On Auslander–Reiten components and splitting trace lattices for integral group rings

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Abstract

Let \( G \) be a finite group and \( \mathcal{O} \) a complete discrete valuation ring of characteristic zero with residue class field \( k = \mathcal{O}/\pi \mathcal{O} \) of characteristic \( p > 0 \). Suppose that \( \mathcal{O} \) is sufficiently large to satisfy certain conditions and the group ring \( \mathcal{O} G \) is of infinite representation type. Let \( \Theta \) be a connected component of the Auslander–Reiten quiver of \( \mathcal{O} G \). We show that if \( \Theta \) contains an \( \mathcal{O} G \)-lattice \( M \) such that \( M/\pi M \) is an indecomposable \( kG \)-module and \( \text{rank}_{\mathcal{O}} M \) is not divisible by \( p \), then the tree class of \( \Theta \) is \( A_\infty \) and \( M \) lies at the end of \( \Theta \).

1. Introduction

Let \( G \) be a finite group, \( p \) a prime number dividing the order of \( G \) and \( (K, \mathcal{O}, k) \) a \( p \)-modular system, that is, \( K \) is a complete discrete valuation field of characteristic zero with multiplicative valuation \( \psi \) and \( \mathcal{O} \) is a valuation ring of \( \psi \) with unique maximal ideal \( \pi \mathcal{O} \) and residue class field \( k = \mathcal{O}/\pi \mathcal{O} \) of characteristic \( p \). We use \( R \) to denote either \( \mathcal{O} \) or \( k \). Let \( B \) be a block of the group ring \( R G \) and \( \Gamma(B) \) the Auslander–Reiten quiver of \( B \). For a connected component \( \Theta \) of \( \Gamma(B) \), we denote by \( \Theta_s \) the stable part of \( \Theta \). Webb [19] showed that the tree class of \( \Theta_s \) is either a Euclidean diagram or one of the infinite trees \( A_\infty, B_\infty, C_\infty, D_\infty \) and \( A_\infty^{\infty} \) if the defect group of \( B \) is not cyclic. In the case where \( R = k \) and \( k \) is algebraically closed, Erdmann showed that the tree class of \( \Theta_s \) is \( A_\infty \) if \( B \) is of wild representation type [9]. It is known that a block of \( kG \) is of wild representation type if its defect group is neither cyclic, dihedral, semidihedral nor generalized quaternion. Concerning the representation type of group rings over \( \mathcal{O} \), we refer to the table due to Dieterich [8].

Throughout this paper, we assume that \( (K, \mathcal{O}, k) \succ (K', \mathcal{O}', k') \) is an extension of \( p \)-modular system satisfying the following two conditions:

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(2.1) The ramification index of $\varphi$ over the valuation $\varphi'$ of $K'$ is greater than two, namely, $\pi' \mathcal{O} \subseteq \pi^2 \mathcal{O}$, where $\pi'$ is a generator of the unique maximal ideal of $\mathcal{O}'$;

(2.11) $k = k'$ and $k$ is algebraically closed.

An $RG$-lattice $X$ is called a splitting trace lattice if the trace map $\text{Tr}: \text{End}_R X \to R$ is a splittable $RG$-epimorphism. Auslander and Carlson [3] showed that under the assumption (2.11), an indecomposable $RG$-lattice $X$ is splitting trace lattice if and only if only if rank$_R X$ is not divisible by $p$. See also Benson and Carlson [5]. In this paper, we consider a connected component $\Theta$ of $\Gamma(OG)$ containing a splitting trace $OG$-lattice $M$ satisfying one of the conditions (A) or (B) mentioned in Section 3. Assuming that $OG$ is of infinite representation type, we show that the tree class of $\Theta$ is $A_\infty$ under the hypotheses (2.1) and (2.11), see Theorems 3.1 and 3.3. In Section 4, we discuss the tensor product of $M$ with the connected component $\Delta$ of $\Gamma(OG)$ containing the trivial $OG$-lattice $O_G$. It will be shown that tensoring with $M$ induces a graph isomorphism from $\Delta$ onto $\Theta$, see Theorem 4.1. A similar assertion for $R = k$ in Theorem 4.1 was shown in [11, Proposition 3.3].

For the basic facts and terminology used here, see the books of Assem, Simson and Skowroński [1], Auslander, Reiten and Smalø [2], Benson [4] and Nagao and Tsushima [15].

2. Preliminaries

All $RG$-modules are assumed to be finitely generated right modules. An $RG$-lattice means an $RG$-module which is free as an $R$-module. If $L$ and $M$ are $RG$-lattices, then $L \otimes_R M$ is an $RG$-lattice with the operation of $G$ given by $(x \otimes y)g = xg \otimes yg$ for all $x \in L$, $y \in M$ and $g \in G$. Throughout this paper, $L \otimes M$ means $L \otimes_R M$. For a short exact sequence $S : 0 \to L_1 \to L_2 \to L_3 \to 0$ of $RG$-lattices, a tensor product sequence $S \otimes M : 0 \to L_1 \otimes M \to L_2 \otimes M \to L_3 \otimes M \to 0$ is exact since $RG$-lattices are $R$-free. We write $L^*$ for the $R$-dual $\text{Hom}_R(L, R)$ of $L$. Let $X$ be an $OG$-lattice. We denote by $\overline{X}$ the factor module $X/\pi X$, so that $\overline{X}$ is regarded as a $kG$-module. If $S : 0 \to X_1 \to X_2 \to X_3 \to 0$ is a short exact sequence of $OG$-lattices, then we have a short exact sequence $\overline{S} : 0 \to \overline{X}_1 \to \overline{X}_2 \to \overline{X}_3 \to 0$ of $kG$-modules.

It is known that the Auslander–Reiten translate of $\Gamma(OG)$ is the Heller operator $\Omega$ in the $OG$-lattice category. See, for example, [16]. For a non-projective indecomposable $OG$-lattice $X$, we write $A(X)$ for the almost split sequence $0 \to OGX \to m(X) \to X \to 0$, where we denote by $m(X)$ the middle term of $A(X)$. $A(X)$ is constructed as a pullback of the projective cover $P_X$ of $X$ along an almost projective $OG$-endomorphism $\rho$ of $X$:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega X & \longrightarrow & m(X) & \longrightarrow & X & \longrightarrow & 0 \\
& & | & & | & \downarrow \text{pull back} & | & & \\
0 & \longrightarrow & \Omega X & \longrightarrow & P_X & \longrightarrow & X & \longrightarrow & 0 \\
\end{array}
\]

Here, an almost projective $OG$-endomorphism of $X$ is a generator of the simple socle $\text{Soc}((\text{End}_{OG}(X))$ of $\text{End}_{OG}(X) = \text{End}_{OG}(X)/\text{End}_{OG}(X)^C$, where $\text{End}_{OG}(X)^C$ is the set of all projective $OG$-endomorphisms of $X$ (see, for example, [18, (34.11) Theorem]). Since $\text{End}_{OG}(O_G) = O\text{id}_{O_G}$ and $\text{End}_{OG}(O_G)^C = [G]O\text{id}_{O_G}$ for the trivial $OG$-lattice $O_G$, $[G]O\text{id}_{O_G}$ is almost projective. Consider the tensor product sequence of $A(O_G)$ with an absolutely indecomposable $OG$-lattice $X$:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega O_G \otimes X & \longrightarrow & m(O_G) \otimes X & \longrightarrow & O_G \otimes X & \longrightarrow & 0 \\
& & | & & | & \downarrow \text{id}_{O_G} \otimes \text{id}_X & | & & \\
0 & \longrightarrow & \Omega O_G \otimes X & \longrightarrow & P_{O_G} \otimes X & \longrightarrow & O_G \otimes X & \longrightarrow & 0 \\
\end{array}
\]
Carlson and Jones showed that \( \frac{|G|}{\pi} \cdot \text{id}_X \) is almost projective if and only if \( \text{rank}_O X \) is not divisible by \( p \) [6, Proposition 4.7], and \( \frac{|G|}{\pi} \cdot \text{id}_X \) is projective if \( p \mid \text{rank}_O X \). Hence, we have the following fact due to Auslander and Carlson [3] and Benson and Carlson [5].

**Proposition 2.1.** (See [3, Theorem 3.6] and [5, Proposition 2.15].) Assume \((\Omega)\). Let \( X \) be an indecomposable \( OG \)-lattice and \( A(O_G) : 0 \to \Omega O_G \to m(O_G) \to O_G \to 0 \) the almost split sequence terminating in the trivial \( OG \)-lattice \( O_G \).

1. If \( \text{rank}_O X \) is not divisible by \( p \), then the tensor product sequence \( A(O_G) \otimes X : 0 \to \Omega O_G \otimes X \to m(O_G) \otimes X \to O_G \otimes X \to 0 \) is written as a direct sum of the almost split sequence \( A(X) \) and some projective \( OG \)-lattice \( I \):

   \[
   0 \longrightarrow \Omega X \longrightarrow m(X) \longrightarrow X \longrightarrow 0
   \]

   \[
   \oplus \oplus
   \]

   \[
   I \longrightarrow I
   \]

2. If \( \text{rank}_O X \) is divisible by \( p \), then the tensor product sequence \( A(O_G) \otimes X \) is split.

Also, we will need the following fact due to Auslander and Carlson [3] and Benson and Carlson [5].

**Proposition 2.2.** Assume \((\Omega)\), and let \( X \) be an indecomposable \( RG \)-lattice.

1. ([3, Theorems 3.6, 4.7] and [5, Theorem 2.1]) The following are equivalent:
   (i) \( X \) is a splitting trace lattice, namely, the trace map \( \text{Tr} : \text{End}_R X \to R \) is splittable;
   (ii) \( p \mid \text{rank}_R X \);
   (iii) \( R_G \mid \text{End}_R(X) \cong X \otimes X^* \).

   If \( X \) is a splitting trace lattice, then the multiplicity of \( R_G \) in \( \text{End}_R(X) \cong X \otimes X^* \) is one.

2. ([3, Corollary 4.3] and [5, Proposition 2.2]) If the \( R \)-rank of \( X \) is divisible by \( p \), then so are all \( R \)-ranks of the indecomposable direct summands of \( X \otimes Y \) for any \( RG \)-lattice \( Y \).

3. ([3, Proposition 4.1] and [5, Theorem 2.1]) Let \( Y \) be an indecomposable \( RG \)-lattice. If \( R_G \mid X \otimes Y^* \), then \( Y \cong X \).

   For the proof of the following fact, see [17, Proposition 2.10].

**Lemma 2.3.** Let \( X \) be a non-projective indecomposable \( RG \)-lattice and \( Q \) a subgroup of \( G \). Then an almost split sequence \( A(X) \) splits on restriction to \( Q \) if and only if \( Q \) does not contain a vertex of \( X \).

For a \( kG \)-module \( M \), the kernel \( Z \) of the projective cover \( P_M \) of \( M \) viewed as an \( OG \)-module is called the \( Heller \ OG \)-lattice of \( M \): \( 0 \to Z \to P_M \to M \to 0 \) (exact).

**Lemma 2.4.** Assume \((\Omega)\) and \((\Omega)\). Suppose that a non-projective indecomposable \( OG \)-lattice \( L \) is not isomorphic to any Heller \( OG \)-lattice of a \( kG \)-module. Let \( A(L) : 0 \to \Omega L \to m(L) \to L \to 0 \) be the almost split sequence terminating in \( L \). Then the reduced short exact sequence \( \overline{A(L)} : 0 \to \Omega L/\pi \Omega L \to m(L)/\pi m(L) \to L/\pi L \to 0 \) of \( kG \)-modules is split.

**Proof.** See [12, Proposition 4.5]. \( \square \)

Let \( Z \) be an indecomposable non-projective Heller \( OG \)-lattice of a \( kG \)-module \( M \). Then \( \text{rank}_O Z = \text{rank}_O P_M \), where \( P_M \) is a projective cover of \( M \) as \( OG \)-modules, and we see that \( p \mid \text{rank}_O Z \). Suppose that \( Z \) belongs to a block \( B \) of infinite representation type, and let \( \Theta \) be the connected component of \( \Gamma(B) \) containing \( Z \). Then, by [13, Theorem], the tree class of \( \Theta \) is \( A_\infty \) and \( Z \) lies at the end of \( \Theta \). Thus, the first part of the following lemma holds.
Lemma 2.5. Assume (z1) and (zII), and suppose that a block \( B \) of \( OG \) is of infinite representation type. Let \( \Theta \) be a connected component of \( \Gamma(B) \).

(1) If \( \Theta \) contains a Heller \( OG \)-lattice, then the tree class of \( \Theta \) is \( A_\infty \) and all \( O \)-ranks of the \( OG \)-lattices in \( \Theta \) are divisible by \( p \).

(2) If \( \Theta \) contains an indecomposable \( OG \)-lattice whose \( O \)-rank is not divisible by \( p \), then \( \Theta \) contains neither Heller \( OG \)-lattices nor projectives.

Proof. The second part (2) follows by (1) and [13, Remark 5.5(1)]. □

Lemma 2.6. Assume (z1) and (zII). Let \( M \) be an indecomposable \( OG \)-lattice and \( \Theta \) a connected component of \( \Gamma(OG) \) containing \( M \). Suppose that \( M/\pi M \) has an indecomposable direct summand \( W \) whose vertex is a Sylow \( p \)-subgroup \( P \) of \( G \) and that \( \Theta \) does not contain any Heller \( OG \)-lattice. Then, for any \( OG \)-lattice \( X \) in \( \Theta \), the following hold.

(1) \( X/\pi X \) has some syzygy of \( W \) as a direct summand. In particular, \( X \) has \( P \) as vertex.

(2) The almost split sequence \( A(X) \downarrow Q \) restricted to any proper subgroup \( Q \) of \( P \) splits.

Proof. Let \( M = X_1 - X_2 - \cdots - X_n = X \) be a walk in \( \Theta \), so that \( X_i \) is a direct summand of the middle term of the almost split sequence \( A(X_{i+1}) \) or \( A(\Omega^{-1}X_{i+1}) \). We proceed by induction on \( n \). Assume that \( X_{n-1}/\pi X_{n-1} \) has some syzygy of \( W \) as a direct summand. As \( X_n \) is not a Heller lattice, \( A(X_n) \) splits by Lemma 2.4. Hence we see that some syzygy of \( W \) is a direct summand of \( X_n/\pi X_n \) and (1) follows. By (1) and Lemma 2.3, (2) follows. □

Choose and fix a non-projective indecomposable \( kG \)-module \( V \). For an \( OG \)-lattice \( X \), we denote by \( \tilde{d}_V(X) \) the number of indecomposable direct summands isomorphic to syzygies of \( V \) in an indecomposable decomposition of \( X/\pi X \).

Also, fix a subgroup \( Q \) of \( G \) and a non-projective indecomposable \( OQ \)-lattice \( U \). Let us denote by \( d_{Q,U}(X) \) the number of indecomposable direct summands isomorphic to syzygies of \( U \) in an indecomposable decomposition of \( X \downarrow Q \).

Lemma 2.7. Assume (z1) and (zII). Let \( \Theta \) be a connected component of \( \Gamma(OG) \) and suppose that \( \Theta \) does not contain any Heller \( OG \)-lattice. Choose and fix an \( OG \)-lattice \( X \) in \( \Theta \).

(1) Let \( V \) be a non-projective indecomposable \( kG \)-module. If \( V \) is a direct summand of \( X/\pi X \), then \( \tilde{d}_V : \Theta \to \mathbb{N} \) is an \( \Omega \)-periodic additive function on \( \Theta \). If no syzygy of \( V \) appears as a direct summand of \( X/\pi X \), then \( \tilde{d}_V(Y) = 0 \) for every \( OG \)-lattice \( Y \) in \( \Theta \).

(2) Suppose that all lattices in \( \Theta \) have a Sylow \( p \)-subgroup \( P \) of \( G \) as vertex. Let \( Q \neq 1 \) be a proper subgroup of \( P \) and \( U \) a non-projective indecomposable \( OQ \)-lattice. If \( U \) is a direct summand of \( X \downarrow Q \), then \( d_{Q,U}(X) \) is an \( \Omega \)-periodic additive function on \( \Theta \). If no syzygy of \( U \) appears as a direct summand of \( X \downarrow Q \), then \( d_{Q,U}(Y) = 0 \) for every \( OG \)-lattice \( Y \) in \( \Theta \).

In particular, if the \( O \)-rank of \( X \) is not divisible by \( p \), then \( \tilde{d}_V \) (resp. \( d_{Q,U} \)) is an \( \Omega \)-periodic additive function on \( \Theta \) for an indecomposable direct summand \( V \) of \( X/\pi X \) (resp. an indecomposable direct summand \( U \) of \( X \downarrow Q \) whose \( O \)-rank is not divisible by \( p \)).

Proof. Lemmas 2.4 and 2.3 imply (1) and (2), respectively. If \( \Theta \) has an \( OG \)-lattice \( X \) whose \( O \)-rank is not divisible by \( p \), then \( \Theta \) does not contain any Heller \( OG \)-lattice by Lemma 2.5. Note that \( X \) is not projective and so \( X/\pi X \) is projective-free. □

Recall that if \( X \) is a non-projective indecomposable \( OG \)-lattice, then \( m(X) \) denotes the middle term of the almost split sequence \( A(X) \).

Lemma 2.8. Assume (z1) and (zII). Suppose that a connected component \( \Theta \) of \( \Gamma(OG) \) is of type \( \mathbb{Z}D_\infty \). Let \( X \) and \( X' \) be non-projective indecomposable \( OG \)-lattices lying at the end of \( \Theta \) such that \( m(X) \equiv m(X') \mod p \). Then we have \( \text{rank}_O X \equiv \text{rank}_O X' \mod p \).
Proof. Suppose that

\[ X/\pi X = \bigoplus_{\lambda \in A} a_{\lambda,i} \Omega^i W_{\lambda} \oplus Y \]

and

\[ X'/\pi X' = \bigoplus_{\lambda \in A} a'_{\lambda,i} \Omega^i W_{\lambda} \oplus Y' \]

are direct decompositions as \( kG \)-modules satisfying the following:

(i) \( W_{\lambda} (\lambda \in A) \) are indecomposable \( kG \)-modules with \( p \nmid \dim_k W_{\lambda} \);

(ii) If \( \lambda_1 \neq \lambda_2 \), then the \( \Omega \)-orbits \( \{ \Omega^j W_{\lambda_1} \}_{j \in \mathbb{Z}} \) of \( W_{\lambda_1} \) and \( \{ \Omega^j W_{\lambda_2} \}_{j \in \mathbb{Z}} \) of \( W_{\lambda_2} \) are distinct;

(iii) All \( k \)-dimensions of the indecomposable direct summands of \( Y \oplus Y' \) are divisible by \( p \);

(iv) For each \( \lambda \in A \), the multiplicity \( a_{\lambda,i} \) of \( \Omega^i W_{\lambda} \) as a direct summand of \( X/\pi X \) is zero if \( i < 0 \).

Note that \( W_{\lambda} (\lambda \in A) \) are not \( \Omega \)-periodic: Indeed, if \( W_{\lambda} \) is \( \Omega \)-periodic, then a Sylow \( p \)-subgroup of \( G \) is cyclic or a generalized quaternion 2-group and all the \( OG \)-lattices are \( \Omega \)-periodic. [10] implies that \( \Theta \) is an infinite tube. (See also [20, Theorem 1].)

Since \( \Theta \) does not contain any Heller \( OG \)-lattice by Lemma 2.5, both \( \mathcal{A}(X) \) and \( \mathcal{A}(X') \) are split by Lemma 2.4. Hence we have

\[ \Omega X/\pi \Omega X \oplus X/\pi X \cong m(X)/\pi m(X) \]

\[ \cong m(X')/\pi m(X') \cong \Omega X'/\pi \Omega X' \oplus X'/\pi X' \]

and

\[ \bigoplus_{\lambda \in A} a_{\lambda,i} \Omega^{i+1} W_{\lambda} \oplus \bigoplus_{\lambda \in A} a_{\lambda,i} \Omega^i W_{\lambda} \]

\[ \cong \bigoplus_{\lambda \in A} a'_{\lambda,i} \Omega^{i+1} W_{\lambda} \oplus \bigoplus_{\lambda \in A} a'_{\lambda,i} \Omega^i W_{\lambda} \].

For each \( \lambda \), the multiplicity of \( W_{\lambda} \) as a direct summand of \( m(X)/\pi m(X) \) is \( a_{\lambda,0} \), and that as a direct summand of \( m(X')/\pi m(X') \) is \( a'_{\lambda,0} \). Hence, we have \( a_{\lambda,0} = a'_{\lambda,0} \) for each \( \lambda \). Also, considering the multiplicity of \( \Omega^i W_{\lambda} \), we see \( a_{\lambda,i-1} + a_{\lambda,i} = a'_{\lambda,i-1} + a'_{\lambda,i} \) for \( 1 \leq i \). Using induction on \( i \), we have \( a_{\lambda,i} = a'_{\lambda,i} \) for all \( i \in \mathbb{N} \) and \( \lambda \in A \). \( \square \)

Lemma 2.9. (1) Let \( X \) be an indecomposable \( RG \)-lattice whose \( R \)-rank is not divisible by \( p \). Suppose that \( \text{rank}_R X = \text{rank}_G \Omega X \). Then \( p = 2 \) and a Sylow 2-subgroup \( P \) of \( G \) is a cyclic group \( C_2 \) of order 2.

(2) Let \( Q \) be a \( p \)-group and \( U \) an indecomposable \( OQ \)-lattice whose \( O \)-rank is not divisible by \( p \). Then \( U \cong \Omega U \).

Proof. (1) Since \( \text{rank}_R X = \text{rank}_G \Omega X = s|P| - \text{rank}_R X \) for some integer \( s \), we have \( 2(\text{rank}_R X) = s|P| \). Since \( \text{rank}_R X \) is not divisible by \( p \), we conclude that \( p = 2 \) and \( |P| = 2 \).

(2) Assume that \( U \cong \Omega U \). By (1), we see that \( p = 2 \) and \( Q \) is a cyclic group \( C_2 \) of order 2. From [7, Proposition 3.1], \( OC_2 \) is of finite representation type and \( U \cong OQ \) or \( U \cong \Omega OQ \) since \( 2 \nmid \text{rank}_O U \). But this is absurd since \( OQ \cong \Omega OQ \). \( \square \)

If \( Q \) is a cyclic \( p \)-group, then the \( \Omega \)-periodicity of \( OQ \) is 2. It is also known that if \( p = 2 \) and \( Q \) is a generalized quaternion 2-group then the \( \Omega \)-periodicity of \( OQ \) is 4.
Lemma 2.10. (1) Suppose that $p = 2$ and $Q$ is a generalized quaternion 2-group. Let $U$ be an indecomposable $O_Q$-lattice of odd $O$-rank. Then neither $\Omega O_Q$ nor $\Omega^2 O_Q$ is a direct summand of $U \otimes U^*$, and $\Omega^2 O_Q$ is a direct summand of $U \otimes U^*$ if and only if $U \cong \Omega^2 U$.

(2) Let $Q$ be a cyclic $p$-group and $U$ an indecomposable $O_Q$-lattice whose $O$-rank is not divisible by $p$. Then $\Omega O_Q$ is not a direct summand of $U \otimes U^*$.

Proof. (1) Note that for an integer $t$, $\Omega^t O_Q \mid U \otimes U^*$ if and only if $\Omega^t U \cong U$ by Proposition 2.2(3). In particular, $\Omega^2 O_Q \mid U \otimes U^*$ if and only if $\Omega^2 U \cong U$. Now we claim that $\Omega O_Q \not\mid U \otimes U^*$: Indeed, if $\Omega O_Q$ is a direct summand of $U \otimes U^*$, then $U \cong \Omega U$, which contradicts Lemma 2.9(2). Likewise, we see that $\Omega^3 O_Q \not\mid U \otimes U^*$.

(2) Since $\Omega U \not\cong U$ by Lemma 2.9(2), $\Omega O_Q$ is not a direct summand of $U \otimes U^*$ by Proposition 2.2(3).

3. Auslander–Reiten components containing splitting trace lattices

In this section, we consider an indecomposable $O_G$-lattice $M$ satisfying one of the following two conditions:

(A) The multiplicity of $k_G$ as a direct summand of $(M/\pi M) \otimes (M/\pi M)^*$ is one.

(B) The multiplicity of $O_Q$ as a direct summand of $M \downarrow Q \otimes (M \downarrow Q)^*$ is one for some proper subgroup $Q$ of a Sylow $p$-subgroup of $G$.

Note that if $k$ is algebraically closed, the condition (A) (resp. (B)) is equivalent to the following (A') (resp. (B')) by Proposition 2.2(3):

(A') $M/\pi M$ has an indecomposable decomposition

$$M/\pi M = V \oplus \left( \bigoplus_i W_i \right)$$

as $kG$-modules, where $p \nmid \dim_k V$ and $p \mid \dim_k W_i$ for all $i$. (Possibly $\bigoplus_i W_i$ may be 0.)

(B') For some proper subgroup $Q$ of a Sylow $p$-subgroup of $G$, $M \downarrow Q$ has an indecomposable decomposition

$$M \downarrow Q = U \oplus \left( \bigoplus_i W_i \right)$$

as $O_Q$-lattices, where $p \nmid \rank O U$ and $p \mid \rank O W_i$ for all $i$. (Possibly $\bigoplus_i W_i$ may be 0.)

The following theorem is the main result of this section.

Theorem 3.1. Assume (♯I) and (♯II). Let $M$ be a non-projective indecomposable $O_G$-lattice satisfying the condition (A), and let $\Theta$ be a connected component of $\Gamma(O_G)$ containing $M$. Suppose that $M$ belongs to a block of infinite representation type. Then the tree class of $\Theta$ is $A_\infty$ and $M$ lies at the end of $\Theta$.

Note that $\Theta$ does not contain any Heller $O_G$-lattice and we see that $\Theta = \Theta_2$ by Lemma 2.5.

In order to prove the above theorem, we need the following lemma.

Lemma 3.2. Assume (♯I) and (♯II). Suppose that the group ring $O_G$ is of infinite representation type and a connected component $\Theta$ of $\Gamma(O_G)$ contains an $O_G$-lattice whose $O$-rank is not divisible by $p$. Then the tree class of $\Theta$ is not $D_\infty$. 
**Proof.** Assume to the contrary that the tree class of $\Theta$ is $D_\infty$. Then a part of $\Theta$ is as follows for some non-projective indecomposable $OG$-lattices $X$ and $X'$:

\[ \begin{array}{ccc}
\Omega m(X) & \rightarrow & m(X) \\
\Omega X & \rightarrow & X \\
\Omega^{-1} m(X) & \rightarrow & X' \\
\end{array} \]

where $m(X)$ is isomorphic to the middle term $m(X')$ of $\mathcal{A}(X')$.

First, we claim that both $\text{rank}_O X$ and $\text{rank}_O X'$ are not divisible by $p$: Indeed, if $\text{rank}_O X$ is divisible by $p$, then so is $\text{rank}_O X'$ by Lemma 2.8 and hence all $O$-ranks of the $OG$-lattices in $\Theta$ are divisible by $p$.

Let $A(O_G): 0 \rightarrow \Omega O_G \rightarrow m(O_G) \rightarrow O_G \rightarrow 0$ be the almost split sequence terminating in $O_G$. By Proposition 2.1, the tensor product sequences $A(O_G) \otimes X$ and $A(O_G) \otimes X'$ are the almost split sequences $A(X)$ modulo projectives and $A(X')$ modulo projectives, respectively. In particular, $m(O_G) \otimes X \cong m(O_G) \otimes X'$ (mod projectives), and we have

\[ m(O_G) \otimes X \otimes X^* \cong m(O_G) \otimes X' \otimes X^* \quad (\text{mod projectives}). \]

Note that $m(O_G) \otimes X \otimes X^*$ and $m(O_G) \otimes X' \otimes X^*$ are the middle terms of $A(O_G) \otimes X \otimes X^*$ and $A(O_G) \otimes X' \otimes X^*$, respectively.

Let $\Delta$ be the connected component of $\Gamma(O_G)$ containing the trivial $OG$-lattice $O_G$. Then the tree class of $\Delta$ is $A_\infty$ and $O_G$ lies at the end of $\Delta$ by our assumption and [14, Theorem 3.1]. Put

\[ X \otimes X^* = O_G \oplus \bigoplus_i L_i \oplus \bigoplus_j L_j' \oplus N \]

where $L_i$ are indecomposable $OG$-lattices lying in $\Delta$ with $p \nmid \text{rank}_O L_i$ and $L_j'$ are indecomposable $OG$-lattices in $\Delta$ with $p \nmid \text{rank}_O L_j'$ and $N$ has no indecomposable direct summand lying in $\Delta$. Then, by Proposition 2.1, it follows that

\[ m(O_G) \otimes X \otimes X^* \cong m(O_G) \oplus \bigoplus_i m(L_i) \oplus \bigoplus_j (\Omega L_j' \oplus L_j') \oplus N' \]

for some $OG$-lattice $N'$ which does not have any direct summand lying in $\Delta$. Note that $L_i$ are not isomorphic to any syzygy of $O_G$ by Proposition 2.2 as $X$ is not $\Omega$-periodic. Hence the number of indecomposable direct summands of $m(O_G) \otimes X \otimes X^*$ lying in $\Delta$ is odd. On the other hand, since no syzygy of $O_G$ is a direct summand of $X' \otimes X^*$, the number of indecomposable direct summands of $m(O_G) \otimes X' \otimes X^*$ lying in $\Delta$ is even, a contradiction. \qed

**Proof of Theorem 3.1.** Note that the tree class is one of the infinite trees $A_\infty$, $D_\infty$ and $A_\infty^n$ by our assumption and [13, Corollary 5.6]. The tree class of $\Theta$ is not $D_\infty$ by Lemma 3.2. Assume to the contrary that the tree class of $\Theta$ is $A_\infty^n$. Then any $\Omega$-periodic additive function on $\Theta$ takes a constant value by [4, Proposition 4.5.7]. By Lemma 2.7, $d_\psi$ is an $\Omega$-periodic additive function on $\Theta$. As
\[ \tilde{d}_V(M) = 1 \] if \( \tilde{d}_V(X) = 1 \) for all indecomposable \( OG \)-lattice \( X \) in \( \Theta \). Also, by Lemma 2.7, if \( V' \) is an indecomposable \( kG \)-module whose \( k \)-dimension is not divisible by \( p \) and \( V' \) is not isomorphic to any syzygy of \( V \), then it follows that \( \tilde{d}_V(X) = 0 \) for all \( X \) in \( \Theta \). Therefore, we see that \( \text{rank}_{OG}X \equiv \pm \text{rank}_{OG}M \pmod{p} \) and \( p \mid \text{rank}_{OG}X \) for all \( X \in \Theta \). On the other hand, \( A(O_G) \otimes M \) is \( A(M) \) modulo projectives by Proposition 2.1. However, since \( m(O_G) \) is indecomposable and \( p \mid \text{rank}_{OG}m(O_G) \), all \( O \)-ranks of the indecomposable direct summands of \( m(M) \) \((\equiv m(O_G) \otimes M \text{ modulo projectives})\) are divisible by \( p \) from Proposition 2.2(2), a contradiction.

Now, \( \tilde{d}_V \) is an \( \Omega \)-periodic additive function on \( \Theta \) with tree class \( A_\infty \). Since \( \tilde{d}_V(M) = 1 \), \( M \) must lie at the end of \( \Theta \). \( \square \)

**Theorem 3.3.** Assume \((zI)\) and \((zII)\). Let \( M \) be a non-projective indecomposable \( OG \)-lattice satisfying the condition \((B)\), and let \( \Theta \) be a connected component of \( \Gamma(OG) \) containing \( M \). Suppose that \( M \) belongs to a block of infinite representation type. Then the tree class of \( \Theta \) is \( A_\infty \) and \( M \) lies at the end of \( \Theta \).

**Proof.** Considering the restriction to \( Q \), an \( OQ \)-lattice \( U \) and \( d_{Q,U} \) instead of the reduction modulo \((\pi)\), a \( kG \)-module \( V \) and \( \tilde{d}_V \), we see that a similar argument in the proof of Theorem 3.1 is also valid in order to prove Theorem 3.3. \( \square \)

We close this section with a remark for the case where \( p = 2 \).

**Proposition 3.4.** Assume that \((K, O, k)\) is a 2-modular system satisfying the hypotheses \((zI)\) and \((zII)\). Let \( M \) be a non-projective indecomposable \( OG \)-lattice of odd \( O \)-rank, and let \( \Theta \) be a connected component of \( \Gamma(OG) \) containing \( M \). Suppose that \( M \) belongs to a block of infinite representation type. Then the tree class of \( \Theta \) is \( A_\infty \).

**Proof.** For \( X \in \Theta \), let \( d(X) \) be the number of indecomposable direct summands of odd \( k \)-dimension in an indecomposable decomposition of \( kG \)-module \( X/\pi X \). Then \( d \) is an \( \Omega \)-periodic additive function on \( \Theta \) by Lemmas 2.5(2) and 2.7. Note that \( d(M) \) is odd.

The tree class of \( \Theta \) is \( A_\infty \) or \( A_\infty^C \) by \([13, \text{Corollary 5.6}]\) and Lemma 3.2. Now assume that the tree class of \( \Theta \) is \( A_\infty^C \). Then \( d \) is constant by \([4, \text{Proposition 4.5.7}]\) and, in particular, \( d(X) \) \((= d(M))\) is odd for any \( X \in \Theta \). Hence we see that all \( O \)-ranks of the \( OG \)-lattices in \( \Theta \) are odd. However, since the middle term \( m(M) \) of \( A(M) \) is isomorphic to \( m(O_G) \otimes M \) modulo projectives by Proposition 2.1(1) and \( m(O_G) \) is an indecomposable \( OG \)-lattice of even \( O \)-rank, all \( O \)-ranks of the indecomposable direct summands of \( m(M) \) are even by Proposition 2.2(2), a contradiction. \( \square \)

### 4. Tensor products with splitting trace lattices

In this section, we continue to assume the hypotheses \((zI)\) and \((zII)\) in the Introduction and to consider an indecomposable \( OG \)-lattice \( M \) satisfying the condition \((A)\) or \((B)\) mentioned in Section 3. The aim of this section is to show the following.

**Theorem 4.1.** Assume \((zI)\) and \((zII)\). Suppose that an indecomposable \( OG \)-lattice \( M \) satisfies the condition \((A)\) or \((B)\), and that \( M \) belongs to a block of \( OG \) of infinite representation type. Let \( \Delta \) be the connected component of \( \Gamma(OG) \) containing the trivial \( OG \)-lattice \( O_G \) and \( \Theta \) the connected component of \( \Gamma(OG) \) containing \( M \). Suppose that \( \Delta \) is of type \( ZA_\infty \). Then tensoring with \( M \) induces a graph isomorphism from \( \Delta \) onto \( \Theta \).

Note that \( \Theta \) of type \( ZA_\infty \) since \( M \) is not \( \Omega \)-periodic and both \( O_G \) and \( M \) lie at the ends of \( \Delta \) and \( \Theta \), respectively (Theorems 3.1 and 3.3). We prepare some notation in order to prove Theorem 4.1. Let

\[ T : O_G = L_1 \rightarrow L_2 \rightarrow \cdots \rightarrow L_n \rightarrow \cdots. \]
Lemma 4.3. We show the assertion (1) by induction on $\Theta$ such that $M_n$ is a direct summand of the middle term $m(M_{n+1})$ of $A(M_{n+1})$ and $L_n$ lies in the $n$-th row from the end of $\Delta$ ($\Delta = \Delta \geq ZT$). Also, take a sequence

$$T': M = M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_n \rightarrow \cdots$$

in $\Theta$ such that $M_n$ is a direct summand of the middle term $m(M_{n+1})$ of $A(M_{n+1})$ and $M_n$ lies in the $n$-th row from the end of $\Theta$ ($\Theta = \Theta \geq \Omega T'$). Here, rank$_O L_{2n-1}$ and rank$_O M_{2n-1}$ ($n \in \mathbb{N}$) are not divisible by $p$, and rank$_O M_{2n}$ ($n \in \mathbb{N}$) are divisible by $p$.

Lemma 4.2. (1) $L_n/\pi L_n \cong k_G \oplus \Omega^{-1}k_G \oplus \cdots \oplus \Omega^{-n+1}k_G$ for $n \in \mathbb{N}$. For a proper subgroup $Q$ of a Sylow $p$-subgroup of $G$, $L_n \downarrow Q \cong O_Q \oplus \Omega^{-1}O_Q \oplus \cdots \oplus \Omega^{-n+1}O_Q$ for $n \in \mathbb{N}$.

(2) If $M$ satisfies the condition (A), then

$$M_n/\pi M_n \cong V \oplus \left( \bigoplus_i W_i \right) \oplus \Omega^{-1} \left( V \oplus \left( \bigoplus_i W_i \right) \right) \oplus \cdots \oplus \Omega^{-n+1} \left( V \oplus \left( \bigoplus_i W_i \right) \right)$$

for $n \in \mathbb{N}$, where $M/\pi M = V \oplus \left( \bigoplus_i W_i \right)$ is an indecomposable decomposition (A').

(3) If $M$ satisfies the condition (B), then

$$M_n \downarrow Q \cong U \oplus \left( \bigoplus_i W_i \right) \oplus \Omega^{-1} \left( U \oplus \left( \bigoplus_i W_i \right) \right) \oplus \cdots \oplus \Omega^{-n+1} \left( U \oplus \left( \bigoplus_i W_i \right) \right)$$

for $n \in \mathbb{N}$, where $M \downarrow Q = U \oplus \left( \bigoplus_i W_i \right)$ is an indecomposable decomposition (B').

Proof. We show the assertion (1) by induction on $n$. Assume that $L_t/\pi L_t \cong k_G \oplus \Omega^{-1}k_G \oplus \cdots \oplus \Omega^{-t+1}k_G$ and $L_t \downarrow Q \cong O_Q \oplus \Omega^{-1}O_Q \oplus \cdots \oplus \Omega^{-t+1}O_Q$ for $1 \leq t \leq n-1$. The middle term of $A(\Omega^{-1}L_{n-1})$ is isomorphic to $L_2 \oplus \Omega^{-1}L_{n-2}$, and both the reduced sequence $A(\Omega^{-1}L_{n-1})$ of $kG$-modules and the restricted sequence $A(\Omega^{-1}L_{n-1}) \downarrow Q$ of $O_Q$-lattices split by Lemmas 2.4 and 2.6. This implies that $L_n/\pi L_n \cong k_G \oplus \Omega^{-1}k_G \oplus \cdots \oplus \Omega^{-n+1}k_G$ and $L_n \downarrow Q \cong O_Q \oplus \Omega^{-1}O_Q \oplus \cdots \oplus \Omega^{-n+1}O_Q$.

A similar argument as above yields the assertions (2) and (3). $\square$

Let $a(RG)$ be the Green ring of the group ring $RG$ and let $a(RG; p)$ be the linear span in $a(RG)$ of the indecomposable $RG$-lattices whose $R$-ranks are divisible by $p$. Note that $a(RG; p)$ is an ideal of $a(RG)$, see [5].

Lemma 4.3. Suppose that an indecomposable $OG$-lattice $M$ satisfies the condition (A) or (B). Then the following hold for every $n \in \mathbb{N}$.

(1) $M_{2n} \uparrow L_{2n+1} \otimes M$ and $\Omega^{-1}M_{2n} \uparrow L_{2n+1} \otimes M$.

(2) $\Omega^{-1}M \uparrow L_3 \otimes M$. Moreover, if $M$ satisfies the condition (A) and the trivial $OG$-lattice $O_G$ is not $\Omega$-periodic, then

$$\Omega^{-1}M_{2n-1} \uparrow L_{2n+1} \otimes M.$$

Proof. (1) First, we consider the case where $M$ satisfies the condition (A). Assume that $L_{2n+1} \otimes M \cong M_{2n} \oplus N$ for some $OG$-lattice $N$. Since

$$(L_{2n+1} \otimes M)/\pi (L_{2n+1} \otimes M) \cong V \oplus \Omega^{-1}V \oplus \Omega^{-2}V \oplus \cdots \oplus \Omega^{-2n}V$$

and

$$M_{2n}/\pi M_{2n} \cong V \oplus \Omega^{-1}V \oplus \Omega^{-2}V \oplus \cdots \oplus \Omega^{-(2n-1)}V$$

in $a(kG)/a(kG; p)$ by Lemma 4.2, we see that $N/\pi N \equiv \Omega^{-2n}V$ in $a(kG)/a(kG; p)$. Since $O_G \mid M \otimes M^*$ and $L_{2n+1} \mid L_{2n+1} \otimes M \otimes M^*$, it follows that $L_{2n+1} \mid N \otimes M^*$ as $M_{2n} \in a(OG; p)$. Since $L_{2n+1}/\pi L_{2n+1} \cong k_G \oplus \Omega^{-1}k_G \oplus \Omega^{-2}k_G \oplus \cdots \oplus \Omega^{-2n}k_G$ by Lemma 4.2(1) and $(N \otimes M^*)/\pi (N \otimes M^*) \equiv \Omega^{-2n}V \otimes V^*$ in $a(kG)/a(kG; p)$, we have

$$k_G \oplus \Omega^{-1}k_G \oplus \Omega^{-2}k_G \oplus \cdots \oplus \Omega^{-2n}k_G \mid \Omega^{-2n}V \otimes V^*$$

and in particular, we see that $k_G \mid \Omega^{-1}V \otimes V^*$ and $V \cong \Omega V$ by Proposition 2.2(3). Thus $p = 2$ and a Sylow 2-subgroup of $G$ is a cyclic group of order 2 by Lemma 2.9(1), and $O_G$ is of finite representation type (see, for example, [8]), a contradiction.

For the case where $M$ satisfies the condition (B), consider the restriction to $Q$ and $OQ$-lattices $U$ and $OQ$, instead of the reduction mod ($\pi$) and $kG$-modules $V$ and $k_G$. Then a similar argument as above yields $U \equiv \Omega U$, but this contradicts Lemma 2.9(2).

Also, we have $\Omega^{-1}M_{2n} \mid L_{2n+1} \otimes M$ analogously to the arguments above.

(2) Assume that $\Omega^{-1}M_{2n-1} \mid L_{2n+1} \otimes M$. Now $O_G \mid \Omega^{-1}M_{2n-1} \otimes (\Omega^{-1}M_{2n-1})^*$ since $\operatorname{rank}_Q \Omega^{-1}M_{2n-1}$ is not divisible by $p$. Hence we see that $L_{2n+1}^* \mid M \otimes (\Omega^{-1}M_{2n-1})^*$ by Proposition 2.2(3).

Now, we consider the case where $M$ satisfies the condition (A) and the trivial $O_G$-lattice $O_G$ is not $\Omega$-periodic. Then

$$(M \otimes (\Omega^{-1}M_{2n-1})^*)/\pi (M \otimes (\Omega^{-1}M_{2n-1})^*) \equiv V \otimes (\Omega^{-1}V \oplus \Omega^{-2}V \oplus \cdots \oplus \Omega^{-(2n-1)}V)^*$$

in $a(kG)/a(kG; p)$ and $M \otimes (\Omega^{-1}M_{2n-1})^*$ has $\Omega k_G \oplus \Omega^2k_G \oplus \cdots \oplus \Omega^{2n-1}k_G$ as a direct summand but does not have $k_G \oplus \Omega^2k_G$ as a direct summand since $V$ is not $\Omega$-periodic by our assumption. On the other hand, by Lemma 4.2(1), we have $L_{2n+1}^* \mid L_{2n+1}^* \cong k_G \oplus \Omega k_G \oplus \Omega^2k_G \oplus \cdots \oplus \Omega^{2n}k_G$. This forces that $k_G \oplus \Omega^{2n}k_G$ is a direct summand of $M \otimes (\Omega^{-1}M_{2n-1})^*$, a contradiction.

Next, we consider the case where $M$ satisfies the condition (A) and $n = 1$. Then $(M \otimes (\Omega^{-1}M_1))^* \mid Q \equiv U \otimes (\Omega^{-1}U)^*$ in $a(OQ)/a(OQ; p)$. By Lemma 4.2(1), $L_1^* \mid Q \cong O_Q \oplus O_Q \oplus O_Q$. Hence $O_Q$ is a direct summand of $U \otimes (\Omega^{-1}U)^*$. However, this forces $U \equiv \Omega U$ by Proposition 2.2(3), which contradicts Lemma 2.9(2). $\square$

We are now ready to prove Theorem 4.1.

**Proof of Theorem 4.1.** For $n \in \mathbb{N}$, we shall show the following assertions (1), (2), (3) and (4) by induction on $n$.

1. $\Omega^{-1}M_{2n-3} \mid L_{2n-1} \otimes M$ ($n \geq 2$).
2. $L_{2n-1} \otimes M \cong M_{2n-1}$ modulo projectives.
3. $A(L_{2n-1}) \otimes M = A(M_{2n-1})$ modulo projectives.
4. $L_{2n} \otimes M \cong M_{2n}$ modulo projectives.

If $n = 1$, these hold by Proposition 2.1. Assume that the assertions hold for $n - 1$. Let $L_{2n-1} \otimes M = (\bigoplus_i N_i) \oplus (\bigoplus_j Y_j)$ be an indecomposable decomposition with $p \nmid \operatorname{rank}_Q N_i$ and $p \nmid \operatorname{rank}_Q Y_j$. Then, since $A(L_{2n-1}) \otimes M = A(O_G) \otimes L_{2n-1} \otimes M$ modulo projectives by Proposition 2.1(1), we have

$$A(L_{2n-1}) \otimes M = \left( \bigoplus_i A(N_i) \right) \oplus \left( \bigoplus_j (0 \rightarrow \Omega Y_j \rightarrow \Omega Y_j \oplus Y_j \rightarrow Y_j \rightarrow 0) \right)$$

modulo projectives. Note that $M_{2n-2} \cong L_{2n-2} \otimes M$ modulo projectives by the inductive hypothesis, and $L_{2n-2}$ is a direct summand of $m(L_{2n-1})$. By Lemma 4.3(1), $M_{2n-2} \cong Y_j$ for any $j$. Hence $M_{2n-2}$ is a direct summand of the middle term $m(N_{i_0})$ of $A(N_{i_0})$ for some $i_0$. 


Here, we claim that the statement (1) holds. Indeed, if \( n = 2 \) or \( M \) satisfies the condition (A), then (1) follows by Lemma 4.3(2). So we consider the case where \( M \) satisfies the condition (B). Assume to the contrary that \( \Omega^{-1}M_{2n-3} \cong \Omega L_{2n-1} \otimes M \). Then \( L_{2n-1}^* \otimes (\Omega^{-1}M_{2n-3})^* \) by using the same argument in the proof of Lemma 4.3(2). Now, \( M_{2n-3} \cong \Omega L_{2n-3} \otimes M \) modulo projectives by the inductive hypothesis. Hence we also have \( \Omega L_{2n-3}^* \otimes (\Omega^{-1}M_{2n-3})^* \). As \( L_{2n-1}^* \not\cong \Omega L_{2n-3}^* \), we see that

\[
L_{2n-1}^* \oplus \Omega L_{2n-3}^* \otimes (\Omega^{-1}M_{2n-3})^*.
\]

By Lemma 4.2(1), \( L_{2n-1} \downarrow Q \) has \((2n-1)\) syzygies of \( O_Q \) in its indecomposable decomposition and \( L_{2n-3} \downarrow Q \) has \((2n-3)\) syzygies of \( O_Q \) in its indecomposable decomposition. On the other hand, \( U \otimes (M_{2n-3} \downarrow Q)^* \) has at most \( 2(2n-3) \) syzygies of \( O_Q \) in its indecomposable decomposition by Lemma 2.10 and so does \((M \otimes (\Omega^{-1}M_{2n-3})^*)) \downarrow Q \), a contradiction.

The assertion (1) means that \( N_i \not\cong \Omega^{-1}M_{2n-3} \) for all \( i \). Since \( M_{2n-2} \) is a direct summand of \( m(N_{i_0}) \), it follows that \( M_{2n-1} \cong N_{i_0} \). As \((L_{2n-1} \otimes M)/\pi (L_{2n-1} \otimes M) \cong M_{2n-1}/\pi M_{2n-1} \) modulo projectives, we conclude that \( L_{2n-1} \otimes M \cong M_{2n-1} \) modulo projectives and the assertions (2) and (3) hold. Since \( L_{2n} \) is a direct summand of the middle term \( M(L_{2n-1}) \) of \( A_i(L_{2n-1}) \), the assertion (4) holds. □

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References