A combinatorial characterization of geometric spreads

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Abstract


A t-spread in a projective space \( P = \text{PG}(d, q) \) is a set of t-dimensional subspaces which partitions the point set of \( P \). A t-spread \( S \) is called geometric if it induces a spread in any \((2t + 1)\)-dimensional subspace containing at least two elements of \( S \). In this note we characterize the geometric t-spreads \( S \) among all partial spreads in the first nontrivial case \( \text{PG}(3t + 2, q) \) by the property that any subspace of dimension \( 3t \) contains at least one element of \( S \). This is the first instance of a combinatorial characterization of geometric spreads.

1. Introduction

Denote by \( P = \text{PG}(d, q) \) the \( d \)-dimensional projective space of dimension \( d \geq 3 \) and order \( q \). A partial t-spread of \( P \) is a set of mutually skew t-dimensional subspaces of \( P \); a partial t-spread \( S \) is said to be a t-spread if every point of \( P \) is on a (necessarily unique) element of \( S \). It is well known (cf. for instance [4, p. 291]) that \( \text{PG}(d, q) \) has a t-spread if and only if \( t + 1 \) divides \( d + 1 \).

Spreads that have found special interest are the so-called geometric spreads. A t-spread \( S \) of \( P \) is called geometric if any three elements \( V_0, V_1, V_2 \) of \( S \) satisfy the following condition: If \( V_0 \) intersects the subspace \( \langle V_1, V_2 \rangle \) generated by \( V_1 \) and \( V_2 \) nontrivially, then \( V_0 \) is contained in \( \langle V_1, V_2 \rangle \). It is known [1, 5] that \( \text{PG}(d, q) \) has a geometric t-spread if (and only if) \( t + 1 \) divides \( d + 1 \). Geometric t-spreads \( S \) have many interesting algebraic and geometric properties. For instance, the elements of \( S \) form the 1-dimensional subspaces of a vector space over GF\( (q^{t+1}) \) [1]: equivalently, the elements of \( S \) together with the subspaces of the form \( \langle V_1, V_2 \rangle \), where \( V_1, V_2 \in S, V_1 \neq V_2 \), form a Desarguesian projective space of order \( q^{t+1} \) [5].
In this paper we shall prove a combinatorial characterization of geometric spreads inside the class of all partial spreads. In order to motivate the main result, we quote a result from [2].

**Result 1.** Let $S$ be a partial $t$-spread in $\text{PG}(d, q)$, where $d = a(t + 1) - 1$ with the property that every subspace of a certain dimension $r$ contains at least one element of $S$. Then $r \geq d + 1 - a (= at)$.

Now we can formulate the main result.

**Theorem.** Let $S$ be a partial $t$-spread in $\text{PG}(d, q)$, where $d = a(t + 1) - 1$, with the property that every subspace of a certain dimension $d + 1 - a$ contains at least one element of $S$. Then:

(a) $S$ is a (total) spread of $P$;
(b) $S$ is geometric, when $a = 3$.

This theorem will follow from more general results (see Corollaries 1 and 2).

2. Proof of the theorem

Let us denote by $P = \text{PG}(d, q)$ the finite projective space of order $q$ and dimension $d \geq 3$. Then there exist integers $a$ and $b$ with $-1 \leq b \leq t - 1$ satisfying $d = a(t + 1) + b$.

Let $S$ be a partial $t$-spread of $P$ with the property that any subspace of codimension $a$ (that is of dimension $a + b + 1$) of $P$ contains at least one element of $S$.

**Lemma 1.** Every subspace $U$ of codimension $a - 1$ (that is of dimension $a + b + 2$) contains at least $q + 1$ elements of $S$; equality holds if and only if the elements of $S$ in $U$ form a $t$-spread in an appropriate $(2t + 1)$-dimensional subspace of $U$.

**Proof.** Let $S_U$ be the set of elements of $S$ contained in $U$. We distinguish two cases.

**Case 1:** There exists a subspace $X$ of dimension $(a - 2)t + b + 1$ of $U$ which has no point in common with any element of $S_U$.

Since the quotient geometry $U/X$ has dimension $a + b + 2 - ((a - 2)t + b + 1) - 1 = 2t$, and any hyperplane of $U$ through $X$ contains at least one element of $S_U$, we obtain

$$|S_U| \cdot (q^{t-1} + \cdots + q + 1) \geq (q^{2t} + \cdots + q + 1) \cdot 1.$$ (Note that the number of hyperplanes of $U$ through $X$ and an element of $S_U$ equals $q^{t-1} + \cdots + q + 1$.) It follows that $|S_U| \geq q^{t+1} + 1$. 

Case 2: Any subspace of dimension \((a - 2)t + b + 1\) of \(U\) intersects at least one element of \(S_U\).

Then, by the theorem of Tallini [6] and Bose–Burton [3], the number \(v\) of points on the elements of \(S_U\) satisfies
\[
v \geq q^{(ar+2)-(a-2)t-1} + \cdots + q + 1 = q^{2t+1} + \cdots + q + 1;
\]
equality holds if and only if these \(v\) points are the points of a \((2t + 1)\)-dimensional subspace of \(U\). It follows that
\[
|S_U| \geq q^{t+1} + 1,
\]
with equality if and only if \(S_U\) is a \(t\)-spread in a \((2t + 1)\)-dimensional subspace of \(U\).

Cases 1 and 2 together prove the lemma. □

**Lemma 2.** Denote by \(U_i\) a subspace of dimension \(at + b + 1 + i\) of \(P\) \((0 \leq i \leq a - 1)\). Then \(U_i\) contains at least \(q^{(i+1)} + q^{(i-1)(t+1)} + \cdots + q^{t+1} + 1\) elements of \(S\). In particular, \(|S| \geq q^{(a-1)(t+1)} + \cdots + q^{t+1} + 1\).

**Proof.** We proceed by induction on \(i\). The case \(i = 0\) is the hypothesis and the case \(i = 1\) has been handled in Lemma 1.

Suppose now \(i \geq 2\) and assume that the assertion is true for \(i - 1\). Fix a subspace \(W\) of dimension \(at + b + 1 + i\). Since, by induction, any hyperplane of \(W\) contains at least \(q^{(i-1)(t+1)} + \cdots + q^{t+1} + 1\) elements of \(S\), we obtain
\[
|S_U| \cdot (q^{(a+i-1)} + \cdots + q + 1) \geq (q^{(i-1)} + \cdots + q + 1) \cdot (q^{(i-1)(t+1)} + \cdots + q^{t+1} + 1).
\]
Therefore,
\[
|S_U| > q^{t+1} \cdot (q^{(i-1)(t+1)} + \cdots + q^{t+1} + 1) = q^{i(t+1)} + q^{(i-1)(t+1)} + \cdots + q^{t+1}.
\]
This is the first assertion. It follows in particular \((i = a - 1)\) that
\[
|S| = |S_u| \geq q^{(a-1)(t+1)} + q^{(a-2)(t+1)} + \cdots + q^{t+1} + 1. \quad □
\]

**Corollary 1.** Suppose \(a = a(t + 1) - 1\) \((that is, b = -1)\). Then \(S\) is a \((total)\) \(t\)-spread of \(P\).

**Proof.** By Lemma 2, \(|S| \geq q^{(a-1)(t+1)} + \cdots + q^{t+1} + 1\). A trivial counting argument shows that a partial \(t\)-spread of \(PG(a(t + 1) - 1, q)\) with so many elements is in fact a spread. □

Thus, part (a) of the theorem is proved.

**Proposition 1.** Suppose \(a = 3\). If \(|S| \leq q^{2(t+1)} + q^{t+1} + 1\), then \(S\) is a geometric \(t\)-spread of an appropriate \((3t + 2)\)-dimensional subspace \(P'\) of \(P\).
Proof. First we show that $S$ is a $t$-spread in an appropriate subspace $P'$ of dimension $3t + 2$.

In order to show this we assume that there exists a subspace $Y$ of dimension $b + 1$ which intersects no element of $S$. Since by Lemma 1 any hyperplane through $Y$ contains at least $q^{t+1} + 1$ elements of $S$, we would have

$$|S| \cdot (q^{2t} + \cdots + q + 1) \geq (q^{3t+1} + \cdots + q + 1)(q^{t+1} + 1)$$

$$= (q^{2t} + \cdots + q + 1)(q^{2t+2} + q^{t+1} + q) + 1,$$

contradicting $|S| \leq q^{3t+1} + q^{t+1} + 1$. Therefore, by the theorem of Tallini [6] and Bose–Burton [3] it follows that

$$q^{3t+2} + \cdots + q + 1 \geq (q^{t} + \cdots + q + 1) \cdot |S| \geq q^{3t+3+b-(b+1)} + \cdots + q + 1.$$

Hence we have equality, and the quoted theorem implies that $S$ is indeed a $t$-spread in a suitable $(3t + 2)$-dimensional subspace $P'$ of $P$. Therefore, every hyperplane $H$ of $P'$, contains precisely $q^{t+1} + 1$ elements of $S$. Thus, Lemma 1 implies that these elements of $S$ form a $t$-spread in a subspace $U_H$ of dimension $2t + 1$ of $H$.

From this it follows easily that $S$ is geometric. Let $V_1, V_2$ be two distinct elements of $S$. Since they are contained in a common hyperplane $H$, they both belong to $U_H$; so $S$ induces a spread in $(V_1, V_2)$. Since this is true for any two elements of $S$, the spread $S$ is geometric. $\square$

Corollary 2. Let $S$ be a partial $t$-spread in $P = PG(3t + 2, q)$ with the property that any subspace of dimension $3t$ contains at least one element of $S$. Then $S$ is a geometric $t$-spread of $P$.

This proves part (b) of the theorem.

We conclude by an open question. Can one generalize Theorem 1 to an arbitrary dimension $d = a(t + 1) + b$ ($a \geq 3$)? One could do so if Lemma 1 could be proved for subspaces $U$ of higher dimension.

Added in proof. The question above can be answered in the affirmative (joint work with J. Ueberberg).

References