Invariants, Nilpotent Orbits, and Prehomogeneous Vector Spaces

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0. INTRODUCTION

Let \( g \) be a semisimple Lie algebra over the complex number field \( \mathbb{C} \), \( g = \bigoplus_{i \in \mathbb{Z}} g(i) \) its \( \mathbb{Z} \)-gradation, i.e., \([g(i), g(j)] \subseteq g(i+j)\). Let \( h \) be the element of \( g \) such that

\[ g(i) = \{ z \in g \mid [h, z] = iz \} \]

Let \( G \) be the simply connected group such that \( \text{Lie}(G) = g \), and \( G(0) \) the centralizer of \( h \) in \( G \), which is known to be connected. Let \( K \) be the Killing form of \( g \). Then the following facts are known [17]:

(0.1) \( K(g(i), g(j)) = 0 \), if \( i + j \neq 0 \).

(0.2) \( g(-i) \) is the dual space of \( g(i) \). Especially \( g(0) \) is reductive.

(0.3) \( (G(0), \text{ad}, g(i)) \) (\( i \neq 0 \)) has only a finite number of orbits. Especially it is a prehomogeneous vector space (abbreviated PV), i.e., \( g(i) \) has an open \( G(0) \)-orbit.

Provisionally in the Introduction, we call such a triple \((G(0), \text{ad}, g(i))\) (\( i \neq 0 \)) a prehomogeneous vector space of Vinberg type (abbreviated PV of V-type).

The purpose of this paper is to construct relative invariants of PV of V-type in a unified way.

Let us explain the way of construction of invariants. We may assume that the original graded Lie algebra \( \mathfrak{g} \) is \( 2\mathbb{Z} \)-graded, i.e., \( \mathfrak{g} = \bigoplus_{i \in 2\mathbb{Z}} \mathfrak{g}(i) \). We may also assume that \( i = 2 \) without loss of generality (cf. (1.2.4)). Let \((\rho, \nu)\) be a representation of \( \mathfrak{g} \) and

\[ V(i) = \{ v \in V \mid \rho(h)v = iv \} \]
Assume that \( \dim V(i) = \dim V(-i) \). Then, if bases of \( V(i) \) are given, \( f(z) = f_{V,i}(z) = \det(\text{ad}(z)^i : V(i) \to V(-i)), \quad z \in \mathfrak{g}(-2) \)

has a meaning. If we write \( g z g^{-1} \) for \( \text{ad}(g)z \), then

\[
f(g z g^{-1}) = \det(g|_{V(-i)}) f(z) \det(g^{-1}|_{V(i)})
\]

for any \( z \in \mathfrak{g}(-2) \) and \( g \in G(0) \), i.e., \( f \) is a relative invariant of \((G(0), \text{ad}, \mathfrak{g}(-2))\) which corresponds to the character

\[
g \mapsto \det(g|_{V(-i)}) \det(g|_{V(i)})^{-1}.
\]

(This construction is due to M. Kashiwara.)

Although we have thus constructed relative invariants of PV of V-type, in many cases the \( f \)'s are not irreducible and even \( f = 0 \). So the following problem is of our main interest:

**Problem 1.** Find a \( \mathfrak{g} \)-module \( V \) and a positive integer \( i \) such that \( f_{V,i} \) is an irreducible polynomial.

In order to get a simple answer to this problem, assume \( \mathfrak{g}(-2) \) to be an irreducible \( G(0) \)-module. We may further assume that \((G(0), \text{ad}, \mathfrak{g}(-2))\) has a relative invariant other than constants. From (0.3) it follows that there exists a \( G(0) \)-orbit of codimension one in \( \mathfrak{g}(-2) \). Hence we can calculate the degree \( d \) of an irreducible relative invariant by applying the degree formula [15, Sect. 4, Proposition 15]. (The degrees are given in [15].) On the other hand,

\[ \deg f_{V,i} = i \cdot \dim V(i) \]

unless \( f \neq 0 \). Thus it is enough to consider the following two problems:

**Problem 2.** Find a necessary and sufficient condition on the graded Lie algebra \( \mathfrak{g} \) under which \( \dim V(i) = \dim V(-i) \) and \( f_{V,i} \neq 0 \) for any \( V \) and \( i \).

**Problem 3.** Find a \( \mathfrak{g} \)-module \( V \) and a positive integer \( i \) such that \( i \cdot \dim V(i) = d \), where \( d \) is the degree of an irreducible relative invariant.

Problem 3 will be discussed in Section 3.

Problem 2 will be discussed in Section 2 without assuming that \( \mathfrak{g}(-2) \) is irreducible. In order to explain our result, let us introduce a class of PVs. Let \( \{x, h, y\} \) be an \( sl_2 \)-triplet in a semisimple Lie algebra \( \mathfrak{g} \), i.e.,

\[
[x, y] = h, \quad [h, x] = 2x, \quad [h, y] = -2y.
\]
Define a $Z$-gradation of $g$ by

$$g(i) = \{ z \in g \mid [h, z] = iz \}.$$  

Provisionally in the Introduction, we call a triple $(G(0), \text{ad}, g(-2))$ obtained in this way, a \textit{prehomogeneous vector space of Dynkin–Kostant type} (abbreviated PV of DK-type). Such prehomogeneous vector spaces play an important role in the classification theory of nilpotent orbits of complex semisimple Lie algebras [3, Chap. 8, Sect. 11; 16]. Our answer to Problem 3 can be stated as follows.

A prehomogeneous vector space of Vinberg type is of Dynkin-Kostant type if and only if $\dim V(i) = \dim V(-i)$ and $f_{V, i} \neq 0$ for any $V$ and $i$.

As we shall see in (1.12) and (2.3.1), the original graded Lie algebra can be reconstructed from a PV of V-type to some extent. Hence the above statement has a meaning. See (2.6) for a precise statement.

Our answer to Problem 2 can be regarded as a characterization of PV of DK-type. A different characterization is given by V. G. Kac [8, Proposition 1.2]. Our characterization has an advantage that the invariant theoretic meaning is clear.

1. Graded Lie Algebras

1.1. Weighted Dynkin Diagrams

1.1.1. A Dynkin diagram of a generalized Cartan matrix [9] whose vertices are colored white or black is called a \textit{weighted Dynkin diagram}. A weighted Dynkin diagram is called \textit{finite, affine, or indefinite type} if the underlying Dynkin diagram is of finite, affine, or indefinite type [9]. Consider the following two kinds of reductions of weighted Dynkin diagrams:

   - \textbf{(R1)} To delete an edge connecting two black vertices.
   - \textbf{(R2)} To delete an edge connecting a white vertex to a black one with an arrow pointing the black one.

A weighted Dynkin diagram is called \textit{reduced} if it does not admit any reductions of these types. From a weighted Dynkin diagram $\Gamma$, we obtain a reduced one by the above reductions, which we shall denote by $\Gamma_{\text{red}}$.

We shall denote by $(WD)$ (resp. $(WD_0), (WD_f)$) the set of weighted Dynkin diagrams (resp. reduced weighted Dynkin diagrams, weighted Dynkin diagrams of finite type). Put $(WD_0)_f = (WD_0) \cap (WD_f)$. We have a mapping

$$\text{red} : (WD) \rightarrow (WD_0), \quad \Gamma = \Gamma_{\text{red}}.$$
which has a cross section

\[ \text{incl}: (WD_0) \to (WD), \quad \Gamma \to \Gamma. \]

Note that \((\text{red})(WD)_f = (WD_0)_f\).

1.1.2. Example. If \(\Gamma\) is

then \(\Gamma_{\text{red}}\) is

1.2. Graded Lie Algebras

1.2.1. Let \(g = \bigoplus_{i \in \mathbb{Z}} g(i)\) be a semisimple \(\mathbb{Z}\)-graded Lie algebra. Define a one parameter subgroup \(\lambda: GL(1) \to \text{Aut}(g)\) by

\[ \lambda(t)(z) = t'z \]

for \(z \in g(i)\) and \(t \in GL(1)\). Let \(d\lambda: \mathbb{C} \to g\) be the derivative of \(\lambda\) and \(h = h_\lambda = (d\lambda)(1)\). Then

\[ g(i) = \{z \in g \mid [h, z] = iz\}. \]

Note that the element \(h \in g\) is characterized by (1.2.2). Let \(h\) be a Cartan subalgebra of \(g\) containing \(h\). Then \(h \subset g(0)\).

1.2.2. Let \(g = \bigoplus_{i \in \mathbb{Z}} g(i)\) be a semisimple \(\mathbb{Z}\)-graded Lie algebra such that \(g(i) = 0\) for odd \(i\). Let \(h\) be a Cartan subalgebra of \(g\) containing \(h\), \(R\) the root system of \((g, h)\), and \(g(x)\) the root space of \(x \in R\). Then each root space \(g(x)\) is contained in \(g(i(x))\) for some \(i(x) \in 2\mathbb{Z}\). Since the set \(P = \{x \in R \mid i(x) \geq 0\}\) is parabolic, we can find a basis \(B\) of \(R\) contained in \(P\) [2, Chap. 6, No. 1.7]. We assume that \(i(x) = 0\) or 2 for \(x \in B\). We shall denote by \((GrL)\) the set of such quartets \((g, \bigoplus_{i \in \mathbb{Z}} g(i), h, B)\).

1.2.3. For \(i \neq 0\), let \(g_i(j) = g(ij)\) and \(g_i = \bigoplus_{j \in \mathbb{Z}} g(j) = \bigoplus_{j \in \mathbb{Z}} g_i(j)\). Since the restriction of the Killing form \(K\) of \(g\) to \(g_i\) is non-degenerate, \(g_i\) is reductive [1, Chap. 1, No. 6.4]. Let \(g_i' = [g_i, g_i]\), \(g_i'(f) = g_i' \cap g_i(f)\), and
Since \( h' \) is a Cartan subalgebra of \( g_i \), the center \( z_i \) of \( g_i \) is contained in \( g(0) \), and
\[
3_i \oplus g_i'(0) = g_i(0) = g(0).
\]
It is easy to see that \( g_i'(j) \subset g_i' \) if \( j \neq 0 \). Hence \( g_i' = \bigoplus_{j \in \mathbb{Z}} g_i'(j) \). Since root subspaces of \((g_i', h')\) are also those of \((g, h)\), the order of the root system \( R \) of \((g, h)\) determined by the basis \( B \) induces an order of the root system \( R_i \) of \((g_i', h_i')\), which determines a basis \( B_i \) of \( R_i \). Thus we get an element \((g_i', \bigoplus_{j \in \mathbb{Z}} g_i'(j), h_i', B_i)\) of \((GrL)\). Although the triple \((g_i'(0), \text{ad}, g_i'(1))\) is different from \((g(0), \text{ad}, g(i))\), the difference will turn out to be not essential for our purpose.

1.3. Representation Diagram. Let \( g \) be a reductive Lie algebra and \((\rho, V)\) its representation. We shall denote by \((Rp)\) the set of triples \((g, \rho, V)\). We shall associate to \((g, \rho, V) \in (Rp)\) a reduced weighted Dynkin diagram \( D_0(g, \rho, V) \in (WD_0)\) as follows:

(D1) Draw the Dynkin diagram of \( g \), whose vertices \( \{1, 2, \ldots, n\} \) are colored white. Let \( \{\varpi_1, \ldots, \varpi_n\} \) be the fundamental weights.

(D2) Let \( \rho = \bigoplus \rho_i \) be an irreducible decomposition of \( \rho \). Write one black vertex for each irreducible component \( \rho_i \) of \( \rho \).

(D3) If the highest weight of \( \rho_i \) is \( \sum_{j=1}^{r} n_{ij} \varpi_j \), draw \( n_{ij} \) lines connecting the black vertex of \( \rho_i \) and the white vertex of \( \varpi_j \), equipped with an arrow pointing toward the white vertex unless \( n_{ij} < 1 \).

We shall call \( D_0(g, \rho, V) \) the representation diagram of \((g, \rho, V)\). \( D_0 \) gives a mapping \((Rp) \rightarrow (WD_0)\).

1.4. \((GrL) \rightarrow (WD)_f\).

Let \( g = (g_i \bigoplus_{j \in \mathbb{Z}} g(i), h, B) \) be an element of \((GrL)\). Let \( D(g) \) be the weighted Dynkin diagram whose underlying diagram is the Dynkin diagram of \( g \) and whose vertex \( j \) is colored white or black if \( i(x_j) = 0 \) or 2, respectively. (See (1.2.3) for \( i(x) \).) Thus we get a mapping \( D: (GrL) \rightarrow (WD)_f \). We can naturally construct the inverse of \( D \).

1.5. We define a mapping \( E: (GrL) \rightarrow (Rp) \) by \( E(g) = (g(0), \text{ad}, g(-2)) \).

1.6. Lemma. The following diagram is commutative.
Proof. For \( g \in (GrL) \), let
\[
\{ j_1, \ldots, j_p \} = \{ j \mid i(\xi_j) - 2 \},
\]
and \( V_k \) \((1 \leq k \leq p)\) the \( g(0) \)-submodule of \( g(-2) \) generated by \( g(-\xi_k) \). Note that \([g(\xi), g(\beta)] = g(\xi + \beta)\) if \( \xi, \beta \), and \( \xi + \beta \) are roots. Then we can prove that

1. each \( V_k \) is an irreducible \( g(0) \)-module with the highest weight
\(-\xi_k\), and
2. \( g(-2) = \bigoplus_{k=1}^p V_k \).

Hence we can draw the representation diagram \( D_0(E(g)) \) of \((g(0), \text{ad}, g(-2)) = E(q)\). It is easy to see that the resulting diagram is equal to \( D(g)_{\text{red}} \).

1.7. Let \( G \) be a reductive group and \((\rho, V)\) its representation. We denote by \((Rp')\) the set of such triples \((G, \rho, V)\). Let \( \text{Lie} : (Rp') \rightarrow (Rp) \) be the mapping defined by
\[
(G, \rho, V) \rightarrow (\text{Lie}(G), \text{Lie}(\rho), V).
\]
We shall write \( \rho \) for \( \text{Lie}(\rho) \). We shall write \( D_0(G, \rho, V) \) for \( D_0(\text{Lie}(G, \rho, V)) \) and call it the representation diagram of \((G, \rho, V)\).

An element \((G, \rho, V) \in (Rp')\) is called a prehomogeneous vector space (abbreviated PV) if \( V \) has an open dense \( G \)-orbit. We denote by \((PV)\) the set of PVs.

1.8. Let \((G, \rho, V)\) be an element of \((Rp')\), and \((\rho, V) = \bigoplus_i (\rho_i, V_i)\) an irreducible decomposition. Let \( G^* \) be the subgroup of \( GL(V) \) generated by \( \rho(G) \) and \( GL(1)^t \), where the \( i \)-th factor of \( GL(1)^t \) acts on \( V_i \) as scalars. Denote by \( \rho^* \) the natural representation of \( G^* \) on \( V \). We shall call \((G^*, \rho^*, V)\) the associated triple of \((G, \rho, V)\).

1.9. Lemma. Let \((G, \rho, V)\) be a prehomogeneous vector space and \((G^*, \rho^*, V)\) its associated triple. A polynomial function \( f \) on \( V \) is a relative invariant of \((G, \rho, V)\) if and only if it is a relative invariant of \((G^*, \rho^*, V)\).

Proof. Assume that \( f \) is a relative invariant of \((G, \rho, V)\) corresponding to the character \( \phi \) of \( G \). For \( c = (c_1, \ldots, c_t) \in GL(1)^t \) and \( v = (v_1, \ldots, v_t) \in \bigoplus_i V_i \), let
\[
f_c(v) = f(c_1 v_1, \ldots, c_t v_t).
\]
Then \( f_c \) is also a relative invariant of \((G, \rho, V)\) which corresponds to \( \phi \).
Hence $f_i$ is a non-zero constant multiple of $f$; i.e., $f$ is a relative invariant of $(G^*, \rho^*, V)$. The converse is trivial.

1.10. LEMMA. If $(G_i, \rho_i, V_i)$ $(i = 1, 2)$ has the same representation diagram, then their associated triples are isomorphic to each other. In this case, we say that $(G_1, \rho_1, V_1)$ is $D$-equivalent to $(G_2, \rho_2, V_2)$.

Since we are mainly interested in the relative invariants of PVs, we do not need to distinguish PVs in the same $D$-equivalence class (in view of the above two lemmas). We also say that two elements of $(GrL)$, $(Rp)$, etc., are $D$-equivalent to each other if their images in $(WD_0)$ via the mappings in the diagram of (1.6) are the same.

1.11. DEFINITION. We say that $(G, \rho, V) \in (PV)$ is of Vinberg type (abbreviated of V-type) if $\text{Lie}(G, \rho, V)$ is $D$-equivalent to an element of $E(GrL)$.

1.12. PROPOSITION. (1) A triple $(G, \rho, V) \in (PV)$ is of V-type if and only if $D_0(G, \rho, V) \in (WD_0)$. (2) If $\text{Lie}(G, \rho, V)$ is $D$-equivalent to $E(g)$ $(g \in (GrL))$, then $D^{-1} D_0(G, \rho, V)$ is $D$-equivalent to $g$. Hence $g$ can be reconstructed from $(G, \rho, V)$ up to $D$-equivalence.

Proof. If $(G, \rho, V)$ is of V-type, there exists an element $(g', \rho', V') \in (Rp)$ such that $D_0(G, \rho, V) = D_0(g', \rho', V')$ and $(g', \rho', V')$ is in the image of $E$. Hence $D_0(G, \rho, V) \in D_0 E(GrL) = (\text{red}) D(GrL) = (WD_0)$.

Conversely, if $D_0(G, \rho, V) \in (WD_0)$, let $(g', \rho', V') = E D^{-1} D_0(G, \rho, V)$. Then $D_0(G, \rho, V) = D_0 E D^{-1} D_0(G, \rho, V) = D_0(g', \rho', V')$. Hence $\text{Lie}(G, \rho, V)$ is $D$-equivalent to $(g', \rho', V')$, which is in the image of $E$.

1.13. Remark. For a triple $(G, \rho, V) \in (Rp)$, let us consider the following two conditions:

(A) $G$ is a reductive group whose derived group is simple.

(B) $\rho$ is irreducible.

In terms of the representation diagram, these conditions can be said as follows:

(A') The full subgraph with the white vertices is connected.

(B') There is only one black vertex.
In terms of the representation diagram, a part of the results of [15] and the result of [10] can be stated as follows:

**Theorem.** (1) The representation diagrams of \((G, \rho, V)\) which are PVs satisfying (A) and (B) are the reduced weighted Dynkin diagram of finite type satisfying (A') and (B') or the following diagrams.

![Diagram](image)

(2) The representation diagrams of \((G, \rho, V)\) such that \(\left[ [G, G], \rho, V \right]\) are coregular (called colibre in [10]) and satisfy (A) and (B) are the reduced weighted Dynkin diagram of finite or affine type satisfying (A') and (B') or the following diagrams.

![Diagram](image)

1.14. *Remark.* We have mainly considered reduced weighted Dynkin diagrams. In some cases it is natural to consider non-reduced ones. For example, let us state a result of V. G. Kac [9, Chap. 8] in terms of weighted Dynkin diagrams. Consider a weighted Dynkin diagram of affine type such that the sum of numerical labels in [9, Tables Aff \(k = 1, 2, 3\)] which are attached to black vertices, is at most \(2/k\). Such diagrams are in one to one correspondence with the complex symmetric spaces, which we list in Table 1. The reduced weighted Dynkin diagrams which are obtained

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
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<tbody>
<tr>
<td>A1 ( A_{2l}^{(2)} )</td>
</tr>
<tr>
<td>A1 ( A_{2l-1}^{(2)} )</td>
</tr>
<tr>
<td>A11 ( A_{2l}^{(2)} )</td>
</tr>
<tr>
<td>AIII ( A_{l+1}^{(1)} )</td>
</tr>
<tr>
<td>BDI ( D_{l+1}^{(2)} )</td>
</tr>
</tbody>
</table>

*Table Continued*
<table>
<thead>
<tr>
<th>Table I—Continued</th>
</tr>
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<tbody>
<tr>
<td>$B_i^{(1)}$</td>
</tr>
<tr>
<td>$(l = p + q, p \geq 2)$</td>
</tr>
<tr>
<td>$D_i^{(1)}$</td>
</tr>
<tr>
<td>$(l = p + q, p \geq 2)$</td>
</tr>
<tr>
<td>$B_i^{(1)}$</td>
</tr>
<tr>
<td>$D_i^{(1)}$</td>
</tr>
</tbody>
</table>

| $B_{II}$ $B_i^{(1)}$ | $D_i$ | $2l$ |
| $D_i^{(2)}$      | $B_i$ | $2l + 1$ |
| $D_{i+1}$      | $A_{i-1}$ | $k(l - 1)$ |
| $CI$ $C_i^{(1)}$ | $A_{i-1}$ | $k(l + 1)$ |
| $CII$ $C_i^{(1)}$ | $C_p C_q$ | $4pq$ |
| $(l = p + q)$ |
| $EI$ $E_i^{(2)}$ | $C_4$ | $42$ |
| $EII$ $E_i^{(1)}$ | $A_1 A_5$ | $40$ |
| $EIH$ $E_i^{(1)}$ | $D_5$ | $32$ |
| $EIV$ $E_i^{(2)}$ | $F_4$ | $26$ |
| $EV$ $E_i^{(1)}$ | $A_7$ | $70$ |
| $EVII$ $E_i^{(1)}$ | $A_1 D_6$ | $64$ |
| $EVII$ $E_i^{(1)}$ | $E_6$ | $54$ |
| $EVIII$ $E_i^{(1)}$ | $D_8$ | $128$ |
| $EIX$ $E_i^{(1)}$ | $A_4 E_7$ | $112$ |
| $FI$ $F_{i}^{(1)}$ | $A_1 C_3$ | $28$ |
| $FII$ $F_i^{(1)}$ | $B_4$ | $16$ |
| $G$ $G_2^{(1)}$ | $A_1 A_1$ | $8$ |
from these diagrams are the representation diagrams of the isotropy representations of the corresponding symmetric spaces. Diagrams corresponding to Hermitian symmetric spaces are those with two black vertices.

2. GRADED LIE ALGEBRAS OF DYNKIN–KOSTANT TYPE

2.1. DEFINITION. We say that an element $g \in (GrL)$ is of Dynkin–Kostant type (abbreviated DK-type) if there exists an $sl_2$-triplet $\{x, h, y\}$ in $g$ with $h = h_8$. (See (1.2.1) for $h_8$.) Let $(GrL)_{DK}$ be the set of elements of $(GrL)$ which are of DK-type. Let $(Rp)_{DK}$, $(WD)_{DK}$, and $(WD_0)_{DK}$ be the subsets of $(Rp)$, $(WD)$, and $(WD_0)$, respectively which are images of $(GrL)_{DK}$ via the mappings in the diagram of (1.6). The elements of $(Rp)_{DK}$, etc., are said to be of Dynkin Kostant type. We say that a triple $(G, \rho, V) \in (Rp')$ is of Dynkin–Kostant type if $Lie(G, \rho, V)$ is $D$-equivalent to an element of $(Rp)_{DK}$. We shall denote by $(Rp')_{DK}$ the set of triples $(G, \rho, V)$ which are of DK-type.

2.2. LEMMA. $(WD_0)_{DK} = (WD_0) \cap (WD)_{DK}$.

Proof: If $\Gamma \in (WD_0) \cap (WD)_{DK}$, $\Gamma = \Gamma_{\text{red}} \in (WD_0)_{DK}$. Conversely, if $\Gamma \in (WD_0)_{DK}$, there exists $\Gamma' \in (WD)_{DK}$ such that $\Gamma = \Gamma'_{\text{red}}$. Hence it is enough to prove that

$$(2.2.1) \quad (\text{red})(WD)_{DK} = (WD)_{DK}.$$  

This fact can be checked by looking over the list of even nilpotent obits [16, 4]. The author does not know an intrinsic proof without using the classification.

2.3. PROPOSITION. For a triple $(G, \rho, V) \in (Rp')$, the following conditions are equivalent:

(i) $(G, \rho, V)$ is of DK-type.

(ii) $D_0(G, \rho, V) \in (WD_0)_{DK}$.

(iii) $D_0(G, \rho, V) \in (WD_0)$, and $D^{-1} D_0(G, \rho, V) \in (GrL)_{DK}$.

Proof. The implications (i) $\iff$ (ii) $\Rightarrow$ (iii) are trivial.

(iii) $\Rightarrow$ (ii). By (2.2),

$$D_0(G, \rho, V) \in (WD_0) \cap D(GrL)_{DK} = (WD_0) \cap (WD)_{DK} = (WD_0)_{DK}.$$  

2.3.1. Remark. If $g \in (GrL)$ is of Dynkin–Kostant type, and $Lie(G, \rho, V)$ is $D$-equivalent to $E(g)$, then $(G, \rho, V)$ is also of Dynkin–Kostant type. Conversely, assume that $(G, \rho, V) \in (Rp')_{DK}$ is given. Then, as we have seen
in (1.12), \( g \) can be reconstructed from \((G, \rho, V)\) up to \(D\)-equivalence; i.e., \( g \) is \(D\)-equivalent to \(D^{-1}D_0(G, \rho, V)\). The above proposition means that, if \((G, \rho, V)\) is of \(DK\)-type, then \( g \) is also of \(DK\)-type.

2.4. Let \( g \in (\text{GrL}), h = h_g, (\rho, V) \) a representation of the underlying Lie algebra of \( g \) and 

\[ V(i) = \{ v \in V \mid \rho(h)v = iv \}. \]

**Theorem.** The following conditions are equivalent:

(i) \( g \in (\text{GrL})_{DK} \).

(ii) For any representation \((\rho, V)\) and any positive integer \( i \), there exists an element \( y \in g(-2) \) such that \( \rho(y)^i \) induces an isomorphism \( V(i) \cong V(-i) \). Moreover, for any \((\rho, V), V = \bigoplus_{i \in \mathbb{Z}} V(i) \).

(iii) There exists an element \( y \in g(-2) \) such that for any positive integer \( i \), \( (\text{ad} \ y)^i \) induces an isomorphism \( g(i) \cong g(-i) \).

*Proof.* The implication (i) \( \Rightarrow \) (ii) is an easy exercise of the representation theory of \( sl_2 \).

(iii) \( \Rightarrow \) (i). Fix an element \( y \) as in (iii). Take bases \( B_i = \{ e_{i-1}, \ldots, e_{i,d(i)} \} \) of \( g(i) \) so that \( \text{ad}(y)^i B_i \subset B_{i-2j} \) for \( 0 \leq j \leq i \). The centralizer \( Z = Z_g(y) \) of \( y \) is the linear span of the union of \( (\text{ad} \ y)^{2i} B_{2j} - (\text{ad} \ y)^{2i+1} B_{2j+2} \) for \( i \geq 0 \). Hence \( Z \) is contained in \( \bigoplus_{i=0} g(i) \). Let \( Z(0) = Z \cap g(0) \) and \( Z(<0) = Z \cap \bigoplus_{i<0} g(i) \). Then

\[ Z = Z(0) \oplus Z(<0), \quad Z(<0) = Z \cap (\text{ad} \ y)g. \]

Let \( K \) be the Killing form of \( g \). Since 

\[ K(z, (\text{ad} \ y)g) = 0 \iff K([y, z], g) = 0 \iff z \in Z, \]

\( Z \) is the orthogonal complement of \( (\text{ad} \ y)g \). Hence \( Z(<0) \) is the radical of \( K|_Z \) and

(2.4.1) \( K|_{Z(0)} \) is non-degenerate.

Since 

\[ g(-2) = (\text{ad} \ y)^2 g(2) \subset (\text{ad} \ y)g(0) - (\text{ad} g(0)) y \subset g(-2), \]
we have

\[(2.4.2) \quad (\text{ad } g(0)) y = g(-2).\]

For each \(i \geq 0\), \(K|_{g(i) \times g(-i)}\) is a complete pairing. Hence \((Z(0), \text{ad}, g(-i))\) is the contragradient representation of \((Z(0), \text{ad}, g(i))\), and

\[\text{Tr}(z | g(-i)) = - \text{Tr}(z | g(i)),\]

for \(z \in Z(0)\). On the other hand, the linear isomorphism \((\text{ad } y)^t: g(i) \rightarrow g(-i)\) commutes with the \(Z(0)\)-actions on \(g(\pm i)\). Hence \(g(i) \cong g(-i)\) as \(Z(0)\)-modules and

\[\text{Tr}(z | g(-i)) = \text{Tr}(z | g(i)) = 0\]

for \(z \in Z(0)\). Hence, for \(h = h_a\) and \(z \in Z(0)\).

\[(2.4.3) \quad K(h, z) = \text{Tr}((\text{ad } z)(\text{ad } h) | g) = \sum_{i \in Z} \text{Tr}((\text{ad } z)(\text{ad } h) | g(i))
\]

\[= \sum_{i \in Z} i \cdot \text{Tr}(\text{ad } z | g(i)) = 0.\]

Thus it is enough to prove the following proposition.

2.5. PROPOSITION [8, Proposition 1.2]. For \(g \in (GrL)\) and \(h = h_a\), the following conditions are equivalent:

(i) \(g \in (GrL)_{DK}\).

(ii) There exists an element \(y \in g(-2)\) such that \(K|_{Z(0) - g(0), y)}\) is non-degenerate, \([g(0), y] = g(-2),\) and \(K(Z(0), z) = 0\).

Proof. (i) \(\Rightarrow\) (ii). If \(g \in (GrL)_{DK}\), then the condition (iii) of (2.4) is satisfied. Hence by (2.4.1)--(2.4.3), we get (ii).

(i) \(\Leftarrow\) (ii). Assume that for any \(x \in g(2), [x, y] \neq h\). Then \([x, y] \neq h\) for any \(x \in g\). If we put

\[Z'_g(y) = \{z \in g | [z, y] \in Ch\},\]

then \(Z'_g(y) = Z_g(y)\). Of course, we may assume that \(g \neq g(0)\). Then

\[K(h, h) = \sum_{i \in Z} \text{Tr}(\text{ad}(h) \text{ad}(h) | g(i)) = \sum_{i \in Z} i^2 \dim g(i) > 0.\]

Hence \(g = Ch \oplus (Ch)^\perp\). Since the conditions

\[K(z, [(Ch)^\perp, y]) = 0,\]

\[K([z, y], (Ch)^\perp) = 0,\]

\(z \in Z'_g(y)\)
are equivalent,

\[(C_h) \perp, y \] is the orthogonal complement of \(Z_{\rho}(y) = Z_{\rho}(y).\)

Since \(Z_{\rho}(y) \subset \bigoplus_{i \in \mathbb{O}} \mathfrak{g}(i)\) and \(y \in \mathfrak{g}(-2),\)

\[(2.5.2) \quad K(Z_{\rho}(y), y) = 0.\]

By (2.5.1) and (2.5.2), \(y \in [(C_h) \perp, y].\) Since \(y \in \mathfrak{g}(-2),\)

\[y \in [(C_h) \perp \cap \mathfrak{g}(0), y].\]

Hence there exists an element \(k\) of \(\mathfrak{g}\) such that

\[k \in \mathfrak{g}(0), \quad K(k, h) = 0, \quad -2y = [k, y].\]

Since \(K|_{Z_{\rho}(0)(y)}\) is non-degenerate,

\[(2.5.3) \quad \mathfrak{g}(0) = Z_{\rho(0)}(y) \oplus Z_{\rho(0)}(y) \perp.\]

Since \((\text{ad} \ y) \mathfrak{g}(0) = \mathfrak{g}(-2),\)

\[(2.5.4) \quad \dim Z_{\rho(0)}(y) \perp = \dim \mathfrak{g}(-2).\]

On the other hand, since \(k \in (C_h) \perp \cap \mathfrak{g}(0)\) and \((\text{ad} \ y)(h) = (\text{ad} \ y)(k),\)

\[(2.5.5) \quad (\text{ad} \ y)((C_h) \perp \cap \mathfrak{g}(0)) = \mathfrak{g}(-2).\]

By assumption

\[(2.5.6) \quad (C_h) \perp \supset Z_{\rho(0)}(y).\]

From (2.5.3) and (2.5.6),

\[(2.5.7) \quad (C_h) \perp \cap \mathfrak{g}(0) = Z_{\rho(0)}(y) \oplus ((C_h) \perp \cap Z_{\rho(0)}(y) \perp).\]

By (2.5.5) and (2.5.7),

\[(2.5.8) \quad \dim((C_h) \perp \cap Z_{\rho(0)}(y) \perp) = \dim \mathfrak{g}(-2).\]

By (2.5.4) and (2.5.8), \((C_h) \perp \supset Z_{\rho(0)}(y) \perp,\) which contradicts (2.5.6). Thus we have completed the proof.

2.6. Corollary. For a prehomogeneous vector space \((G, \rho, V),\) the following conditions are equivalent:

(i) \((G, \rho, V)\) is a prehomogeneous vector space of Dynkin–Kostant type.

(ii) \(D_0(G, \rho, V) \in (WD_0)_{DK}.\)
(iii) \( D_0(G, \rho, V) \) is of finite type, and \( D^{-1} D_0(G, \rho, V) \) satisfies the equivalent conditions of (2.4).

(See (2.1) for DK-type, (1.3) for \( D_0 \), (1.1) and (2.1) for \((W D_0)_{DK}\), and (1.4) for \( D \).)

### 3. Relative Invariants

#### 3.1. Let \( g \) be a complex semisimple Lie algebra, \( h \) a Cartan subalgebra, \( h^\vee \) the dual space of \( h \), \( \langle \ , \ \rangle \) the natural pairing of \( h^\vee \) and \( h \), and \( R \) the root system. Fix an order in \( R \) and let \( \{ \alpha_i \} \), \( \{ \alpha_i^\vee \} \), \( \{ \varpi_i \} \), and \( \{ \varpi_i^\vee \} \) be the sets of fundamental roots, fundamental coroots, fundamental weights, and fundamental coweights, respectively. Let \( G \) be the simply connected group such that \( \text{Lie}(G) = g \) and \( G(0) \) the centralizer of \( h \) in \( G \), which is known to be connected.

Let \( \Gamma \) be a weighted Dynkin diagram whose underlying diagram is the Dynkin diagram of \( g \). Then \( D^{-1}(\Gamma) \) is an element of \((Gr^L)\) whose underlying Lie algebra is \( g \). The gradation is defined by the element

\[
h = h_\gamma = 2 \sum_{k=1}^{\rho} \varpi_{i_k}^\vee,
\]

where \( \varpi_{i_1}^\vee, \ldots, \varpi_{i_p}^\vee \) correspond to the black vertices of \( \Gamma \). For a representation \((\rho, V)\) of \( g \), let

\[
V(i) = \{ i \in V \mid \rho(h)v = i v \}.
\]

Assume that \( \Gamma \) is of Dynkin–Kostant type. Then \( V = \bigoplus_{i \in \mathbb{Z}} V(i) \). Fix a basis of each \( V(i) \), and let

\[
f_{\nu, i}(z) = \det((\text{ad} z)^i : V(i) \to V(-i))
\]

for \( z \in g(-2) \). Then \( f_{\nu, i} \) is a relative invariant of \((G(0), \text{ad}, g(-2))\) which corresponds to the character

\[
g \to \det(g|_{V(-i)}) \det(g|_{V(i)})^{-1}.
\]

Note that \( f_{\nu, i} \neq 0 \) by (2.4). Assume further that \((G(0), \text{ad}, g(2))\) is irreducible; i.e., \( \Gamma \) has only one black vertex. Then it has a unique irreducible relative invariant up to scalar multiple, whose degree we shall denote by \( d \).

The purpose of this section is to show that, except for a few cases, we can get an irreducible relative invariant in this way.

Let \( \varpi_{i_0} \) be the fundamental weight corresponding to the unique black
vertex of $\Gamma$. Then $h = 2\alpha_{i_0}$. Let $(\rho, V)$ be the irreducible representation of \(\mathfrak{g}\) whose highest weight is $\alpha_{i_0}$, $v$ a highest weight vector of $V$, and $V(\lambda)$ the weight space of weight $\lambda$.

3.2. LEMMA. If we put $p = \langle \alpha_{i_0}, \alpha_{i_0} \rangle$, then $\dim V(2p) = \dim V(-2p) = 1$. Especially $V(2p) = V(\alpha_{i_0})$.

Proof. Let $\lambda$ be a weight of $V$ other than $\alpha_{i_0}$, and

$$\alpha_{i_0} - \lambda = \sum_{j=1}^{l} c_j \alpha_j.$$

Then $V(\lambda)$ is contained in $V(2p - 2c_{i_0})$. Since $\dim V(2p) = \dim V(-2p)$, it is enough to prove that $c_{i_0} \neq 0$. Assume that $c_{i_0} = 0$. There exist positive roots $\beta_1, \ldots, \beta_N$ and root vectors $X_{-\beta_1}, \ldots, X_{-\beta_N}$ such that

$$\rho(X_{-\beta_1}) \cdots \rho(X_{-\beta_N}) v \in V(\lambda) - \{0\}.$$

Then

$$\sum_{k=1}^{N} \beta_k = \sum_{j=1}^{l} c_j \alpha_j$$

and each $\beta_k$ (resp. $\beta_k^\vee$) is contained in the linear span of $\{\alpha_j \mid j \neq i_0\}$ (resp. $\{\alpha_j^\vee \mid j \neq i_0\}$). Let $\beta = \beta_N$ and $s$ be the Lie algebra generated by $\{X_{\beta}, \beta^\vee, X_{-\beta}\}$, which is isomorphic to $s_{l_2}$. Since

$$\rho(\beta^\vee) v = \langle \beta^\vee, \alpha_{i_0} \rangle v = 0$$

and

$$\rho(X_{\beta}) v = 0,$$

the $s$-module generated by $v$ is equal to zero. Hence

$$\rho(X_{-\beta}) v = 0,$$

which is a contradiction.

3.3. THEOREM. (1) Let $\Gamma$ be a reduced weighted Dynkin diagram of Dynkin–Kostant type with only one black vertex $i_0$, $\mathfrak{g} = D^{-1}(\Gamma)$, $p = \langle \alpha_{i_0}, \alpha_{i_0} \rangle$, $(\rho, V)$ the irreducible $\mathfrak{g}$-module with the highest weight $\alpha_{i_0}$, and $v_+$ (resp. $v_-$) a highest (resp. lowest) weight vector of $V$. For $z \in \mathfrak{g}(-2)$, define a polynomial function $f$ on $\mathfrak{g}(-2)$ by

$$\rho(z)^{2p} v_+ - f(z) v_-.$$
Then $f(z)$ is an irreducible relative invariant of the irreducible prehomogeneous vector space $(G(0), \text{ad}, g(-2))$ unless $\Gamma$ is one of the following diagrams:

(A) \[ \begin{array}{cccccccc}
1 & 2 & \cdots & 2m & \cdots & 2m + n \\
\circ & \circ & \cdots & \bullet & \cdots & \circ
\end{array} \]

$n \geq 2m \geq 4$

(B)

(C)

(D)

In the above four cases, $f(z)$ is a square of an irreducible relative invariant.

(2) In the above four cases, $f = f_{v,i}$ is not irreducible for any representation $(\rho, V)$ and any positive integer $i$.

Proof of (1). By (2.4), $f \neq 0$. Hence $\deg f = 2p$. If

$$\sigma_i = \sum_{j=1}^{l} p_{ij} \alpha_j,$$

$p = p_{\alpha_i \alpha_i}$. Since $p_{ij}$ are given in the table of [2], we can find the value of $p$. On the other hand, the degree $d$ of an irreducible relative invariant of a reduced irreducible PV is given in table of [15, Sect. 7]. (See (3.4) below.) The case (D) is only one exception where $(G(0), \text{ad}, g(-2))$ is not reduced. In this case it is obtained from a reduced one by applying the castling transformation [15] two times:

Hence $d = 12$. The assertion follows from these data.

The proof of (2) is given in the next section.
<table>
<thead>
<tr>
<th>G</th>
<th>V</th>
<th>dim V</th>
<th>deg f</th>
</tr>
</thead>
<tbody>
<tr>
<td>GL_n × SL_m</td>
<td>V_m ⊗ V_m</td>
<td>m^2</td>
<td>m(n+1)/2</td>
</tr>
<tr>
<td>GL_n (n even)</td>
<td>S^1V_m</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>GL_n (n odd)</td>
<td>m(2m-1)</td>
<td>20</td>
<td></td>
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<tr>
<td>GL_n</td>
<td>S^1V_2m</td>
<td>7</td>
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<td>GL_n</td>
<td>A^2V_4</td>
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<td>16</td>
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<td>GL_n</td>
<td>A^3V_4</td>
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<td>GL_n</td>
<td>A^4V_4</td>
<td>40</td>
<td>40</td>
</tr>
<tr>
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<td>S^1V_2</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>SL_n × GL_m</td>
<td>S^1V_3 ⊗ V_2</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>SL_n × GL_m</td>
<td>A^2V_3 ⊗ V_2</td>
<td>40</td>
<td>40</td>
</tr>
<tr>
<td>SL_n × GL_m</td>
<td>A^3V_3 ⊗ V_2</td>
<td>56</td>
<td>56</td>
</tr>
<tr>
<td>SL_n × GL_m</td>
<td>A^4V_3 ⊗ V_2</td>
<td>66</td>
<td>66</td>
</tr>
<tr>
<td>SL_n × GL_m</td>
<td>A^5V_3 ⊗ V_2</td>
<td>80</td>
<td>80</td>
</tr>
<tr>
<td>SL_n × GL_m</td>
<td>A^6V_3 ⊗ V_2</td>
<td>96</td>
<td>96</td>
</tr>
</tbody>
</table>

**Table II**
(12) $SL_2 \times SL_3 \times GL_2$  
$V_3 \otimes V_3 \otimes V_2$  
18 12  $S_3$

(13) $Sp_{2n} \times GL_{2m}$  
$(n \geq 2m \geq 4)$  
$V_{2n} \otimes V_{2m}$  
$4mn$ 2$m$  
$S_1$

(14) $Sp_6 \times GL_1$  
$V_{14}$  
14 4  $S_2$

(15) $SO_n \times GL_m$  
$V_n \otimes V_m$  
$nm$ 2$m$  
$S_1$ ($n$ even, $m$ odd)

$(n \geq 2m \geq 4)$  
$S_2$ (otherwise)

(20) $Spin_{10} \times GL_2$  
$half-spin_{16} \otimes V_2$  
32 4  $S_1$

(21) $Spin_{10} \times GL_3$  
$half-spin_{16} \otimes V_3$  
48 12  $S_1$

(23) $Spin_{12} \times GL_1$  
$half-spin_{12}^{12}$  
32 4  $S_2$

(24) $Spin_{14} \times GL_1$  
$half-spin_{14}^{14}$  
64 8  $S_2$

(27) $E_6 \times GL_1$  
$V_{27}$  
27 3  $S_1$

(28) $E_7 \times GL_2$  
$V_{27} \otimes V_2$  
54 12  $S_3$

(29) $E_7 \times GL_1$  
$V_{56}$  
56 4  $S_2$
3.4. Remark. Let \((G, \rho, V)\) be an irreducible PV with an irreducible relative invariant \(f\). Then \(GL(V)\) acts on \((\mathbb{C}[V] - \{0\})/GL(V)\). Denote by \(G'\) the isotropy subgroup of \(GL(V)\) at the image of \(f\), and by \(\rho'\) the natural representation of \(G'\) on \(V\). Since \(G'\) contains \(\rho(G)\), \((G', \rho'V)\) is also an irreducible PV with the irreducible relative invariant \(f\). Consequently \(G'\) is reductive. Since we are mainly interested in the invariants, we may replace \((G, \rho, V)\) by \((G', \rho', V')\). Table II contains all the reduced, irreducible, regular PV \((G, \rho, V)\) such that \(\rho(G)\) is the identity component of \(G'\). (The group of connected components \(\pi_0(G')\) is isomorphic to the automorphism group of the representation diagram of \((G, \rho, V)\).) The 5th column contains the \(\pi_0\) of the isotropy subgroup of \(\rho(G)\) at a generic point. The 6th column contains the representation diagram. Cf. [14].

3.5. Remark. Here we have not constructed irreducible relative invariants of the four exceptional cases in (3.3). But we can construct the irreducible relative invariants in any cases by ad hoc methods. See [15] for all the cases except for (6), (7), (10), (20), (21), and (24) in Table II. See [12] for (6) and (7), [5] for (10), [11] (and [6]) for (20), [6] for (21) and [7] for (24). See [15] for the relation between the castling transformation and the relative invariants.

3.6. Remark. Between the classes of PVs of V-type and PVs of DK-type, there lies the class of regular PVs of V-type, which A. Mortajine [18] and the present author have classified independently. The result is quite complicated.

4. Proof of Theorem 3.3(2)

Let \(g \in (GrL)\), \(h = h_\mu\), and for a representation \((\rho, V)\) of \(g\), \(V(\mu)\) be the \(\mu\)-weight space and

\[ V(i) = \{ v \in V | \rho(h)v = iv \}. \]

4.1. Lemma. Let \(\lambda_i (i = 1, 2)\) be a dominant integral weight of \(g\), and \(V_i\) (resp. \(V\)) an irreducible \(g\)-module with the highest weight \(\lambda_i\) (resp. \(\lambda_1 + \lambda_2\)). If \(V_i(p_i) \neq 0\) for some \(p_i \geq 0\),

\[ \dim V(p_1 + p_2) \geq \max(\dim V_1(p_1), \dim V_2(p_2)). \]

Proof. Let \(\mu_i\) be a weight of \(V_i\). It is known that

\[ \dim V(\mu_1 + \mu_2) \geq \max(\dim V_1(\mu_1), \dim V_2(\mu_2)) \]
[13, Theorem 4.1] Since $V_2(p_2) \neq 0$, there exists a weight $\mu_2$ of $V_2$ such that $\langle h, \mu_2 \rangle = p_2$. For such a weight $\mu_2$, we have

$$\dim V(p_1 + p_2) \geq \sum_{\langle h, \mu_1 \rangle = p_1} \dim V(\mu_1 + \mu_2) \geq \sum_{\langle h, \mu_1 \rangle = p_1} \dim V_1(\mu_1) = \dim V_1(p_1).$$

Suppose that $V$ is irreducible. Let $\stackrel{\circ}{\lambda} = \sum_{j=1}^l c_j \omega_j$ be its highest weight and $p = \langle \lambda, h \rangle / 2$.

4.2. **Lemma.**

1. $\dim V(i) = \dim V(-i)$.
2. $\dim V(2i) \geq \dim V(2j)$, if $0 \leq i < j$.
3. $\{ i \mid V(2i) \neq 0 \} - [ -p, p ]$.
4. Let $V'$ be an irreducible $g$-module whose highest weight $\lambda' = \sum_{j=1}^l c'_j \omega_j$ satisfies $c_j \geq c'_j$ for any $j$. Put $p' = \langle \lambda', h \rangle / 2$. Then $\dim V(2p) \geq \dim V'(2p')$.

4.3. **Remark.** In the cases (A)–(D), $V(i) = 0$ for odd $i$. Assume that $f_{V,2q}$ is irreducible. Then

$$d = \deg f_{V,2q} = 2q \cdot \dim V(2q).$$

Since $f_{V_1 \oplus V_2,2q} = f_{V_1,2q} \circ f_{V_2,2q}$, $V$ is irreducible.

Henceforth, we shall use the same notations as in [2], e.g., $\alpha_1$ and $\alpha_3$ are connected by an edge in the Dynkin diagram of $E_7$.

4.4. **Case (A).** In this case $i_0 = 2m$, $p = 2m$, and $d = 2m$. Let us denote by $(\lambda)$ the irreducible $g$-module with the highest weight $\lambda = \sum c_j \omega_j$. Then $(\omega_1)$ is the natural representation of $g = sp_{2l}$ ($l = 2m + n$) on $C^{2l}$. Let $\wedge^0 (\omega_1)$ be the one dimensional representation of $g$ and $\wedge^{-1} (\omega_1) = 0$. Then $\wedge^i (\omega_1) = \wedge^{i-2} (\omega_1) \oplus \omega_1$, for $1 \leq i \leq l$ [15, Sect. 1, Example 25]. Hence

(4.4.1) $\dim (\omega_i)(2i) = \binom{2m}{i}$, if $1 \leq i \leq 2m$,

(4.4.2) $\dim (\omega_i)(4m) = \binom{2n}{i-2m} \cdot \binom{2n}{i-2-2m}$, if $2m \leq i \leq 2m + n$,

(4.4.3) $\dim (\omega_{2m})(4m - 2) = 4mn$.

By (4.4.1), (4.4.2), and (2) of (4.2),

$$\dim (\omega_i)(0) \geq 2m,$$
for \( i \neq 2m \). (Recall that \( n \geq 2m \). Hence if \( c_i \neq 0 \) for some \( i \neq 2m \), by (2) and (4) of (4.2)

\[
2q \cdot \dim(\lambda)(2q) \geq 2q \cdot \dim(\lambda)(2p) \geq 2q \cdot \dim(\sigma_i)(0) \geq 4mq > 2m = d,
\]

for any \( q \in [1, p] \). Hence \( \lambda = c_{2m}\sigma_{2m} \). Let \( c = c_{2m} \). By (1) of (3.3), \( c \geq 2 \). By (4.4.3) and (4.2),

\[
\dim(c\sigma_{2m})(4m) \geq \dim(c\sigma_{2m})(4mc - 2) \\
\geq \max(\dim((c - 1)(\sigma_{2m}))(4m(c - 1)), \dim(\sigma_{2m})(4m - 2)) \\
\geq 4mn > 2m.
\]

Hence

\[
2q \cdot \dim(c\sigma_{2m})(2q) \geq 2q \cdot \dim(c\sigma_{2m})(4m) > 4mq > 2m = d,
\]

for \( q \leq p = 2m \). Hence \( f_{V,q} \) cannot be irreducible.

In order to consider the cases (B), (C), and (D), we need a lemma. Assume that \( h = 2m_0 \). Let \( R_{s,i} \) be the set of roots \( \alpha = \sum_{j=1}^{l} a_j \alpha_j \) such that \( a_{i_0} = s \) and \( a_i > 0 \). Let \( R_{s} = \bigcup_{i \geq 0} R_{s,i} \) and \( p_{ij} = \langle \sigma_i, \sigma_j \rangle \).

4.5. LEMMA. (1) If \( c_i > 0 \) and \( 0 \leq s < c_i p_{i,0} - q \) for given \( i \) and \( s \), then \( d \geq 2q \left| R_{s} \right| \). Here \(| \cdot |\) denotes the cardinality.

(2) If \( 2s \leq \langle \lambda, h \rangle - 2q \), then \( d \geq 2q \left| R_{s} \right| \).

Proof. (2) If \( c_i > 0 \) and \( \alpha \in R_{s,i} \),

\[
\langle \lambda, \alpha \rangle \geq \langle \sigma_i, \alpha \rangle \geq 1.
\]

Hence \( \{ \lambda - \alpha \mid \alpha \in R_{s} \} \) are weights of \( V \). Since

\[
\langle h, \lambda - \alpha \rangle = \langle h, \lambda \rangle - 2s \quad (\alpha \in R_{s}),
\]

it follows that

\[
\dim V(2q) \geq \dim V(\langle \lambda, h \rangle - 2s) \geq \left| R_{s} \right|.
\]

Hence

\[
d = 2q \cdot \dim V(2q) \geq 2q \left| R_{s} \right|.
\]

Thus we have proved (2). The first statement can be deduced from (2).

4.6. Case (B). In this case, \( i_0 = 6 \) and \( d = 4 \). Assume that the coefficient \( c_i \) is positive. Since \( 4 = 2q \cdot \dim V(2q) \), \( q = 1 \) or \( 2 \). Since \( p_{i,6} \geq 2 \), by taking
$s = 0$ in (4.5), we get $4 \geq 2q |R_{0i}|$. If $i \neq 6, 7$, then $|R_{0i}| \geq 3$, which is a contradiction. Assume that $i = 6$. Since $p_{66} = 4$, by taking $s = 1$ in (4.5), we get $4 \geq 2q |R_{16}|$. But $|R_{16}| \geq 3$, which is a contradiction. Assume that $c_7 \geq 2$ or $q = 1$. Since $p_{76} = 2$, by taking $s = 1$ in (4.5), we get $4 \geq 2q |R_{17}|$. But $|R_{17}| \geq 3$, which is a contradiction. Hence $\lambda = \sigma_7$, $\dim V = 56$, $q = 2$ and $\dim V(4) = 1$. But a direct calculation shows that $\dim V(4) = \dim V(-4) = 2$, $\dim V(2) = \dim V(-2) = 16$, and $\dim V(0) = 20$. (Note that $\dim V(2i)$ is equal to the number of $E_8$-roots $x = \sum_{j=1}^{8} a_j x_j$ such that $a_6 = 2 - i$ and $a_8 = 1$.) Hence $f_{V,2q}$ cannot be irreducible in this case.

4.7. Case (C). In this case $i_0 = 6$ and $d = 12$. Assume that the coefficient $c_i$ is positive. Since $12 = 2q \cdot \dim V(2q)$, $q = 1, 2, 3, \text{ or } 6$. Assume that $i \neq 6, 7, 8$. Since $p_{i6} \geq 6$, by taking $s = 0$ in (4.5), we get $12 \geq 2q |R_{i6}|$. But $|R_{0i}| \geq 7$, which is a contradiction. Assume that $i = 6$ or $7$. Since $p_{66} \geq 8$, by taking $s = 1$ in (4.5), we get $12 \geq 2q |R_{16}|$. But $|R_{16}| \geq 7$, which is a contradiction. Assume that $c_8 \geq 2$ or $q \neq 6$. Since $p_{86} = 4$, by taking $s = 1$ in (4.5), we get $12 \geq 2q |R_{17}|$. But $|R_{17}| \geq 7$, which is a contradiction. Hence $\lambda = \sigma_8$, $q = 6$, and $\dim V(12) = 1$. Since $V$ is the adjoint representation of $E_8$, $\dim V(i)$ can be calculated easily and $\dim V(12) = 0$, which is a contradiction. Hence $f_{V,2q}$ cannot be irreducible in this case.

4.8. Case (D). In this case $i_0 = 4$ and $d = 12$. Assume that the coefficient $c_i$ is positive. Since $12 = 2q \cdot \dim V(2q)$, $q = 1, 2, 3, \text{ or } 6$. Assume that $i \neq 1, 6, 7$. Since $p_{i4} \geq 8$, we get $12 \geq 2q |R_{i4}|$. But $|R_{14}| \geq 7$. Hence $i = 1, 2, 6, \text{ or } 7$. Assume that $8c_1 + 12c_2 + 12c_6 + 6c_7 \geq 14$. Then

$$\langle h, \lambda \rangle = \langle 2\omega_4, c_1 \omega_1 + c_2 \omega_2 + c_6 \omega_6 + c_7 \omega_7 \rangle = 8c_1 + 12c_2 + 12c_6 + 6c_7 \geq 14.$$ By taking $s = 1$ in (4.5), we get $12 \geq 2q |R_{i4}|$. But $|R_{i4}| \geq 7$. Hence $\lambda = \sigma_1$, $\sigma_2$, $\sigma_7$, or $2\sigma_7$. Assume that $\lambda = \sigma_1$. Since $(\sigma_1)$ is the adjoint representation of $E_7$, $\dim(\sigma_1)(2i)$ can be easily calculated; $\dim(\sigma_1)(2j) = 27, 24, 18, 8, 3$ for $j = 0, 1, 2, 3, 4$, respectively. Hence $\lambda \neq \sigma_1$. Assume that $\lambda = \sigma_2$. Then $\dim(\sigma_2)(2j)$ is equal to the number of $E_8$-roots $x = \sum_{k=0}^{8} a_k x_k$ such that $a_4 = 3 - j$ and $a_8 = 1$; $\dim(\sigma_2)(2j) = 12, 12, 6, 4$ for $j = 0, 1, 2, 3$, respectively. Hence $\lambda \neq \sigma_2$. Assume that $\lambda = 2\sigma_7$. If $q \neq 6$, then

$$\dim(2\sigma_7)(2q) \geq \max(\dim(\sigma_7)(2q), \dim(\sigma_7)(0)) = 12.$$ Hence we get a contradiction. Hence $q = 6$ and $\dim(2\sigma_7)(12) = 1$. But

$$\dim(2\sigma_7)(12) \geq \dim(\sigma_7)(6) = 4.$$ Assume that $\lambda = \sigma_2$. We get $q = 6$ and $\dim(\sigma_2)(12) = 1$. But $(\sigma_2)(12)$ has weights $\omega_2$ and $\omega_2 - \alpha_2$. Hence $f_{V,2q}$ cannot be irreducible.
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