Determination of poles of sectionally meromorphic functions

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Abstract: The numerical methods of Abd-Elall, Delves and Reid for locating poles of meromorphic functions inside smooth closed contours $C$ in the complex plane are generalized to apply to sectionally meromorphic functions satisfying a homogeneous or nonhomogeneous Riemann–Hilbert boundary value problem on the simple closed contours of their discontinuity (which may intersect $C$). In the first case, the zeros of the meromorphic functions can be determined as well. Two simple illustrations of the present generalization are also presented.

Keywords: Analytic and meromorphic functions, poles, Riemann–Hilbert boundary value problem, sectionally analytic and meromorphic functions, zeros.

1. Introduction

Abd-Elall, Delves and Reid proposed two numerical methods for locating the zeros and poles of meromorphic functions (that is, analytic functions with poles) inside a simple smooth closed contour $C$ [1]. For such a function $\Phi(z)$, meromorphic in the domain $S$ inside $C$ and on $C$, one has to evaluate integrals of the form

$$G_k = \frac{1}{2\pi i} \int_C t^k \frac{\Phi'(t)}{\Phi(t)} \, dt, \quad t = x + iy,$$

in the first method of Abd-Elall, Delves and Reid [1] and integrals of the form

$$G_k^* = \frac{1}{2\pi i} \int_C t^k [\Phi(t)]^{-1} \, dt$$

in the second method. In (2) the plus sign concerns poles and the minus sign zeros. It is also assumed that $\Phi(z)$ does not have zeros and poles on $C$. We will not enter into details concerning the method of Abd-Elall, Delves and Reid, which can be found with sufficient clarity in [1]. We just mention that if $\kappa$ is the index of $\Phi(z)$ along $C$, defined by [3, §12.1]

$$\kappa = \frac{1}{2\pi} \arg \Phi(t) \big|_C,$$

then an integration by parts in (1) leads directly to [5]

$$G_k = -\frac{k}{2\pi i} \int_C t^{k-1} \log[(t-a)^{-}\Phi(t)] \, dt + \kappa a^k,$$

where $a$ is an arbitrary point in the domain $S$ surrounded by $C$. 
It is also clear that locating zeros and poles of meromorphic functions is a frequently appearing problem in physics and engineering. For example, points of application of concentrated loads or points of appearance of edge dislocations in plane elasticity constitute poles of the complex potential $\Phi(z)$ of Muskhelishvili [6]. These points can be determined by the methods of Abd-Elall, Delves and Reid [1], which are the only methods available for finding poles of meromorphic functions. Probably, the method described in [2] can also be generalized to apply to the same problem. Finally, another problem where we have to find poles of meromorphic functions is that of locating crack tips in plane elasticity [4]. The method of [1] was used for the solution of this problem in a special case [4].

The aim of this note is to generalize the results of [1], based on (2) and (4) to apply to sectionally meromorphic functions, that is, to meromorphic functions discontinuous along one or more closed contours $L_k$ ($k = 1(1)n$), probably lying in $S$ or intersecting $C$. This situation appears, for example, in plane elasticity problems with inclusions ($L_k$ being in this case the boundaries of the inclusions). In this problem we assume that $\Phi(z)$ satisfies along $L_0 = \bigcup_{k=1}^{n} L_k$ a Riemann–Hilbert boundary value problem [3]. The approach that we will follow consists in replacing the sectionally meromorphic function $\Phi(z)$ (discontinuous along $L_0$) by a meromorphic function $R(z)$ (continuous along $L_0$), for which the method of [1] is directly applicable.

2. Analysis

We consider a sectionally meromorphic function $\Phi(z)$ in the domain $S_0$ of its definition inside a closed contour $C_0$ (Fig. 1). $\Phi(z)$ is assumed to be discontinuous on $n$ smooth closed contours $L_k$ lying in $S_0$ (Fig. 1). On these contours the boundary values $\Phi^\pm(t)$ of $\Phi(z)$ (where $S_k^+$ denotes the region surrounded by $L_k$) are assumed satisfying a Riemann–Hilbert boundary value problem [3]

$$
\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in L_0.
$$

(5)

For convenience, both the coefficient $G(t)$ and the free term $g(t)$ of this problem are assumed to be Hölder-continuous functions [3]; moreover, $G(t)$ is assumed without zeros and poles. We wish to determine the poles of $\Phi(z)$ in the region $S$ surrounded by a smooth closed contour $C$ lying
in $S_0$ (Fig. 1) on the basis of the boundary values $\Phi(t)$ of this function along $C$, as well as the values of $G(t)$ and $g(t)$ along $L_0$.

Clearly, we can ignore the discontinuities of $\Phi(z)$ on the contours $L_k$ lying completely outside $C$. We restrict our attention to the contours $L_k$ ($k = 1(1)m$, $m \leq n$) lying inside $C$ or intersecting $C$. The positive direction along both $L_k$ and $C$ is assumed to be the anticlockwise one (Fig. 1). The whole set of points of the contours $L_k$ ($k = 1(1)m$) is denoted by $L$, that is, $L = \bigcup_{k=1}^{m} L_k$.

Finally, we denote by $\kappa_k$ the indices of $G(t)$ along $L_k$ [3], that is,

$$\kappa_k = \frac{1}{2\pi} \left[ \arg G(t) \right]_{L_k}, \quad k = 1(1)m. \quad (6)$$

Under these conditions, we obtain easily (by slightly modifying the method described in [3, §16] for the solution of the Riemann-Hilbert boundary value problem for multiply-connected domains) the solution of our problem (5) in $S$:

$$\Phi(z) = \Psi(z) + R(z), \quad (7)$$

where

$$X(z) = \exp[\Gamma(z)], \quad z \in S^+, \quad S^+ = \bigcup_{k=1}^{m} S_k^+. \quad (8a)$$

$$X(z) = \Pi(z) \exp[\Gamma(z)], \quad z \in S^-, \quad S^- = S - S^+, \quad (8b)$$

with

$$\Pi(z) = \prod_{k=1}^{m} (z - a_k)^{-\kappa_k} \quad (9)$$

($a_k$ being an arbitrary point in $S_k^+$) and

$$\Gamma(z) = \frac{1}{2\pi i} \int_{L} \frac{\log[\Pi(t)G(t)]}{t - z} \, dt. \quad (10)$$

Moreover, $\Psi(z)$ is defined by

$$\Psi(z) = \frac{1}{2\pi i} \int_{L} \frac{g(t)}{X(t)(t - z)} \, dt \quad (11)$$

and $R(z)$ is a meromorphic function analytic along $L$, that is, analytic in $S$ with the exception of its poles. (In [3] $R(z)$ was restricted to be a polynomial of a sufficiently low degree; this restriction does not hold here.)

From (7) or from the equivalent formula

$$R(z) = \left[ \Phi(z)/X(z) \right] - \Psi(z) \quad (12)$$

it is clear that $\Phi(z)$ and $R(z)$ have the same poles in $S$ (with the same orders), since $X(z)$ has neither zeros nor poles, as is clear from (8), and $\Psi(z)$ has no poles as is clear from (11). Therefore, the method of Abd-Elall, Delves and Reid [1] is now directly applicable and (2) or (4) can be used for $R(z)$ with boundary values along $C$ determined by

$$R(t) = \left[ \Phi(t)/X(t) \right] - \Psi(t) \quad (13)$$

on the basis of the available corresponding values of the original meromorphic function $\Phi(z)$ and the values of $G(t)$ and $g(t)$ along $L$ (because of (10) and (11)). (Clearly, there is no need
that $\Phi(z)$ be known analytically in $S$. Just its boundary values $\Phi(t)$ on $C$ are required to be available even numerically.

Unfortunately, as is clear from (12), because of $\Psi(z)$ the zeros of $R(z)$ and $\Phi(z)$ do not coincide in general so it is only for poles that the above approach is always valid. But in the special case when we have $g(t) \equiv 0$ in (5), that is, when we have a homogeneous Riemann–Hilbert boundary value problem, whence $\Psi(z) \equiv 0$ because of (11), not only the poles but also the zeros of $\Phi(z)$ coincide with the poles and zeros of $R(z)$, respectively (since $X(z)$ has neither zeros nor poles as already mentioned). Of course, in this particular case, the method is also applicable to simple analytic functions $\Phi(z)$ (without poles in $S$) for the determination of their zeros.

3. Applications

Let us now illustrate the above approach by means of two very simple applications. Consider at first the problem of an isotropic plane elastic medium with misfitting inclusions (consisting of the same elastic material). This problem is described in detail in [6, §109]. The complex potential $\Phi(z)$ will be discontinuous along the boundaries $L_k$ of the inclusions and the following simple discontinuity boundary value problem is valid:

$$\Phi^+(t) - \Phi^-(t) = g(t), \quad t \in L = \bigcup_{k=1}^{m} L_k,$$

where $g(t)$ is a known quantity proportional to the derivative of the complex discontinuity of displacements along $L$ as is clear from the results of [6]. For this boundary value problem, (6) to (12) yield

$$\kappa_k = 0, \quad k = 1(1)m, \quad \Pi(z) = 1, \quad \Gamma(z) \equiv 0, \quad X(z) \equiv 1$$

and

$$R(z) = \Phi(z) - \Psi(z) \quad \text{with} \quad \Psi(z) = \frac{1}{2\pi i} \int_{L} \frac{g(t)}{t-z} \, dt.$$

With $R(z)$ the discontinuities of $\Phi(z)$ along the boundaries $L_k$ of the misfitting inclusions disappear and the points of the elastic medium associated with concentrated loads or edge dislocations can be determined as poles of $R(z)$.

As a second simple application, we consider the case when (5) reduces to the following homogeneous Riemann–Hilbert boundary value problem

$$\Phi^+(t) = c_k \Phi^-(t), \quad t \in L_k, \quad c_k \neq 0, \quad k = 1(1)m.$$

In this case, $G(t)$ takes constant values, $c_k$, along the contours $L_k$. In this particular case, we obtain directly from (6) to (12)

$$\kappa_k = 0, \quad k = 1(1)m, \quad \Pi(z) \equiv 1, \quad \Psi(z) \equiv 0,$$

$$\Gamma(z) = \log c_k, \quad X(z) = c_k, \quad z \in S_k^+, \quad (18)$$

$$\Gamma(z) = 0, \quad X(z) = 1, \quad z \in S^-,$$

and

$$R(z) = \Phi(z)/c_k, \quad z \in S^+_k \quad R(z) = \Phi(z), \quad z \in S^-.$$

$$\text{(19)} \quad \text{(20)}$$
This means that $\Phi(z)$ coincides with $R(z)$ along $C$ and, therefore, both the zeros and poles of $\Phi(z)$ completely ignore its discontinuity along $L$ and can be determined as if $\Phi(z)$ were a meromorphic function in $S$ by the method of [1] without any modification. (Alternatively, the first equation of (20) indicates that the division of $\Phi(z)$ by $c_k$ in $S_k^+$ is sufficient for this function to become analytic on $L_k^+$.)

References