$	$-\nu$	$-\nu$	$-\nu$	a

where $a = -\nu$ or 1 according as $b \geqslant Tg_1$ or $g_1 < b < Tg_1$. If $a = -\nu$, then $E(b,\tau_1) = -\nu < 0$. If $a = 1$, then $B_1 E(b,\tau_1) = -(\nu+1)(\nu^2+2\nu) + 1 = -H_1(\nu)$, which is < 0 for $\nu > \mu_0$ (but > 0 for ν immediately below μ_0). Next, let $n > 1$. Then the g_1 and b rows are as follows:

	e_1	$\cdots$	u	f_1	f_2	$\cdots$	f_n	g_1
g_1	$-\nu$	$\cdots$	$-\nu$	1	1	$\cdots$	1	0
b	$-\nu$	$\cdots$	$-\nu$	$-\nu$	a_1	$\cdots$	a_{n-1}	a_n

where each a_i is $-\nu$ or 1. If $b \in (g_1, Tf_2)$, then each a_i is 1, and $E(b,\tau_1)B_n = -(\nu+1)q_{n+1} + q_1 = -(\nu+1)G_{n-1} + F_{n-2} = -(\nu+1)(\nu^2 + 2\nu)F_{n-1} + (\nu+1)F_{n-1} - F_n = -H_n(\nu) < 0$ for $\nu > \mu_{n-1}$. If $b \geqslant Tf_2$, then $a_1 = -\nu$ and $E(b,\tau_1)B_n \leqslant -(\nu+1)(q_{n+1} + q_n) + q_1 = -(\nu+1)(G_{n-1} + G_{n-3}) + F_{n-2} \leqslant -(\nu+1)(\nu^2+2\nu)(F_{n-1} + F_{n-3}) + F_{n-2} = [1 - (\nu+1)^2(\nu^2+2\nu)]F_{n-2}$. Now, $F_{n-2} > 0$ for $\nu > \nu_{n-3}$, and $1 - (\nu+1)^2 (\nu^2+2\nu) < 0$ for $\nu > \alpha \doteq 0.2720$. Thus for $b \geqslant Tf_2$, $E(b,\tau_1) \leqslant 0$ for all $\nu \geqslant \alpha$

when $n = 2$ and for all $\nu \geqslant \nu_{n-2}$ when $n > 2$. If $n = 2$ and S has no elements in $(f_2, Tu) \cup (g_1, Tg_1)$, then $E(b, \tau_1) = -\nu < 0$, so τ_1 is optimal for the full game as described in the statement of the theorem.

It remains only to prove the uniqueness statement for τ_1. For this it suffices to show that for all $\nu > 0$, the nullity of $\overline{M}_n$ is 1. Assume that for some $\nu > 0$ and $n \geqslant 1$ the nullity of $\overline{M}_n$ exceeds 1. Then since this nullity is odd, there is a nonzero vector U_n in the null space of $\overline{M}_n$ whose middle and last components are both zero. Let U_n' be the $2n+1$ by 1 vector obtained from U_n by deleting the middle and last components. The matrix obtained from $\overline{M}_n$ by deleting its middle and last rows and columns is M_n. Since $\overline{M}_n U_n = 0$, we have $M_n U_n' = 0$. By (3.1) and Lemma 1, we therefore have, without loss of generality, $U_n' = V_n$. Thus $U_n^T = (—, 0, —, 0)$, where the first blank is filled by the first $n+1$ components of V_n and the second by the last n components of V_n. We have

$$F_{n-2}(\nu) = 0, \tag{7.2}$$

since

$$0 = \overline{M}_n(n+2)U_n = F_n + \textstyle\sum_{r=0}^{n-2} F_r - \nu F_{n-1} - \sum_{r=0}^{n-1} F_r = F_n - (\nu+1) F_{n-1}$$
$$= -F_{n-2}.$$

It follows readily from (2.1) that

$$(1-\nu) \sum_{r=0}^{n-1} F_r(\nu) = F_{n-2}(\nu) - \nu F_{n-1}(\nu). \tag{7.3}$$

By (7.2) and (7.3), $0 = \overline{M}_n(2n+3)U_n = -\nu \sum_{r=0}^{n} F_r + \sum_{r=0}^{n-1} F_r = -\nu F_n + (1-\nu)\sum_{r=0}^{n-1} F_r = -\nu F_n - \nu F_{n-1} = -\nu(\nu+2)F_{n-1}$. Since $\nu > 0$, we thus have $F_{n-1}(\nu) = 0$, which contradicts (7.2). ■

Remark. If $\nu = 0$, it is not true that $\overline{M}_n$ always has nullity 1; for example, $\overline{M}_3$ has nullity 3.

8. THE CASES WHERE $U = \varnothing$

As remarked in Section 6, we always have $f_n \leqslant g_1 \leqslant g_2$, and when $U = \varnothing$ equality is possible in either place, leading to three cases. When $n = 1$, g_2 is undefined, and we use $h_1 = \langle Tg_1 \rangle$ in place of g_2. We begin with the case of strict inequalities. Theorem 6 deals with $n = 1$, while Theorem 7 deals with $n > 1$.

THEOREM 6. *Assume that* $n = \deg(S,T) = 1$, $U = \varnothing$, *and* $f_1 < g_1 < h_1$. *Let* $\tilde{Q}_1^T = (-F_2, G_0, F_{-1}, G_0, -F_2)$ *and* $W_2 = (e_1, e_2, f_1, g_1, h_1)$. *Let* τ_2 *be the strategy which assigns probabilities to* W_2 *in proportion to* $\tilde{Q}_1$. *Then*:

(a) *For* $\mu_0 < \nu < \nu_1$, τ_2 *is the unique optimal strategy.*

(b) *If* $S \cap (f_1, h_1/T] = \varnothing$, *then* τ_2 *is optimal for* $\rho_0 \leqslant \nu \leqslant \nu_1$, *where* $\rho_0 \doteq 0.2470$ *is the positive zero of* $x^3 + 4x^2 + 3x - 1$.

(c) *If S has no elements in* $(f_1, h_1/T] \cup (h_1, Th_1)$, *then* τ_2 *is optimal for* $0 < \nu \leqslant \nu_1$.

Proof. We show first that τ_2 is optimal for the subgame on W_2. The matrix $\tilde{M}_1$ of this subgame is

	e_1	e_2	f_1	g_1	h_1
e_1	0	-1	ν	ν	ν
e_2	1	0	-1	ν	ν
f_1	$-\nu$	1	0	-1	ν
g_1	$-\nu$	$-\nu$	1	0	-1
h_1	$-\nu$	$-\nu$	$-\nu$	1	0

It is easily checked that $\tilde{M}_1 \tilde{Q}_1 = 0$.

Next we show that τ_2 is optimal on the full game by showing that $E(b, \tau_2) \leqslant 0$ for every b in S. If $b < e_2$, we have $E(b, \tau_2) < 0$ as in proof of Theorem 3. If $e_2 < b < f_1$ then $b > g_1/T$ because $U = \varnothing$. The payoff rows for b and f_1 then are

	e_1	e_2	f_1	g_1	h_1
b	$-\nu$	1	-1	-1	ν
f_1	$-\nu$	1	0	-1	ν

so $E(b, \tau_2) < 0$. For $f_1 < b_1 < g_1 < b_2 < h_1 < b_3$ the payoff rows are

	e_1	e_2	f_1	g_1	h_1
b_1	$-\nu$	$-\nu$	1	-1	x
g_1	$-\nu$	$-\nu$	1	0	-1
b_2	$-\nu$	$-\nu$	$-\nu$	1	-1
h_1	$-\nu$	$-\nu$	$-\nu$	1	0
b_3	$-\nu$	$-\nu$	$-\nu$	$-\nu$	y

where x is -1 or ν and y is 1 or $-\nu$. If $b_1 \leqslant h_1/T$ then $x=\nu$, and $[E(b_1,\tau_2)-E(g_1,\tau_2)]B = -(\nu^2+\nu)+(\nu+1)(-\nu^2-\nu+1) = -\nu^3-3\nu^2-2\nu+1 \leqslant 0$ when $\nu \geqslant \mu_0$, where B is the sum of the components of $\tilde{Q}_1$. If $h_1/T < b_1$ then $x=-1$, and $E(b_1,\tau_2)<E(g_1,\tau_2)=0$. Also, $E(b_2,\tau_2)<E(h_1,\tau_2)=0$. If $b_3<Th_1$ then $y=1$ and $BE(b_3,\tau_2)=BE(h_1,\tau_2)-(\nu+1)(\nu^2+2\nu)+(-\nu^2-\nu+1)\leqslant 0$ when $\nu \geqslant \rho_0$. If $b_3 \geqslant Th_1$ then $y=-\nu=E(b_3,\tau_2)$.

It remains only to prove the uniqueness, and this follows from the fact that $\tilde{M}_1$ has nullity 1 for all real ν. ■

THEOREM 7. *Assume* $n>1$, $U=\varnothing$, *and* $f_n<g_1<g_2$. *Let* $W_2=(e_1,e_2,\ldots,e_{n+1},f_1,\ldots,f_n,g_1,g_2)$. *Let* $\tilde{Q}_n^T=(q_{n+2},q_{n+1},\ldots,q_2,q_1,q_2,\ldots,q_{n+1},q_{n+2})$, *where* q_i *is the* i*th component of* Q_n *(defined in Section 7), and let* τ_2 *be the strategy which assigns probabilities to* W_2 *in proportion to* $\tilde{Q}_n$. *Then*:

(a) *For* $\mu_{n-1}<\nu<\nu_n$ *and* $n>2$, τ_2 *is the unique optimal strategy.*

(b) *If S has no elements in* $(f_1,g_2/T)$, *then for* $n=2$, τ_2 *is optimal for* $\rho_1 \leqslant \nu \leqslant \nu_2$, *where* $\rho_1 \doteq 0.3406$ *is the positive zero of* $2x^3+5x^2+x-1$, *and for* $n>2$, τ_2 *is optimal for* $\nu_{n-2}\leqslant\nu\leqslant\nu_n$.

(c) *For* $n=2$, *if S has no elements in* $(f_1,g_2/T)\cup(g_2,Tg_1)$, *then* τ_2 *is optimal for* $\sigma_1\leqslant\nu\leqslant\nu_2$, *where* $\sigma_1\doteq 0.2888$ *is the positive zero of* $x^4+5x^3+7x^2+x-1$.

(d) *For* $n=2$, *if S has no elements in* $(f_1,g_2/T)\cup(g_2,Tg_2)$, *then* τ_2 *is optimal for* $0=\nu_0<\nu<\nu_2$.

Proof. We first show that τ_2 is optimal for the subgame on W_2. The matrix $\tilde{M}_n$ of this subgame is the $2n+3$ by $2n+3$ skew-symmetric matrix with each entry in the first n subdiagonals equal to 1 and each entry below this equal to $-\nu$. It is easily checked (cf. the proof of Lemma 1) that $\tilde{M}_n\tilde{Q}_n=0$.

Next we show that τ_2 is optimal on the full game. If $b<e_{n+1}$, we have $E(b,\tau_2)<0$ as in the proof of Theorem 3. If $e_{n+1}<b<f_1$ then $b>g_1/T$ because $U=\varnothing$, and one finds $E(b,\tau_2)<E(f_1,\tau_2)=0$. If $f_1<b<f_2$, the payoff rows for f_1, b, and f_2 are

	e_1	e_2	e_3	$\cdots$	e_{n+1}	f_1	f_2	f_3	$\cdots$	g_1	g_2
f_1	$-\nu$	1	1	$\cdots$	1	0	-1	-1	$\cdots$	-1	ν
b	$-\nu$	$-\nu$	1	$\cdots$	1	1	-1	-1	$\cdots$	-1	x
f_2	$-\nu$	$-\nu$	1	$\cdots$	1	1	0	-1	$\cdots$	-1	-1

where x is -1 or ν. Let $\tilde{B}_n$ be the sum of the components of $\tilde{Q}_n$. If $b \leqslant g_2/T$, then $x = \nu$ and $\tilde{B}_n E(b, \tau_2) = \tilde{B}_n E(f_1, \tau_2) - (\nu + 1)q_{n+1} + q_1 = (-\nu + 1)G_{n-1} + F_{n-2} = -(\nu + 1)(\nu^2 + 2\nu)F_{n-1} + (\nu + 1)F_{n-1} - F_n = -H_n(\nu) \leqslant 0$ for $\nu \geqslant \mu_{n-1}$. If S has no elements in $(f_1, g_2/T]$, then $x = -1$ and $E(b, \tau_2) < E(f_2, \tau_2) = 0$.

Suppose $f_i < b < f_{i+1}$ for some i, $2 \leqslant i \leqslant n-1$. Then $E(b, \tau_2) < E(f_{i+1}, \tau_2) = 0$, as one sees by comparing the b and f_{i+1} payoff rows, and for $f_n < b < g_1$ as for $g_1 < b < g_2$, the situation is similar. Finally, suppose that $b > g_2$. The g_2 and b payoff rows are

	$\cdots$	f_1	f_2	f_3	$\cdots$	g_1	g_2
g_2	$\cdots$	$-\nu$	1	1	$\cdots$	1	0
b	$\cdots$	$-\nu$	$-\nu$	y	$\cdots$		

where y is $-\nu$ or 1. Then $\tilde{B}_n E(b, \tau_2) \leqslant \tilde{B}_n E(g_2, \tau_2) - (\nu + 1)q_2 + q_{n+2} = -(\nu + 1)G_{n-2} - F_{n+1} = -(\nu + 1)\,[(\nu + 1)G_{n-1} - G_n] - [(\nu + 1)F_n - F_{n-1}] = [1 - (\nu + 1)^2(\nu^2 + 2\nu)]F_{n-1} + (\nu + 1)(\nu^2 + 2\nu - 1)F_n = K_n(\nu)$, say. Now $K_2(\nu) = -2\nu^3 - 5\nu^2 - \nu + 1 < 0$ for $\nu > \rho_1$. For $n > 2$ and $\nu > \nu_{n-2}$, we will show that

(i) $K_n(\nu) < -H_n(\nu)$, *so that* $K_n(\nu) < 0$ *for* $\nu \geqslant \mu_{n-1}$, *and*
(ii) $K_n(\nu) < 0$ for ν in (ν_{n-2}, ν_{n-1}).

It will follow, by Lemma 4, that $K_n(\nu) < 0$ for $\nu > \nu_{n-2}$.

To see (i) note first that $(\nu + 1)^2(\nu^2 + 2\nu) - 1 > (\nu + 1)(\nu^2 + 2\nu) - 1 > (\nu + 1)(\nu^2 + 2\nu - 1)$. Since $F_{n-1} > 0$ for $\nu > \nu_{n-2}$, the F_{n-1} term in the definition of K_n is less than the F_{n-1} term in the definition of $-H_n$. Moreover, $(\nu + 1)(\nu^2 + 2\nu - 1)F_n < 0 < -F_n$ for $\nu > \max\{-1 + \sqrt{2}, \nu_{n-2}\} = \nu_{n-2}$. As for (ii), $1 - (\nu + 1)^2(\nu^2 + 2\nu) < 0 < \nu^2 + 2\nu - 1$ when $\nu > \nu_1$. For ν in (ν_{n-2}, ν_{n-1}), $F_n < 0 < F_{n-1}$, so $K_n < 0$.

If $n = 2$ and S has no elements in (g_2, Tg_1), then the g_1-column takes the place of the f_3-column, $y = -\nu$, and $B_2 E(b, \tau_2) \leqslant -(\nu + 1)(G_0 + G_1) - F_3 = -(\nu + 1)(\nu^3 + 3\nu^2 + 2\nu) - (\nu^3 + 2\nu^2 - \nu - 1) < 0$ when $\nu > \sigma_1$. If, further, S has no elements in (g_2, Tg_2), then $E(b, \tau_2) = -\nu$.

It remains only to prove the uniqueness statement for τ_2. For this, it suffices to show that for all $\nu > 0$, the nullity of $\tilde{M}_n$ is 1. It is easily checked that the nullity of $\tilde{M}_1$ is 1. Assume that for some $n \geqslant 2$, $\tilde{M}_{n-1}$ has nullity 1 but $\tilde{M}_n$ has nullity > 1. Then there is a nonzero vector U_n in the null space of $\tilde{M}_n$ whose $(n + 1)$th and $(2n + 2)$th entries are both zero. Let U_n' be the $2n + 1$ by 1 vector obtained from U_n by deleting the $(n + 1)$th and $(2n + 2)$th entries. The matrix obtained from $\tilde{M}_n$ by deleting the $(n + 1)$th and $(2n + 2)$th rows and columns is $\tilde{M}_{n-1}$. Since $\tilde{M}_n U_n = 0$, we have

$$\tilde{M}_{n-1} U_n' = 0 = \tilde{M}_{n-1} \tilde{Q}_{n-1},$$

so by the induction hypothesis, we have, without loss of generality, $U_n' = \tilde{Q}_{n-1}$. Thus, $U_n^T = (t_{n+1}, t_n, \ldots, t_2, 0, t_1, t_2, \ldots, t_n, 0, t_{n+1})$, where t_i is the ith component of Q_{n-1}.

Now,

$$\begin{aligned} 0 = \tilde{M}_n(n+1)U_n &= \sum_{r=2}^{n+1} t_r - \sum_{r=1}^{n} t_r + \nu t_{n+1} \\ &= -t_1 + (\nu+1)t_{n+1} = -F_{n-3} - (\nu+1)F_n \\ &= -\nu(\nu+2)F_{n-1}. \end{aligned} \tag{8.1}$$

Since $\nu > 0$, it follows that

$$F_{n-1}(\nu) = 0. \tag{8.2}$$

Also,

$$\begin{aligned} 0 = \tilde{M}_n(2n+2)U_n &= -\nu \sum_{r=2}^{n+1} t_r + \sum_{r=1}^{n} t_r - t_{n+1} \\ &= -(\nu+1)t_{n+1} + t_1 + (1-\nu) \sum_{r=2}^{n} t_r = (1-\nu) \sum_{r=2}^{n} t_r, \end{aligned}$$

where the last equality follows from (8.1). Therefore, since $\nu^2 + 2\nu \neq 0$,

$$0 = (1-\nu) \sum_{r=0}^{n-2} G_r = (1-\nu) \sum_{r=0}^{n-2} F_r. \tag{8.3}$$

From (7.3), (8.2) and (8.3), we have $F_{n-2}(\nu) = 0$, which contradicts (8.2). ■

THEOREM 8. *If $g_1 = f_n$, then ($U = \varnothing$ and) the strategy τ of Theorem 3 is optimal also for $\nu_{n-1} \leqslant \nu \leqslant \nu_n$. This optimal strategy is unique for $\nu > \nu_{n-1}$.*

Proof. As noted after the proof of Theorem 3, that proof remains valid for $\nu > \nu_{n-1}$ up to the case $b > w_{2n+1}$. For $b > w_{2n+1} = f_n$ we now have $b > g_1$, so $b \geqslant Tf_1$, and $-\nu$ occurs at least $n+2$ times in the b-row. Then $E(b,\tau)A_n \leqslant -F_{n+1} - (\nu+1)F_{n-2} = -(\nu+1)F_n + F_{n-1} - (\nu+1)F_{n-2} = -(\nu^2+2\nu)F_{n-1} < 0$ where A_n is the sum of the entries of V_n. The uniqueness argument in the proof of Theorem 3 remains valid for $\nu > \nu_{n-1}$. ■

THEOREM 9. *Suppose that $U = \varnothing$ and $f_n < g_1$. If*

(a) $n = 1$ *and* $g_1 = h_1$, *or*
(b) $n > 1$ *and* $g_1 = g_2$,

then the strategy τ of Theorem 3, with W replaced by $W^\circ = \{e_2, \ldots, e_{n+1}, f_1, \ldots, f_n, g_1\}$, is optimal for $\nu_{n-1} \leqslant \nu \leqslant \nu_n$. For $\nu_{n-1} < \nu < \nu_n$ this optimal strategy is unique.

Proof. Denote the modified strategy by τ°. Rename the elements of W° as follows. For $i = 1, \ldots, n$, $e_i^\circ = e_{i+1}$ and $f_{i-1}^\circ = f_i$, $f_n^\circ = g_1$. Also define $g_1^\circ = \langle Tf_1^\circ \rangle$. Then the elements of $W^\circ = \{e_1^\circ, e_2^\circ, \ldots, e_n^\circ, f_0^\circ, \ldots, f_n^\circ\}$ are related to one another exactly as $\{e_1, e_2, \ldots, e_n, f_0, \ldots, f_n\}$ are, namely $f_i^\circ = \langle Te_{i+1}^\circ \rangle$. That $g_1 = h_1$ in (a), or $g_1 = g_2$ in (b), means that $g_1^\circ = f_n^\circ$. The set corresponding to U in Section 6 is $U^\circ = \{c \in S : e_{n+1}^\circ < c \leqslant g_1^\circ / T\} = \varnothing$, and the proof of Theorem 8 shows that $E(b,\tau^\circ) \leqslant 0$ for all $b \geqslant e_1^\circ$. We next show that $E(e_1, \tau^\circ) \leqslant 0$. The e_1 and e_2 payoff rows are as follows:

	e_1	e_2 (F_{n-1})	e_3	$\cdots$	e_{n+1} (F_{n-2})	f_1 (F_n)	f_2	$\cdots$	g_1
e_1	0	-1	-1	$\cdots$	-1	ν	ν	$\cdots$	ν
e_2	1	0	-1	$\cdots$	-1	-1	ν	$\cdots$	ν

Then $[E(e_1,\tau^\circ) - E(e_2,\tau^\circ)]A_n = -F_{n-1} + (\nu+1)F_n = F_{n+1} \leqslant 0$. If $b < e_1$ or $e_1 < b < e_2$, familiar arguments show that $E(b,\tau^\circ) \leqslant E(e_1,\tau^\circ)$ or $E(b,\tau^\circ) \leqslant E(e_2,\tau^\circ)$, respectively. The uniqueness for $\nu_{n-1} < \nu < \nu_n$ follows as in the proof of Theorem 8. ■

9. CONCLUDING REMARKS

The methods used to find the solutions described above will yield solutions for further values of ν. The condition $\nu > \nu_n$ corresponds to the polynomial conditions $F_k(\nu) > 0$, $k = 0, 1, \ldots, n+1$, but as ν decreases a plethora of additional polynomial conditions enter the picture, and we do not know of a reasonably concise way to describe the solutions in general for all $\nu > 0$.

It seems likely that there are always solutions of finite type. In [6] it is shown that if $c = \min S$, then for $\nu \geqslant 1$ every pure strategy $\geqslant T^2 c$ is dominated. Perhaps it is realistic to try to obtain, as a function of ν, a similar upper bound for the essential set for values of ν in $(0, 1)$.

APPENDIX

Theorem 10 below gives explicit optimal strategies for the game (S, T, ν) for all $\nu > 0$, where $S = \{T^{k/2} : k = 0, 1, 2, 3, \ldots]$.

For $r \geqslant 0$, define the polynomials $A_r(\nu), B_r(\nu)$ recursively by

$$A_0 = 1, \quad A_1 = 1, \qquad A_{r+2} = (\nu+2)A_{r+1} - (\nu+1)^2 A_r \tag{10.1}$$

and

$$B_0 = 0, \quad B_1 = \nu, \qquad B_{r+2} = (\nu+2)B_{r+1} - (\nu+1)^2 B_r. \tag{10.2}$$

For $m \geqslant 1$, $1 \leqslant r \leqslant 2m+1$, define polynomials $C_{r,m}(\nu)$ by

$$C_{r,m} = \nu^{-1} B_k B_{m+1-k} \qquad (r = 2k, \quad 1 \leqslant k \leqslant m), \tag{10.3}$$

$$C_{r,m} = A_k A_{m-k} \qquad (r = 2k+1, \quad 0 \leqslant k \leqslant m). \tag{10.4}$$

Define α_r for $r \geqslant 1$ by

$$\alpha_r = \frac{2\tan^2\left(\dfrac{\pi}{2r+1}\right) - 2 + 2\left[1 + \tan^2\left(\dfrac{\pi}{2r+1}\right)\right]^{1/2}}{3 - \tan^2\left(\dfrac{\pi}{2r+1}\right)}. \tag{10.5}$$

Observe that $\infty = \alpha_1 > \alpha_2 > \alpha_3 > \cdots > 0$ and $\alpha_r \to 0$ as $r \to \infty$. Thus $\alpha_{m+1} \leqslant \nu < \alpha_m$ for some $m \geqslant 1$. For this m, let τ denote the strategy which assigns probabilities to $1, T^{1/2}, T, T^{3/2}, \ldots, T^m$ in proportion to $C_{1,m}(\nu)$, $C_{2,m}(\nu), \ldots, C_{2m+1,m}(\nu)$. It can be shown that when $\nu < \alpha_m$, $C_{r,m}(\nu) > 0$ for each r $(1 \leqslant r \leqslant 2m+1)$, so τ is well defined for any $\nu > 0$.

THEOREM 10. *For $\alpha_{m+1} \leqslant \nu < \alpha_m$, τ is an optimal strategy for the game (S,T,ν). If $\alpha_{m+1} < \nu < \alpha_m$, then τ is in fact the unique optimal strategy.*

EXAMPLES. If $\alpha_2 = (\sqrt{5} - 1)/2 < \nu < \infty$, then the unique optimal strategy is to choose $1, T^{1/2}, T$ with probabilities in proportion to $1, \nu, 1$. This is consistent with Theorem 3. If $\alpha_3 \doteq 0.24698 < \nu < (\sqrt{5} - 1)/2 = \alpha_2 \doteq 0.618$, then the unique optimal strategy is to choose $1, T^{1/2}, T, T^{3/2}, T^2$ with probabilities in proportion to $1 - \nu - \nu^2, \nu^2 + 2\nu, 1, \nu^2 + 2\nu, 1 - \nu - \nu^2$. This is consistent with Theorem 6(b). For the "boundary value" $\nu = (\sqrt{5} - 1)/2$, one optimal strategy is to choose $1, T^{1/2}, T$ with probabilities in proportion to $1, \nu, 1$, while another is to choose $T^{1/2}, T, T^{3/2}$ with probabilities in proportion to $1 + \nu, 1, 1 + \nu$. Any convex linear combination of these two strategies is also optimal.

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