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Relaxation Oscillation and Canard Explosion[1]

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We give a geometric analysis of relaxation oscillations and canard cycles in singularly perturbed planar vector fields. The transition from small Hopf-type cycles to large relaxation cycles, which occurs in an exponentially thin parameter interval, is described as a perturbation of a family of singular cycles. The results are obtained by means of two blow-up transformations combined with standard tools of dynami-

1. INTRODUCTION

Relaxation oscillations are a type of periodic solutions found in singularly perturbed systems and ubiquitous in systems modelling chemical and biological phenomena [13, 18, 20]. A prototypical system where they occur is the van der Pol equation. Relaxation oscillations consist of long periods of quasi-static behavior interspersed with short periods of rapid transition (see [13, Definition 2.1.2] for a mathematical definition). *Canard explosion* is a term used in chemical and biological literature [5] to denote a very fast transition, upon variation of a parameter, from a small amplitude limit cycle to a relaxation oscillation. This phenomenon is related to the presence of a family of *canard cycles* and has been thoroughly investigated in the context of the van der Pol equation [4, 8, 11].

In this work we analyze relaxation oscillations and canard explosion in the context of two-dimensional systems of the type

$$\begin{aligned} \varepsilon\dot{x} &= f(x, y, \varepsilon), \\ \dot{y} &= g(x, y, \varepsilon), \end{aligned} \qquad x \in \mathbb{R}, \quad y \in \mathbb{R}, \quad 0 < \varepsilon \ll 1, \tag{1.1}$$

where f, g are C^k-functions with $k \geqslant 3$. The van der Pol equation can be transformed to the form (1.1) with $f(x, y) = y - x^2/2 - x^3/3$ and $g(x, y) = (\lambda - x)$, where λ is a parameter. Often systems of higher dimension can be

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reduced to equations of the form (1.1). In particular this is possible for the three dimensional Oregonator system [14]. Let τ denote the independent variable in (1.1). The variable τ is referred to as the *slow* time scale. By switching to the *fast* time scale $t := \tau/\varepsilon$ one obtains the equivalent system

$$\begin{aligned} x' &= f(x, y, \varepsilon), \\ y' &= \varepsilon g(x, y, \varepsilon). \end{aligned} \tag{1.2}$$

One tries to analyze the dynamics of (1.1) by suitably combining the dynamics of the *reduced problem*

$$\begin{aligned} 0 &= f(x, y, 0), \\ \dot{y} &= g(x, y, 0), \end{aligned} \tag{1.3}$$

and the dynamics of the *layer problem*

$$\begin{aligned} x' &= f(x, y, 0), \\ y' &= 0, \end{aligned} \tag{1.4}$$

which are the limiting problems for $\varepsilon = 0$ on the slow and the fast time scales, respectively. The phase space of (1.3) is the *critical manifold* S defined by $S := \{(x, y) : f(x, y, 0) = 0\}$. For (1.4) S corresponds to a set of equilibria. By Fenichel theory [12] normally hyperbolic pieces of S perturb to nearby invariant manifolds S^ε of (1.1) with the flow given approximately by the flow of (1.3). Typically a point of non-hyperbolicity is a fold point of S. At a generic fold point p the reduced problem (1.3) is singular and solutions reach p in finite forward or backward time. This case is known as *jump point* and is an ingredient necessary for the existence of a relaxation oscillator. Figure 1 shows the dynamics of the limiting problems (1.3) and (1.4) at a jump point. A solution given by Fenichel theory may follow closely an attracting branch S_a of S and arrive in the vicinity of a fold point. Then, if the situation is as shown in Fig. 1, the solution continues on following approximately the dynamics of the layer problem (1.4) [16]. This scenario can recur and eventually produce a relaxation oscillation. If the critical manifold is S-shaped with the fast and the slow flow as shown in Fig. 3 then, for small ε, a relaxation oscillation exists. In this work we give a short proof of this well known result using blow-up and Fenichel theory.

A *canard point* is a fold point p satisfying $g(p, 0) = 0$. This degeneracy has the effect of removing the singularity of (1.3) at p. As a result the

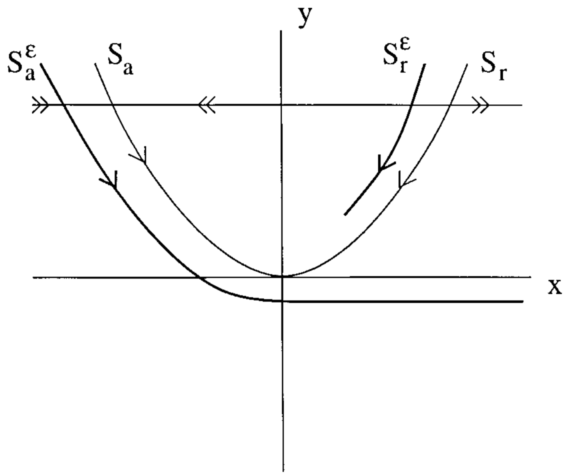

FIG. 1. Critical manifold and slow manifolds for a jump point.

reduced problem (1.3) has a solution passing through p. Hence, it is conceivable that the attracting slow manifold S_a^ε stays close to the repelling slow manifold S_r^ε for a time of $O(1)$; see Fig. 2. In problems depending on an additional parameter λ the attracting slow manifold S_a^ε may connect to the repelling slow manifold S_r^ε at isolated values $\lambda = \lambda(\varepsilon)$. Solutions of the former type are commonly called *canards*, the latter are sometimes called *maximal canards*.

For small $\varepsilon > 0$ system (1.1) has an equilibrium close to the fold point. One of the interesting features of a canard point is the presence of a Hopf

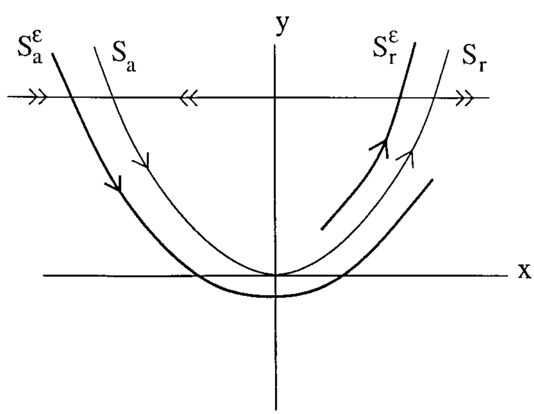

FIG. 2. Critical manifold and slow manifolds for a canard point.

bifurcation in its unfolding, which is known as *singular Hopf bifurcation* [1, 2]. Suppose now that the critical manifold is S-shaped with one of the folds a canard point and the other a jump point; see Fig. 4. In the unfolding of this configuration relaxation oscillations as well as points of Hopf bifurcation must be present. Consequently, one expects a transition from solutions of Hopf type to relaxation oscillations to occur. The surprising and interesting effect is that this transition, called canard explosion, happens in an exponentially small range of the parameter λ. The transition occurs through a family of canard cycles, which pass close to the canard point, follow the repelling slow manifold and subsequently jump to one of the attracting slow manifolds.

The canard phenomenon has been analyzed using three different methods: non-standard analysis, by Benoit *et al.* [4], matched asymptotic expansions, by Eckhaus [11] and Mishchenko *et al.* [19], blow-up combined with tools of dynamical systems, by Dumortier and Roussarie [8]. From the work of Dumortier and Roussarie it became apparent that blow-up was the right tool for extending geometric singular perturbation theory to non-hyperbolic points.

This work complements and extends the results of [8]. Unlike [8], where only the van der Pol equation was considered, we treat a large class of systems, paying much attention to the genericity issues. We feel that it was important to demonstrate how the canard problem fits in the general context of geometric singular perturbation theory. Hence, in much of the analysis, we use typical methods of singular perturbation theory, like Fenichel theory or center manifold theory. We recognize canard explosion as arising through a combination of a local phenomenon, namely the unfolding of a canard point, and a global return mechanism provided by the S-shaped critical manifold. The main technical part of the paper is a complete analysis of the unfolding of a canard point. Here we significantly extend the results of [8], obtaining a description of the transition from cycles of Hopf type to small canard cycles. As a part of the analysis of a canard point we establish an interesting relation between (i) the non-degeneracy condition for the Hopf bifurcation, (ii) the transversality of the intersection of slow manifolds, and (iii) the stability of small canard cycles, i.e. the sign of the same (computable) constant A controls these three phenomena.

The strategy employed in this work is to use blow-up coordinates sparingly and to interpret all the results in the original coordinates. This leads to a significantly different presentation than in [8]. In our minds our presentation is more accessible to the readers familiar with geometric singular perturbation theory. Our approach also results in a simplification of some of the proofs. For example, canard cycles are constructed by matching trajectories and not as intersections of three dimensional center manifolds.

One of our goals was to demonstrate that canard explosion is a phenomenon likely to occur in many contexts, rather than something exotic present only in the van der Pol equation. As a part of this endeavor we analyze canard explosion also in the case when the Hopf bifurcation is subcritical (it is supercritical for the van der Pol equation). This case is known to arise for the Oregonator system [14]. In this case the families of periodic orbits joining small oscillations to relaxation oscillations must have at least one limit point. To show the existence and genericity of the limit point we obtain an approximate formula for the Floquet multipliers of canard cycles. Similar results were obtained in [9], nonetheless we find our approach to be more elementary.

Our main results are somewhat different than the ones in [8]. Our point is that from the practical point of view one is interested in canard explosion as a very fast transition from a small oscillation to a relaxation oscillation. This can be seen as a strong instability of the system, although it has nothing to do with instability in the usual sense of the word. Consequently, we have focused our attention on describing the entire family of canard cycles, starting with small oscillations and ending with relaxation oscillations. In accordance with this philosophy we prove results of the following kind: "for fixed ε there exists a family of limit cycles joining small oscillations to relaxation oscillation and the transition occurs within an $O(e^{-K/\varepsilon})$ small λ interval". This kind of a statement can be seen as a mathematical rephrasing of the description of canard explosion found in applied literature [5].

The present paper is the sequel to the more basic [16] where the blow-up method is explained in detail and used to analyze the local behaviour of slow manifolds near non-hyperbolic points. Now we analyze the more global phenomena of relaxation oscillations and canard explosion. As mentioned, the philosophy of our approach is to treat these global problems in the spirit of manifold theory by patching together local results. We feel that this point of view is well suited to address singular perturbation problems, which by definition are problems that can not be approximated uniformly by a single expansion [3, 17, 21]. From the geometric point of view the different expansions used in a singular perturbation problem can—and should—be regarded as different charts. We are currently investigating a wide range of problems in this spirit.

The article is organized as follows. In Section 2 we state and prove a result on relaxation oscillation for an S-shaped critical manifold. In Section 3 we present background material and our results on canard explosion. Section 4 is devoted to the local analysis near the canard point. In Section 5 we analyze the global phenomena involved in a canard explosion. In particular a novel treatment of the stability of canard cycles is given and a result on saddle-node bifurcation of canard cycles is proved. In Section 6 we briefly discuss canard explosion for the van der Pol equation.

2. RELAXATION OSCILLATION

In this section we discuss relaxation oscillations arising near S-shaped critical manifolds. As noted in the introduction relaxation oscillations can be found in a much more general setting, and here we just describe the prototypical situation. We remark, however, that it may often be possible to reduce the study of a higher dimensional system to the analysis we present here. The following hypothesis is essential for the existence of a relaxation oscillator.

(A1) The critical manifold is S-shaped, i.e., it can be written in the form $y=\varphi(x)$ and the function φ has precisely two critical points, one non-degenerate minimum and one non-degenerate maximum.

With no loss of generality we assume that the minimum is at the origin and the maximum occurs for $x=x_M>0$. It follows that the critical manifold can be broken up into three pieces, S_l, S_m and S_r, separated by the minimum and the maximum. These three pieces are defined as follows:

$$S_l=\{(x,\varphi(x)) : x<0\},$$

$$S_m=\{(x,\varphi(x)) : 0<x<x_M\},$$

$$S_r=\{(x,\varphi(x)) : x>x_M\}.$$

We assume that

(A2) For the layer problem S_l and S_r are attracting, i.e. $\frac{\partial f}{\partial x}<0$ on S_l and S_r, and S_m is repelling, i.e. $\frac{\partial f}{\partial x}>0$ on S_m.

(A3) Both folds are generic, i.e. satisfy the following conditions [16]

$$\frac{\partial^2 f}{\partial x^2}(x_0,y_0,0)\neq 0, \qquad \frac{\partial f}{\partial y}(x_0,y_0,0)\neq 0, \qquad g(x_0,y_0,0)\neq 0. \tag{2.1}$$

Finally we need to assume that the slow flow on S is as shown in Fig. 3. Substituting $y=\varphi(x)$ into (1.3) we obtain

$$\dot{x}=\frac{g(x,\varphi(x),0)}{\varphi'(x)}. \tag{2.2}$$

We make the following assumption.

(A4) The slow flow on S_l satisfies $\dot{x}>0$ and the slow flow on S_r satisfies $\dot{x}<0$.

Assumptions (A1)–(A4) imply that the fast and the slow dynamics are as shown in Fig. 3.

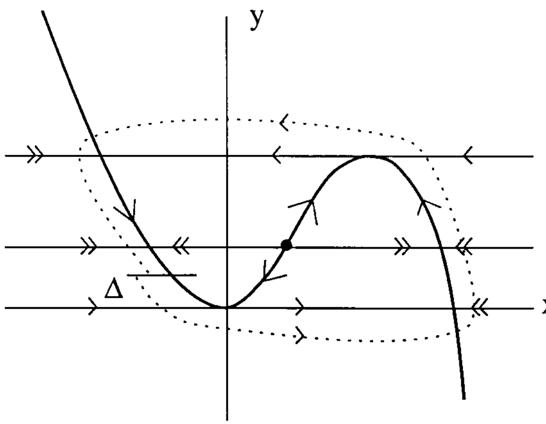

FIG. 3. The singular orbit Γ, the section Δ and a typical trajectory (dotted) for $\varepsilon > 0$.

Let $(x_r, 0)$ be the point of intersection of the x axis with S_r and (x_l, y_M) the point of intersection of the line $\{y = y_M\}$ with S_l. Let Γ be the singular trajectory defined as the union of the critical fibers joining $(0, 0)$ to $(x_r, 0)$ and (x_M, y_M) to (x_l, y_M) and of the two pieces of the critical manifold joining $(x_r, 0)$ to (x_M, y_M) and (x_l, y_M) to $(0, 0)$. Let U be a small tubular neighborhood of Γ. A continuous family of periodic orbits Γ_ε is a family of relaxation oscillators if Γ_ε converges to Γ in the Hausdorff distance as $\varepsilon \to 0$. The following well known result [18, 22] establishes the existence of a family of relaxation oscillators.

THEOREM 2.1. *Assume* (A1)–(A4). *Then for sufficiently small* ε *there exists a unique limit cycle* $\Gamma_\varepsilon \subset U$. *The cycle* Γ_ε *is strongly attracting, i.e. its Floquet exponent is bounded above by* $-K/\varepsilon$ *where* $K > 0$ *is a constant. As* $\varepsilon \to 0$ *the cycle* Γ_ε *approaches* Γ *in the Hausdorff distance.*

We give a proof of Theorem 2.1 using the framework of geometric singular perturbation theory.

Proof of Theorem 2.1. By Fenichel theory S_l and S_r perturb to slow manifolds $S_{l,\varepsilon}$ and $S_{r,\varepsilon}$. By [16, Theorem. 2.1] the manifolds $S_{l,\varepsilon}$ and $S_{r,\varepsilon}$ continue beyond the respective fold points and roughly follow the critical fibers, arriving in the vicinity of S_r and S_l, respectively. Let Δ be a section of the flow defined as a small horizontal interval intersecting S_l at a point between (x_l, y_M) and $(0, 0)$. Consider tracking a trajectory staring in Δ for $0 < \varepsilon \ll 1$. Initially this trajectory will be attracted to $S_{l,\varepsilon}$ and then, by [16, Theorem. 2.1], to the extension of $S_{l,\varepsilon}$ beyond the fold point at $(0, 0)$. As it arrives in the vicinity of S_r it will be attracted to $S_{r,\varepsilon}$ and follow it and its extension until it comes close to S_l. It will then follow $S_{l,\varepsilon}$ until it reaches Δ. Let $\pi: \Delta \to \Delta$ be the return map. It follows from [16, Theorem. 2.1] and Fenichel theory that, for ε sufficiently small, π is a contraction with contraction rate bounded above by $e^{-K/\varepsilon}$, where $K>0$ is a constant. By the implicit function theorem there exists a unique, strongly attracting fixed point of π in Δ. This fixed point gives rise to a limit cycle Γ_ε. It follows that the Floquet exponent of Γ_ε is bounded above by $-K/\varepsilon$. Let $q_\varepsilon = \Delta \cap S_{l,\varepsilon}$ and let $\tilde{\Gamma}_\varepsilon$ denote the segment of the forward trajectory of q_ε until the first return to Δ. By [16, Theorem. 2.1] $\tilde{\Gamma}_\varepsilon$ approaches Γ as $\varepsilon \to 0$. Since Γ_ε is exponentially close to $\tilde{\Gamma}_\varepsilon$ it follows that Γ_ε approaches Γ in the Hausdorff distance as $\varepsilon \to 0$. ∎

3. CANARD EXPLOSION

In this section we formulate precise conditions for the occurrence of a canard explosion and state our main results. We consider a system of the form

$$\begin{aligned} x' &= f(x, y, \lambda, \varepsilon), \\ y' &= \varepsilon g(x, y, \lambda, \varepsilon), \end{aligned} \tag{3.1}$$

where λ is a parameter, and assume that f and g are C^k smooth in $(x, y, \lambda, \varepsilon)$ with $k \geqslant 3$. Further we assume that, for all λ (in some interval), the conditions (A1) and (A2) introduced in Section 2 hold. We assume, without loss of generality, that φ is defined on all of $\mathbb{R}$ and $S = \{(x, y): y = \varphi(x)\}$. A canard explosion occurs when, for some special value of λ, one of the folds becomes a canard point. Without loss of generality we assume that this special value is $\lambda = 0$, the canard point corresponds to the minimum of φ and is located at the origin $(x, y) = (0, 0)$, and that the maximum occurs for a positive value of $x = x_M$. In this context the defining condition for a canard point is $g(0, 0, 0, 0) = 0$. Henceforth we consider $\lambda \in (-\lambda_0, \lambda_0)$ with λ_0 sufficiently small.

We assume without loss of generality that the fold point which, for $\lambda = 0$, becomes a canard point is located at the origin for all $\lambda \in (-\lambda_0, \lambda_0)$. In this context the canard point is *non-degenerate* [16, Conditions 3.4] if

$$\begin{aligned} &\frac{\partial^2 f}{\partial x^2}(0, 0, 0, 0) \neq 0, \qquad \frac{\partial f}{\partial y}(0, 0, 0, 0) \neq 0, \\ &\frac{\partial g}{\partial x}(0, 0, 0, 0) \neq 0, \qquad \frac{\partial g}{\partial \lambda}(0, 0, 0, 0) \neq 0. \end{aligned} \tag{3.2}$$

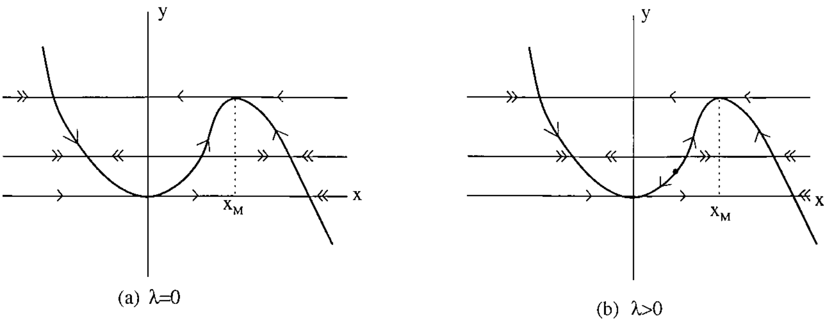

FIG. 4. Fast and slow dynamics leading to canard explosion. The unfolding of the $\lambda = 0$ slow dynamics shown in (b) corresponds to the case $\frac{\partial g}{\partial x}(0, 0, 0, 0)\, \frac{\partial g}{\partial \lambda}(0, 0, 0, 0) < 0$. (a) $\lambda = 0$. (b) $\lambda > 0$.

Consequently, we replace assumptions (A3) and (A4) by

(A3′) For $\lambda = 0$ one of the folds is a non-degenerate canard point and the other fold point is non-degenerate.

Note that at a canard point the expression $g(x, \varphi(x), 0, 0)/\varphi'(x)$ no longer has a singularity. It follows that for $\lambda = 0$ there exists a solution of (2.2) passing through the canard point (see [16, Section 4.1] for more details. We make the following assumption concerning the flow on $S_l \cup \{0\} \cup S_m$ and the flow on S_r.

(A4′) When $\lambda = 0$ then $\dot{x} < 0$ for the slow flow on S_r and $\dot{x} > 0$ for the slow flow on $S_l \cup \{0\} \cup S_m$.

Assumptions (A1)–(A4′) imply that the fast and slow dynamics for $\lambda = 0$ are as shown in Fig. 4a. In Fig. 4b we show the fast and slow dynamics for $\lambda > 0$. Note that this is precisely the situation for which, by Theorem 2.1, a relaxation oscillation, for $\varepsilon > 0$, must exist. On the other hand a direct calculation shows that Eq. (3.1) must have points of Hopf bifurcation limiting on the canard point.

3.1. *Singular Approximation of Canard Cycles*

The essence of canard explosion is that a small cycle coming from a Hopf bifurcation grows through a sequence of *canard cycles* to a relaxation oscillation. Canard cycles are obtained as perturbations of closed singular trajectories consisting of trajectories of the reduced and the layer problems, which contain trajectories lying on the unstable critical manifold S_m. We now define the family of closed singular trajectories giving rise to canard cycles.

Set $\lambda = 0$. Let $y_M = \varphi(x_M)$. For $s \in (0, y_M)$ let $x_l(s) < x_m(s) < x_r(s)$

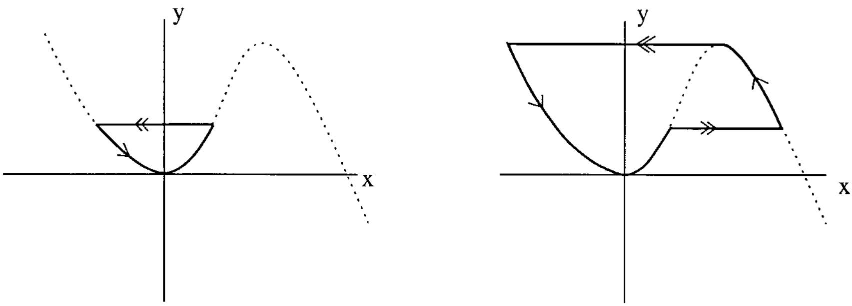

FIG. 5. Singular canard cycles $\Gamma(s)$. (a) $\Gamma(s)$ for $s \in (0, y_M)$. (b) $\Gamma(s)$ for $s \in (y_M, 2y_m)$.

be the three distinct roots of $\varphi(x) = s$. We set $x_l(0) = x_m(0) = 0$ and $x_m(y_M) = x_r(y_M) = x_M$, respectively. We define

$$\Gamma(s) = \{(x, \varphi(x)) : x \in [x_l(s), x_m(s)]\} \cup \{(x, s) : x \in [x_l(s), x_m(s)]\},$$
$$\text{for} \quad s \in [0, y_M],$$

and

$$\begin{aligned} \Gamma(s) = \{&(x, \varphi(x)) : x \in [x_l(y_M), x_m(2y_M - s)]\} \\ &\cup \{(x, 2y_M - s) : x \in [x_m(2y_M - s), x_r(2y_M - s)]\} \\ &\cup \{(x, \varphi(x)) : x \in [x_M, x_r(2y_M - s)]\} \\ &\cup \{(x, y_M) : x \in [x_l(y_M), x_M]\}, \qquad \text{for} \quad s \in [y_M, 2y_M]. \end{aligned}$$

Figure 5 shows two representatives of the family of singular cycles $\Gamma(s)$. Cycles of the family corresponding to $s \in (0, y_M)$ resp. $s \in (y_M, 2y_M)$ are often referred to as "canards without head" resp. "canards with head".

Our goal is to obtain a family $\Gamma(s, \varepsilon)$ of canard cycles existing for corresponding parameter values $\lambda = \lambda(s, \varepsilon)$ as a perturbation of the degenerate family $\Gamma(s)$, $\lambda = 0$ which exists for $\varepsilon = 0$. As s sweeps through a suitable interval the family of canard cycles $\Gamma(s, \varepsilon)$ connects the Hopf bifurcation to relaxation oscillations and canard explosion takes place. In contrast, the approach taken in [8] is to investigate convergence of one parameter families of canard cycles along certain curves in (ε, λ) space to individual singular cycles.

3.2. *Canard Point*

Near the non-degenerate canard point Eq. (3.1) can be transformed to the following canonical form (see [16, Section 3.1]).

$$\begin{aligned} x' &= -yh_1(x, y, \lambda, \varepsilon) + x^2 h_2(x, y, \lambda, \varepsilon) + \varepsilon h_3(x, y, \lambda, \varepsilon), \\ y' &= \varepsilon(xh_4(x, y, \lambda, \varepsilon) - \lambda h_5(x, y, \lambda, \varepsilon) + yh_6(x, y, \lambda, \varepsilon)), \end{aligned} \tag{3.3}$$

where

$$h_3(x, y, \lambda, \varepsilon) = O(x, y, \lambda, \varepsilon)$$

$$h_j(x, y, \lambda, \varepsilon) = 1 + O(x, y, \lambda, \varepsilon), \qquad j = 1, 2, 4, 5.$$

The main tool in the analysis of (3.3) is the blow-up transformation

$$\Phi: S^3 \times [0, \rho] \to \mathbb{R}^4$$

given by

$$x = \bar{r}\bar{x}, \quad y = \bar{r}^2\bar{y}, \quad \varepsilon = \bar{r}^2\bar{\varepsilon}, \quad \lambda = \bar{r}\bar{\lambda}, \qquad (\bar{x}, \bar{y}, \bar{\varepsilon}, \bar{\lambda}) \in S^3. \tag{3.4}$$

Blow-up serves to desingularize the flow near the canard point and thus makes it possible to apply standard tools of dynamical systems [7, 8, 16]. In the context of this work it is convenient to use two charts, denoted by K_1 and K_2, to describe much of the dynamics of the blown-up vector field. Chart K_1 is defined by requiring that the blow-up transformation be given by

$$x = r_1 x_1, \qquad y = r_1^2, \qquad \lambda = r_1 \lambda_1, \qquad \varepsilon = r_1^2 \varepsilon_1. \tag{3.5}$$

Let $\Phi_1(x_1, r_1, \varepsilon_1, \lambda_1)$ be the map defined by (3.5). As the domain of Φ_1 we use the set V_1 given by

$$V_1 = (-x_{1,0}, x_{1,0}) \times (-\rho, \rho) \times [0, 1) \times (-\mu, \mu),$$

with $x_{1,0} > 0$ sufficiently large and $\rho > 0$, $\mu > 0$ sufficiently small. It follows that in the original problem we consider $\varepsilon \in [0, \varepsilon_0)$, with $\varepsilon_0 = \rho^2$. Let

$$V_{1,\varepsilon} = \{(x_1, r_1, \varepsilon_1, \lambda_1) \in V_1 : \varepsilon = r_1^2 \varepsilon_1\}$$

and let $P_{(x, y)}$ denote the projection onto the (x, y) coordinates. It follows that

$$P_{(x, y)}(\Phi_1(V_{1,\varepsilon})) = \{(x, y) \colon y \in (\varepsilon, \rho^2), x \in (-x_{1,0}\sqrt{y}, x_{1,0}\sqrt{y})\}.$$

Chart K_2 is defined by requiring that the blow-up transformation be given by

$$x=r_2x_2, \qquad y=r_2^2y_2, \qquad \lambda=r_2\lambda_2, \qquad \varepsilon=r_2^2. \tag{3.6}$$

Let $\Phi_2(x_2, y_2, r_2, \lambda_2)$ be the map defined by (3.6). As the domain of Φ_2 we use the set

$$V_2=D\times[0,\rho)\times(-\mu,\mu),$$

where D is a disk of large radius centered at the origin and μ is small. Let

$$V_{2,\varepsilon}=\{(x_2, y_2, \varepsilon^{1/2}, \lambda_2)\in V_2\}.$$

It follows that $P_{(x,y)}(\Phi_2(V_{2,\varepsilon}))$ is a neighborhood of $(0,0)$ of size $O(\sqrt{\varepsilon})$ in x-direction and $O(\varepsilon)$ in y-direction. Let $V_\varepsilon=P_{(x,y)}(\Phi_1(V_{1,\varepsilon})\cup\Phi_2(V_{2,\varepsilon}))$. Clearly, $V_\varepsilon\subset V_{\tilde{\varepsilon}}$ for $\varepsilon<\tilde{\varepsilon}$. In the statements of the results on dynamics near a canard point we use $V=V_{\varepsilon_0}$.

For later convenience we restate the formulas for coordinate changes between K_1 and K_2. Let κ_{12} denote the change of coordinates from K_1 to K_2 and let $\kappa_{21}=\kappa_{12}^{-1}$. Then κ_{12} is given by

$$x_2=x_1\varepsilon_1^{-1/2}, \quad y_2=\varepsilon_1^{-1}, \quad r_2=r_1\varepsilon_1^{1/2}, \quad \lambda_2=\varepsilon_1^{-1/2}\lambda_1, \quad \text{for } \varepsilon_1>0, \tag{3.7}$$

and κ_{12} is given by

$$x_1=x_2y_2^{-1/2}, \quad \varepsilon_1=y_2^{-1}, \quad r_1=r_2y_2^{1/2}, \quad \lambda_1=\lambda_2y_2^{-1/2}, \quad \text{for } y_2>0. \tag{3.8}$$

Remark 3.1. While the analysis is carried out in the blow-up space it is instructive to introduce the overlapping neighborhoods $P_{(x,y)}(\Phi_1(V_{1,\varepsilon}))$ and $P_{(x,y)}(\Phi_2(V_{2,\varepsilon}))$, and their union V_ε in (x, y)-space for $\varepsilon>0$. It turns out that, for a given $\varepsilon\in(0,\varepsilon_0)$, the non-trivial dynamics local to the canard point $(0,0)$ takes place in V_ε. It is crucial that V_ε does not shrink to zero in the positive y-direction as $\varepsilon\to 0$. The neighborhood $P_{(x,y)}(\Phi_1(V_{1,\varepsilon}))$ can be viewed as the domain to which Fenichel theory—original valid only up to a fixed distance from the canard point—can be extended. Here the dynamics is relatively simple due to normal hyperbolicity. In the shrinking neighborhood $P_{(x,y)}(\Phi_2(V_{2,\varepsilon}))$ more standard rescaling techniques can be applied to study the dynamics.

Remark 3.2. In the analysis of the transition from large canard cycles to relaxation oscillations in Section 5 we will need two additional charts, K_3 and K_4, corresponding to directional blow-ups obtained by setting $\bar{x}=1$ and $\bar{\lambda}=1$, respectively, in the blow-up transformation (3.4).

3.3. *Contraction and Expansion Rates along S*

It follows from (A4′) that, for $\lambda = 0$, there exists a solution $x_0(t)$ of the reduced problem (2.2) defined on $(-\infty, T_M)$, where $T_M > 0$ is such that $x_0(0) = 0$ and $\lim_{t \to T_M-} x_0(t) = x_M$ (see Fig. 4). Let $t_l(s) \leqslant 0 \leqslant t_m(s)$ be such that

$$\varphi(x_0(t_l(s))) = \varphi(x_0(t_m(s))) = s.$$

For $s \in [0, y_M]$ we define the function $R(s)$ as follows

$$R(s) = \int_{t_l(s)}^{t_m(s)} \frac{\partial f}{\partial x}(x_0(t)), \varphi(x_0(t), 0, 0)\, dt. \tag{3.9}$$

Similarly, there is a solution $\hat{x}_0(t)$ of (2.2) defined on $(-\infty, T_M)$ which corresponds to the reduced flow on S_r and satisfies $\lim_{t \to T_{M-}} \hat{x}_0(t) = x_M$. Let $\hat{t}(s)$ be defined by $\varphi(\hat{x}_0(\hat{t}(s))) = s$. For $s \in [y_M, 2y_M]$ we define $R(s)$ as follows:

$$\begin{aligned} R(s) = & \int_{t_l(y_M)}^{t_m(2y_M - s)} \frac{\partial f}{\partial x}(x_0(t), \varphi(x_0(t)), 0, 0)\, dt \\ & + \int_{\hat{t}(2y_M - s)}^{T_M} \frac{\partial f}{\partial x}(\hat{x}_0(t), \varphi(\hat{x}_0(t)), 0, 0)\, dt. \end{aligned} \tag{3.10}$$

The function $R(s)$ is called the "way in–way out" function in [4]. Using (2.2) and changing variables from t to x in (3.9) and (3.10) we obtain

$$\begin{aligned} R(s) &= \int_{x_l(s)}^{x_m(s)} \frac{\partial f}{\partial x}(x, \varphi(x), 0, 0) \frac{\varphi'(x)}{g(x, \varphi(x), 0, 0)}\, dx, \qquad s \in [0, y_M], \\ R(s) &= \int_{x_l(y_M)}^{x_m(2y_M - s)} \frac{\partial f}{\partial x}(x, \varphi(x), 0, 0) \frac{\varphi'(x)}{g(x, \varphi(x), 0, 0)}\, dx \\ &\quad + \int_{x_r(2y_M - s)}^{x_M} \frac{\partial f}{\partial x}(x, \varphi(x), 0, 0) \frac{\varphi'(x)}{g(x, \varphi(x), 0, 0)}\, dx, \quad s \in [y_M, 2y_M]. \end{aligned} \tag{3.11}$$

Recall the functions $h_1, ..., h_6$ defining the right hand side of (3.3). Let

$$\begin{aligned} &a_1 = \frac{\partial h_3}{\partial x}(0, 0, 0, 0), \quad a_2 = \frac{\partial h_1}{\partial x}(0, 0, 0, 0), \quad a_3 = \frac{\partial h_2}{\partial x}(0, 0, 0, 0), \\ &a_4 = \frac{\partial h_4}{\partial x}(0, 0, 0, 0), \quad a_5 = h_6(0, 0, 0, 0) \end{aligned} \tag{3.12}$$

and define the constant

$$A = -a_2 + 3a_3 - 2a_4 - 2a_5. \tag{3.13}$$

The condition $A \neq 0$ and the sign of A turn out to be very important in various dynamic phenomena related to canard explosion.

The following result states the basic properties of R.

PROPOSITION 3.1. *The function $R(s)$ has the following properties*:

(i) $R(0) = 0$ *and* $R(2y_M) < 0$.

(ii) $R(s)$ *is* C^k *smooth for* $s \in [0, 2y_M]$.

(iii) $R(s) = \frac{4}{3} A s^{3/2} + O(s^2)$ *as* $s \to 0+$.

Proof. Assertions (i) and (ii) follow immediately from (3.11). For (iii) note that

$$R'(s) = \frac{\frac{\partial f}{\partial x}(x_m(s), s, 0, 0)\, g(x_l(s), s, 0, 0) - \frac{\partial f}{\partial x}(x_l(s), s, 0, 0)\, g(x_m(s), s, 0, 0)}{g(x_m(s), s, 0, 0)\, g(x_l(s), s, 0, 0)}$$

We show that $R'(s) = 2As^{1/2} + O(s)$, which implies the result. From the definition of $x_l(s)$ and $x_m(s)$ and from the form of (3.3) it follows that

$$\begin{aligned} &\frac{\partial f}{\partial x}(x_m(s), s, 0, 0)\, g(x_l(s), s, 0, 0) - \frac{\partial f}{\partial x}(x_l(s), s, 0, 0)\, g(x_m(s), s, 0, 0) \\ &\quad = (2\sqrt{s} - a_2 s + 3a_3 s)(-\sqrt{s} + a_4 s + a_5 s) \\ &\qquad - (-2\sqrt{s} - a_2 s + 3a_3 s)(\sqrt{s} + a_4 s + a_5 s) + O(s^2) \\ &\quad = -2As^{3/2} + O(s^2). \end{aligned} \tag{3.14}$$

The result follows from (3.14) and from the expansion

$$g(x_m(s), s, 0, 0)\, g(x_l(s), s, 0, 0) = -s + O(s^{3/2}). \quad \blacksquare$$

3.4. *Statement of the Main Results*

In this section we state our main results on canard explosion. Fix $s_0 > 0$ small and let $U(s)$, $s \in [s_0, 2y_M]$ denote small tubular neighborhoods of $\Gamma(s)$. For $s \in [0, s_0]$ we define $U(s) \equiv V_{\varepsilon_0}$ as a small neighborhood of the origin. We assume that ε_0, λ_0 and $V = V_{\varepsilon_0}$ are chosen such that the description of the flow near a canard point given in Section 4 holds. We begin by stating a result on the existence of Hopf bifurcation.

THEOREM 3.1. *Suppose* (A1)–(A4′) *hold. Then there exist* $\varepsilon_0 > 0$, $\lambda_0 > 0$ *such that for each* $0 < \varepsilon < \varepsilon_0$, $|\lambda| < \lambda_0$ *Eq.* (3.1) *has precisely one equilibrium*

$p_e \in V$ which converges to the canard point as $(\varepsilon, \lambda) \to 0$. Moreover, there exists a curve $\lambda_H(\sqrt{\varepsilon})$ such that p_e is stable for $\lambda < \lambda_H(\sqrt{\varepsilon})$ and loses stability through a Hopf bifurcation as λ passes through $\lambda_H(\sqrt{\varepsilon})$. The curve $\lambda_H(\sqrt{\varepsilon})$ has the expansion

$$\lambda_H(\sqrt{\varepsilon}) = -\frac{a_1 + a_5}{2}\varepsilon + O(\varepsilon^{3/2}). \tag{3.15}$$

The Hopf bifurcation is non-degenerate if the constant A defined in (3.13) is nonzero. It is supercritical if $A < 0$ and subcritical if $A > 0$.

The proof of Theorem 3.1 will be given in Section 4. The cases of a supercritical and a subcritical Hopf bifurcation are quite different. The case $A < 0$ is simpler and is the one that occurs for the well known example of the van der Pol equation. The case $A > 0$ is also known in applications—it occurs, in particular, for the Oregonator model of the Belousov–Zhabotinsky equation [14].

We now fix the slow manifolds $S_{l,\varepsilon}$, $S_{m,\varepsilon}$ and $S_{r,\varepsilon}$, assuming that they satisfy the following requirements:

- $S_{l,\varepsilon}$ continues beyond the boundary of $U(2y_M)$ in the negative x direction and reaches V_{ε_0} in the positive x direction,
- $S_{m,\varepsilon}$ starts in V_{ε_0} and continues to a small neighborhood of the fold point at (x_M, y_M),
- $S_{r,\varepsilon}$ starts in a small neighborhood of the fold point at (x_M, y_M) and continues beyond the boundary of $U(2y_M)$ in the positive x direction.

The existence of slow manifolds satisfying the above requirements is guaranteed by Fenichel theory. The non-uniqueness of these slow manifolds does not pose any problems since they are only used as auxiliary geometric objects in the description and analysis of canard explosion.

We now restate Theorem 3.1 of [16] in the notation of this article.

THEOREM 3.2. *Suppose that (A1)–(A4′) hold. Then there exists a smooth function $\lambda_c(\sqrt{\varepsilon})$ such that a solution starting in $S_{l,\varepsilon}$ connects to $S_{m,\varepsilon}$, if and only if $\lambda = \lambda_c(\sqrt{\varepsilon})$. The function λ_c has the expansion*

$$\lambda_c(\sqrt{\varepsilon}) = -\left(\frac{a_1 + a_5}{2} + \frac{1}{8}A\right)\varepsilon + O(\varepsilon^{3/2}). \tag{3.16}$$

The subscript c in λ_c stands for "canard", more precisely "maximal canard". It follows from the analysis in [16] that, as λ passes through

$\lambda_c(\sqrt{\varepsilon})$, the manifolds $S_{l,\varepsilon}$ and $S_{m,\varepsilon}$ change their relative position, as indicated in Figs. 9 and 10. The curve $\lambda_c(\sqrt{\varepsilon})$ plays a central role in the description of canard explosion, as given in the following theorem.

THEOREM 3.3. *Fix* ε_0 *sufficiently small and* $v \in (0, 1)$. *Suppose* (A1)–(A4′) *hold and* $A < 0$. *For* $\varepsilon \in (0, \varepsilon_0)$ *there exists a family of periodic orbits*

$$s \to (\lambda(s, \sqrt{\varepsilon}), \Gamma(s, \sqrt{\varepsilon})), \qquad s \in (0, 2y_M)$$

which is C^k*-smooth in* $(s, \sqrt{\varepsilon})$, *and such that*:

(i) *for* $s \in (0, \varepsilon^v)$ *the orbit* $\Gamma(s, \sqrt{\varepsilon})$ *is attracting and uniformly* $O(\varepsilon^v)$ *close to the canard point and* $\lambda(s, \sqrt{\varepsilon})$ *is strictly increasing in* s,

(ii) *for* $s \in (2y_M - \varepsilon^v, 2y_M)$ *the orbit* $\Gamma(s, \sqrt{\varepsilon})$ *is a relaxation oscillation and* $\lambda(s, \sqrt{\varepsilon})$ *is strictly increasing in* s,

(iii) *for* $s \in [\varepsilon^v, 2y_M - \varepsilon^v]$

$$|\lambda(s, \sqrt{\varepsilon}) - \lambda_c(\sqrt{\varepsilon})| \leqslant e^{-1/\varepsilon^{1-v}}; \tag{3.17}$$

(iv) *as* $\varepsilon \to 0$ *the family* $\Gamma(s, \sqrt{\varepsilon})$ *converges uniformly in Hausdorff distance to* $\Gamma(s)$;

(v) *any periodic orbit passing sufficiently close to the critical manifold* S *is a member of the family* $\Gamma(s, \sqrt{\varepsilon})$, *or a relaxation oscillation.*

The meaning of smooth dependence of Γ on $(s, \sqrt{\varepsilon})$ is that the corresponding periodic solutions $\gamma(t, s, \sqrt{\varepsilon})$ depend smoothly on $(s, \sqrt{\varepsilon})$. The curves $\lambda_s(\sqrt{\varepsilon}) \overset{\text{def}}{=} \lambda(\varepsilon^v, \sqrt{\varepsilon})$ and $\lambda_r(\sqrt{\varepsilon}) \overset{\text{def}}{=} \lambda(2y_M - \varepsilon^v, \sqrt{\varepsilon})$ mark the beginning and the end of canard explosion, hence subscripts s and r denote "small" and "relaxation" cycles, respectively (see Fig. 6a).

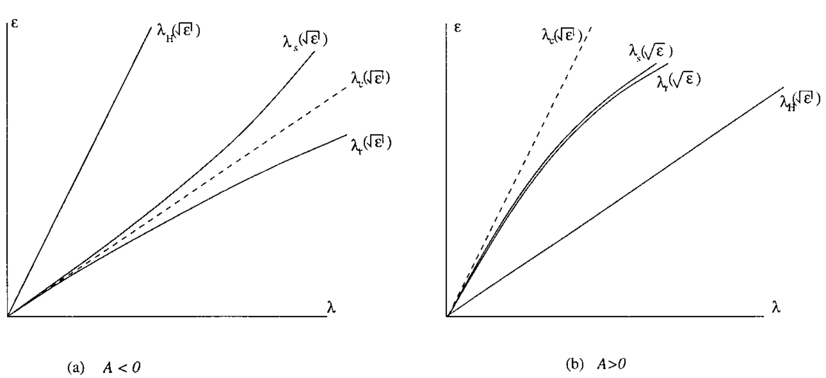

FIG. 6. Bifurcation lines corresponding to a canard explosion. (a) $A < 0$. (b) $A > 0$.

Theorem 3.3 guarantees that a canard explosion takes place. However, we can obtain information on stability and a better characterization of the functions $\lambda(s, \sqrt{\varepsilon})$ based on the knowledge of $R(s)$ which describes the contraction rates along S.

THEOREM 3.4. *Under the assumptions of Theorem 3.3 assume additionally $R(s) < 0$ for all $s \in (0, y_M]$. Then all canard cycles are stable and the functions $\lambda(s, \sqrt{\varepsilon})$ are monotonic in s.*

Under the assumptions of Theorem 3.4 the bifurcation diagram for fixed ε is as shown in Fig. 7a.

Now we discuss the case $A > 0$.

THEOREM 3.5. *Suppose (A1)–(A4′) hold and $A > 0$. Fix ε_0 sufficiently small and $\nu \in (0, 1)$. For $\varepsilon \in (0, \varepsilon_0)$ there exists a family of periodic orbits*

$$s \to (\lambda(s, \sqrt{\varepsilon}), \Gamma(s, \sqrt{\varepsilon})), \qquad s \in (0, 2y_M),$$

which is C^k-smooth in $(s, \sqrt{\varepsilon})$, and such that:

(i) *for $s \in (0, \varepsilon^\nu)$ the orbit $\Gamma(s, \sqrt{\varepsilon})$ is repelling and uniformly $O(\varepsilon^\nu)$ close to the canard point and $\lambda(s, \sqrt{\varepsilon})$ is strictly decreasing in s,*

(ii) *for $s \in (2y_M - \varepsilon^\nu, 2y_M)$ the orbit $\Gamma(s, \sqrt{\varepsilon})$ is a relaxation oscillation and $\lambda(s, \sqrt{\varepsilon})$ is strictly increasing in s,*

(iii) *for $s \in [\varepsilon^\nu, 2y_M - \varepsilon^\nu]$,*

$$|\lambda(s, \sqrt{\varepsilon}) - \lambda_c(\sqrt{\varepsilon})| \leqslant e^{-1/\varepsilon^{1-\nu}},$$

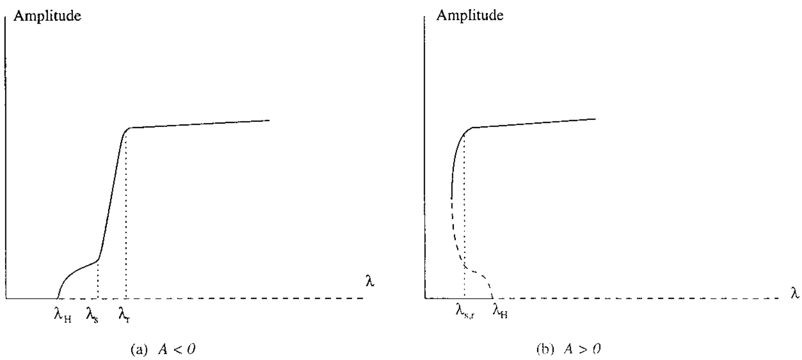

FIG. 7. Bifurcation diagrams for fixed ε corresponding to a canard explosion. (a) $A < 0$. (b) $A > 0$.

(iv) *as* $\varepsilon \to 0$ *the family* $\Gamma(s, \sqrt{\varepsilon})$ *converges uniformly in Hausdorff distance to* $\Gamma(s)$,

(v) *any periodic orbit passing sufficiently close to the critical manifold* S *is a member of the family* $\Gamma(s, \sqrt{\varepsilon})$, *or a relaxation oscillation.*

The functions $\lambda_s(\sqrt{\varepsilon}) \stackrel{\text{def}}{=} \lambda(\varepsilon^\nu, \varepsilon)$ and $\lambda_r(\sqrt{\varepsilon}) \stackrel{\text{def}}{=} \lambda(2y_M - \varepsilon^\nu, \sqrt{\varepsilon})$ corresponding to the onset and the end of canard explosion are shown in Fig. 6b. Note that in this case $\lambda(s, \sqrt{\varepsilon})$ must have at least one *limit point* in $[\varepsilon^\nu, 2y_M - \varepsilon^\nu]$. The simplest situation occurs when $\lambda(s, \sqrt{\varepsilon})$ has just one limit point. This case is described by the following theorem.

THEOREM 3.6. *Suppose* (A1)–(A4′) *hold,* $A > 0$, *and that* $R(s)$ *has exactly one simple zero at* $s_{lp,\,0} \in (0, 2y_M)$. *Then there exists a* C^1 *function* $s_{lp}(\sqrt{\varepsilon})$ *such that, for each* $\varepsilon \in (0, \varepsilon_0)$, *the curve* $(s, \lambda(s, \sqrt{\varepsilon}))$ *has a unique, non-degenerate limit point for* $s = s_{lp}(\sqrt{\varepsilon})$ *and* $s_{lp}(\sqrt{\varepsilon}) \to s_{lp,\,0}$ *as* $\varepsilon \to 0$. *Moreover, the cycles* $\Gamma(s, \sqrt{\varepsilon})$ *are repelling for* $s \in (0, s_{lp}(\sqrt{\varepsilon}))$ *and attracting for* $s \in (s_{lp}(\sqrt{\varepsilon}), 2y_M)$.

The corresponding bifurcation diagram for fixed ε is shown in Fig. 7b. Related results on bifurcations of multiple canard cycles have been obtained in [4, 9, 10].

The results outlined above will be proved using standard singular geometric perturbation theory combined with two blow-ups, one at the canard point, and one at the regular fold point.

4. THE FLOW NEAR THE CANARD POINT

A complete description of the flow near a canard point is necessary for the results on canard explosion and will be obtained in this section. The canard solution, which by Theorem 3.2 exists along a curve $\lambda = \lambda_c(\sqrt{\varepsilon})$, plays a central role in this description. The dynamics of (3.1) in V strongly depends on the criticality of the Hopf bifurcation described in Theorem 3.1, that is on the sign of A. We will see that the sign of A determines the local dynamics and in particular the monotonicity of $\lambda(s, \sqrt{\varepsilon})$ for small values of s.

The following result describes the case of a supercritical Hopf bifurcation.

THEOREM 4.1. *Suppose* ε_0, λ_0 *and* $V = V_{\varepsilon_0}$ *are sufficiently small and* $A < 0$. *Fix* $\varepsilon \in (0, \varepsilon_0]$. *Then the following statements hold:*

(i) *For* $\lambda \in (-\lambda_0, \lambda_H(\sqrt{\varepsilon})]$ *all orbits starting in* V *converge to* p_e *or leave* V.

(ii) *There exists a curve* $\lambda = \lambda_{sc}(\sqrt{\varepsilon})$ *and a constant* $K > 0$, *with*

$$0 < \lambda_c(\sqrt{\varepsilon}) - \lambda_{sc}(\sqrt{\varepsilon}) = O(e^{-K/\varepsilon}), \tag{4.1}$$

such that for each $\lambda \in (\lambda_H(\sqrt{\varepsilon}), \lambda_{sc}(\sqrt{\varepsilon}))$ *Eq.* (3.1) *has a unique, attracting limit cycle* $\Gamma_{(\lambda, \varepsilon)}$ *contained in* V. *All orbits starting in* V, *except for* p_e, *either leave* V *or are attracted to* $\Gamma_{(\lambda, \varepsilon)}$.

(iii) *For* $\lambda \in (\lambda_{sc}(\sqrt{\varepsilon}), \lambda_0]$ *all orbits starting in* V, *except for* p_ε, *leave* V.

In the theorems above and below the subscript in λ_{sc} denotes "small" canard cycle. The case of subcritical Hopf bifurcation is described by the following result.

THEOREM 4.2. *Suppose* ε_0, λ_0 *and* $V = V_{\varepsilon_0}$ *are sufficiently small and* $A > 0$. *Fix* $\varepsilon \in (0, \varepsilon_0)$. *Then the following statements hold*:

(i) *There exists a curve* $\lambda = \lambda_{sc}(\sqrt{\varepsilon})$ *and a constant* $K > 0$, *with*

$$0 < \lambda_{sc}(\sqrt{\varepsilon}) - \lambda_c(\sqrt{\varepsilon}) = O(e^{-K/\varepsilon}),$$

such that for each $\lambda \in (\lambda_{sc}(\sqrt{\varepsilon}), \lambda_H(\sqrt{\varepsilon}))$ *Eq.* (3.1) *has a unique, repelling limit cycle* $\Gamma_{(\lambda, \varepsilon)}$ *contained in* V.

(ii) *For* $\lambda \in (-\lambda_0, \lambda_H(\sqrt{\varepsilon}))$ *all orbits starting in* V, *except for* $\Gamma_{(\lambda, \varepsilon)}$ *either leave* V *or are attracted to* p_e.

(iii) *For* $\lambda \in [\lambda_H(\sqrt{\varepsilon}), \lambda_0)$ *all orbits starting in* V, *except for* p_e, *leave* V.

Remark 4.1. Based on the information contained in the results stated in this section we can draw phase portraits for the dynamics local to the canard point. In the (λ, ε)-plane there are four regions of robust behavior, marked as I, II, III and IV in Fig. 8. The corresponding phase portraits are shown in Fig. 9 for the case $A < 0$ and in Fig. 10 for $A > 0$. The box containing each phase portrait is a schematic representation of the neighborhood V.

4.1. *Chart* K_2

In chart K_2 the transformed and desingularized Eqs. (3.3) have the following form (see [16, Section 3.3] for more details).

$$\begin{aligned} x_2' &= -y_2 + x_2^2 + r_2 G_1(x_2, y_2) + O(r_2(\lambda_2 + r_2)), \\ y_2' &= x_2 - \lambda_2 + r_2 G_2(x_2, y_2) + O(r_2(\lambda_2 + r_2)), \end{aligned} \tag{4.2}$$

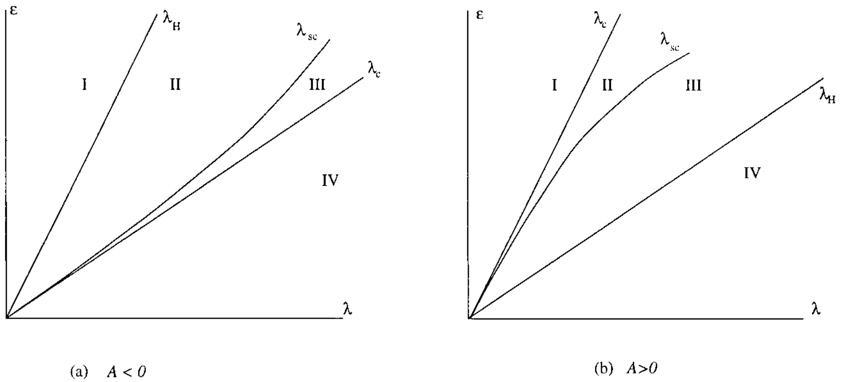

FIG. 8. Bifurcation curves and regions of typical behavior for the flow near the canard point. (a) $A < 0$. (b) $A > 0$.

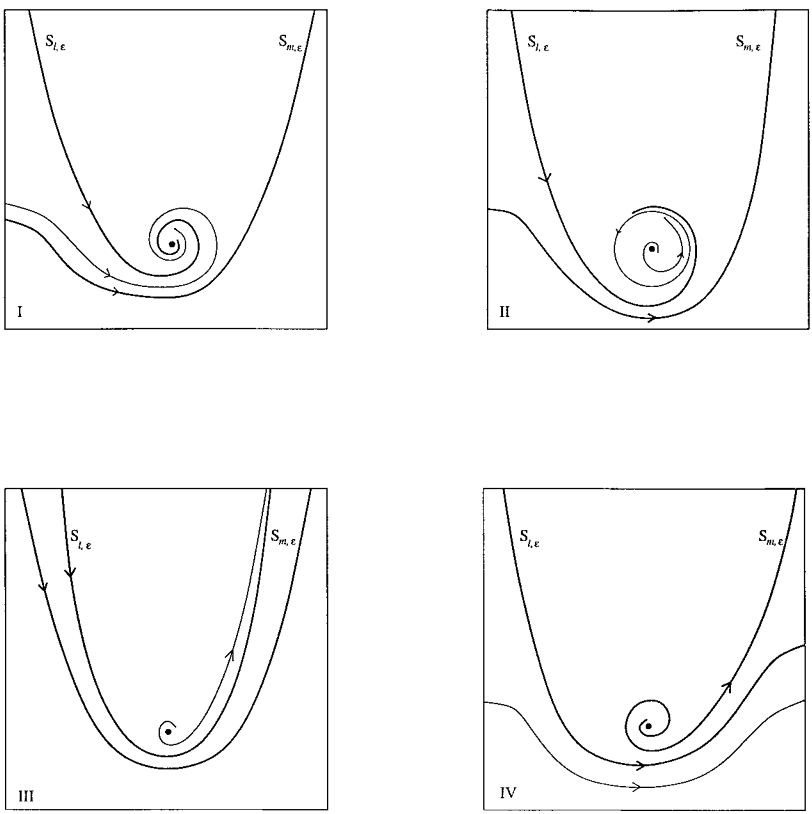

FIG. 9. Phase portraits in V corresponding to the regions I, II, III and IV for $A < 0$.

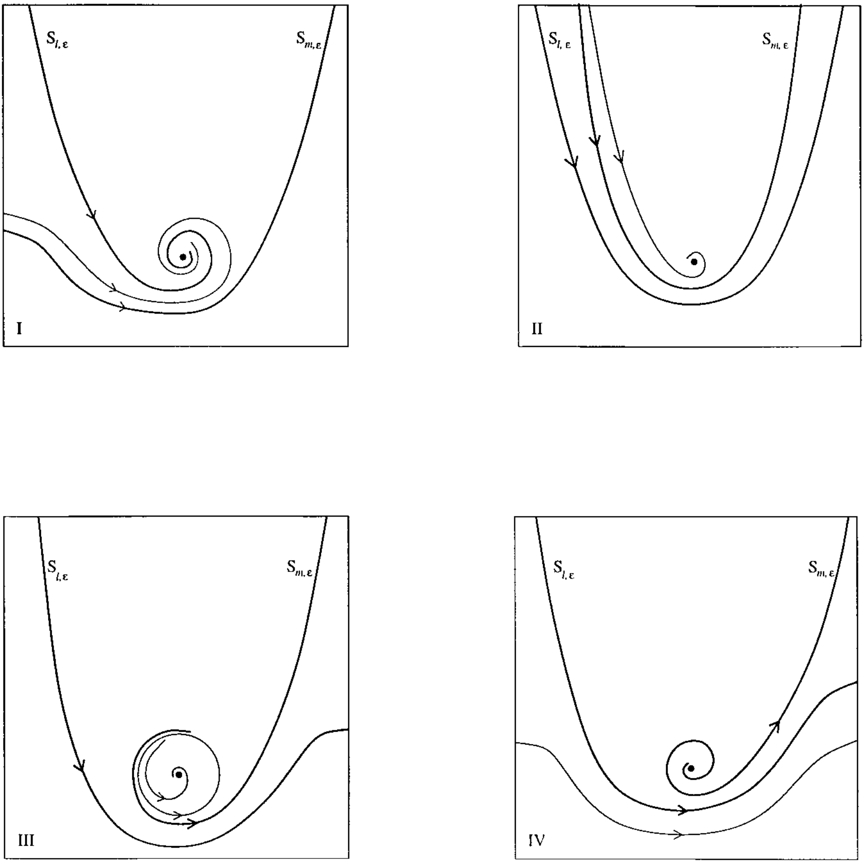

FIG. 10. Phase portraits in V corresponding to the regions I, II, III and IV for $A>0$.

where

$$G(x_2, y_2)=\begin{pmatrix} G_1(x_2, y_2) \\ G_2(x_2, y_2) \end{pmatrix}=\begin{pmatrix} a_1 x_2 - a_2 x_2 y_2 + a_3 x_2^3 \\ a_4 x_2^2 + a_5 y_2 \end{pmatrix}.$$

For $r_2=\lambda_2=0$ the system (4.2) is integrable with

$$H(x_2, y_2)=\tfrac{1}{2} e^{-2y_2}(y_2 - x_2^2 + \tfrac{1}{2}) \tag{4.3}$$

the corresponding constant of motion. The function $H(x_2, y_2)$ has a continuous family of closed level curves

$$\Gamma_2^h=\{(x_2, y_2)\colon H(x_2, y_2)=h\},$$

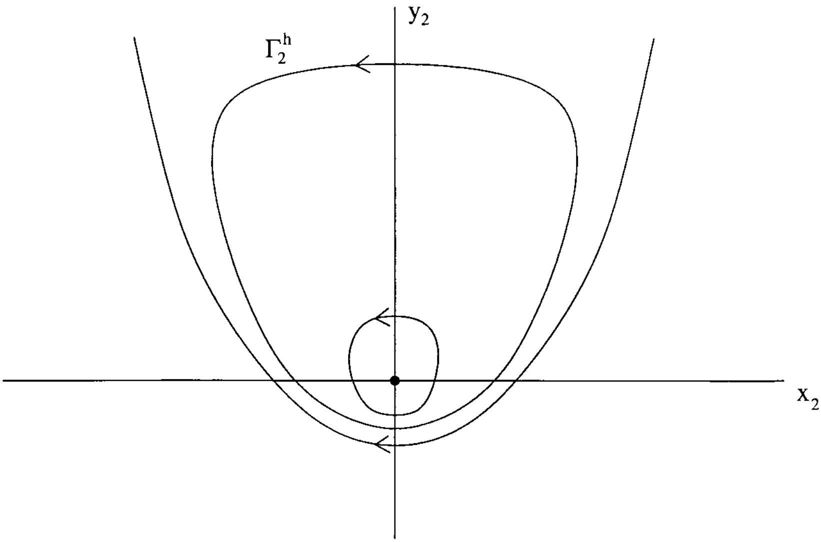

FIG. 11. The level curves Γ_2^h.

$h \in (0, 1/4)$ contained in the interior of the parabola $x_2^2 - y_2 = 1/2$, which corresponds to the level curve for $h = 0$ (see Fig. 11). The corresponding special solution is given by

$$\gamma_2^0(t) = (x_2^0(t), y_2^0(t)) = (\tfrac{1}{2}t, \tfrac{1}{4}t^2 - \tfrac{1}{2}).$$

In [16] we prove that γ_2^0 (denoted by $\gamma_{c,2}$ there) perturbs to a canard solution using a variant of the Melnikov method. Here we study the problem of persistence of periodic orbits Γ_2^h for $(\varepsilon, \lambda) \neq (0, 0)$. The usual approach [6] is to use the function H to measure the separation between the backward and the forward trajectories emanating from a given point. This method will be used later on, with a part of the computation carried out in chart K_2 and the other part in chart K_1. Theorem 3.1 concerns dynamics taking place near the equilibrium $\{h = 1/4\}$ and has to be proved separately, using a version of the Hopf bifurcation theorem. This proof, which is given below, is completely standard and only information on the dynamics in chart K_2 is necessary.

Proof of Theorem 3.1. The proof is based on Theorem 2.6 of [6, Chapter 3], see also the proof Lemma 1.10 of [6, Chapter 4]. Equation (4.2) has an equilibrium $p_{e,2} = (x_{e,2}, y_{e,2})$ with $x_{e,2} = \lambda_2 + O(2)$ and $y_{e,2} = O(2)$, where $O(2) = O(r_2^2 + |r_2 \lambda_2| + \lambda_2^2)$. The linearization of (4.2) at $p_{e,2}$ has the form

$$\begin{pmatrix} 2\lambda_2 + r_2 a_1 + O(2) & -1 + O(2) \\ 1 + O(2) & r_2 a_5 + O(2) \end{pmatrix}. \tag{4.4}$$

It follows that the linearization (4.4) has a purely imaginary eigenvalue for

$$\lambda_{H,2}(r_2) = -\tfrac{1}{2}(a_1 + a_5)\, r_2 + O(r_2^2).$$

We now rescale λ_2 letting $\lambda_2 = r_2 \tilde{\lambda}_2$. With the rescaled parameter system (4.2) has the form of (2.13) in [6, Chapter 3]. A straightforward computation shows that conditions H_1^* and H_2^* of Theorem 2.6 of [6, Chapter 3] hold, namely the imaginary part of the eigenvalue is non-zero and the real part satisfies a suitable crossing condition. It remains to verify the hypothesis H_3^* saying that $\lim_{r_2 \to 0} L_1(r_2) \neq 0$, where $L_1(r_2)$ is the first Liapunov coefficient. Applying formula (2.34) of [6, Chapter 3] to (4.2) we obtain

$$L_1(r_2) = \tfrac{1}{8} A r_2 + O(r_2^2).$$

The result follows. ∎

4.2. *Chart* K_1

In this section we outline the information on the dynamics in chart K_1 necessary for the computation of periodic orbits. Equation (3.1) transformed to K_1 and desingularized (see [16, Sections 2.5 and 3.4]) has the form

$$\begin{aligned} x_1' &= -1 + x_1^2 + r_1(a_1 \varepsilon_1 x_1 - a_2 x_1 + a_3 x_1^3) \\ &\quad - \tfrac{1}{2}\varepsilon_1 x_1 F(x_1, r_1, \varepsilon_1, \lambda_1) + O(r_1(r_1 + \lambda_1)), \end{aligned} \tag{4.5a}$$

$$r_1' = \tfrac{1}{2} r_1 \varepsilon_1 F(x_1, r_1, \varepsilon_1, \lambda_1), \tag{4.5b}$$

$$\varepsilon_1' = -\varepsilon_1^2 F(x_1, r_1, \varepsilon_1, \lambda_1), \tag{4.5c}$$

$$\lambda_1' = -\tfrac{1}{2}\lambda_1 \varepsilon_1 F(x_1, r_1, \varepsilon_1, \lambda_1), \tag{4.5d}$$

where

$$F(x_1, r_1, \varepsilon_1, \lambda_1) = x_1 - \lambda_1 + r_1(a_4 x_1^2 + a_5) + O(r_1(r_1 + \lambda_1)).$$

We will be interested in λ_1 close to 0, i.e. μ small. Observe that the hyperplanes $r_1 = 0$, $\varepsilon_1 = 0$ and $\lambda_1 = 0$ are invariant. The invariant line $l_1 := \{(x_1, 0, 0, 0) : x_1 \in \mathbb{R}\}$ contains two equilibria p_l and p_m. As shown in [16, Section 3.4] the equilibria p_l and p_m are contained in three-dimensional center manifolds $M_{l,1}$ and $M_{m,1}$ which are exponentially attracting and exponentially repelling, respectively. System (4.5) restricted to the invariant plane $r_1 = \lambda_1 = 0$ has the form

$$\begin{aligned} x_1' &= -1 + x_1^2 - \tfrac{1}{2}\varepsilon_1 x_1^2, \\ \varepsilon_1' &= -\varepsilon_1^2 x_1. \end{aligned} \tag{4.6}$$

Let $\tilde{H} = H \circ \kappa_{12}$, i.e.

$$\tilde{H}(x_1, r_1, \varepsilon_1, \lambda_1) = \tilde{H}(x_1, \varepsilon_1) = H\left(\frac{x_1}{\sqrt{\varepsilon_1}}, \frac{1}{\varepsilon_1}\right)$$

$$= e^{-2/\delta_1}\left(\frac{1}{4} + \frac{1}{2\varepsilon_1} - \frac{x_1^2}{2\varepsilon_1}\right). \tag{4.7}$$

System (4.6) is integrable with $\tilde{H}$ being a constant of motion. In particular the curves

$$\Gamma_1^h = \kappa_{21}(\{(x_2, y_2) \in \Gamma_2^h : y_2 > 0\})$$

are level curves of $\tilde{H}$ and are invariant for (4.6). Let $N_{l,1} = M_{l,1} \cap \{r_1 = \lambda_1 = 0\}$ and $N_{m,1} = M_{m,1} \cap \{r_1 = \lambda_1 = 0\}$, see Fig. 12.

The following sections of the flow of (4.5) will be used in the remainder of this paper

$$\Sigma_{l,1}^{in} := \{(x_1, r_1, \varepsilon_1, \lambda_1) \in V_1 : r_1 = \rho, |1 + x_1| < \beta\},$$

$$\Sigma_{l,1}^{out} := \{(x_1, r_1, \varepsilon_1, \lambda_1) \in V_1 : \varepsilon_1 = \delta, |1 + x_1| < \beta\},$$

$$\Sigma_{m,1}^{in} := \{(x_1, r_1, \varepsilon_1, \lambda_1) \in V_1 : \varepsilon_1 = \delta, |1 - x_1| < \beta\},$$

$$\Sigma_{m,1}^{out} := \{(x_1, r_1, \varepsilon_1, \lambda_1) \in V_1 : r_1 = \rho, |1 - x_1| < \beta\},$$

where β and δ are small positive constants.

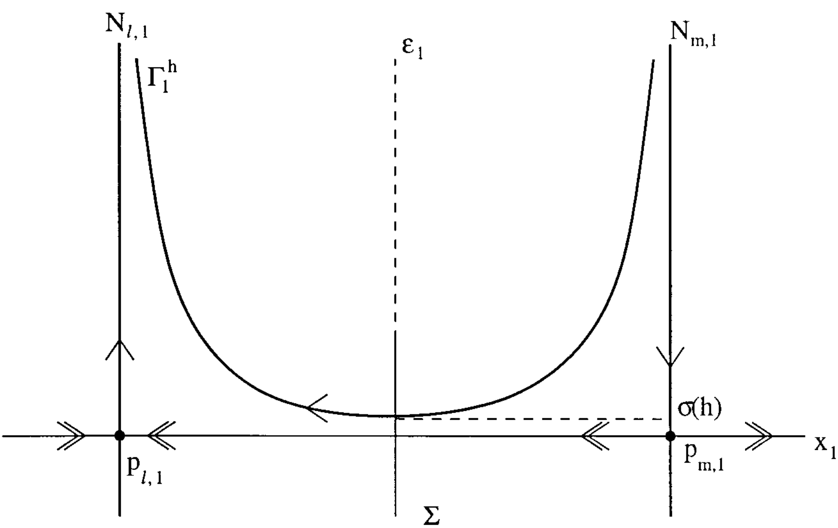

FIG. 12. The curves Γ_1^h and the section Σ.

4.3. *Periodic Orbits*

A standard approach for finding periodic orbits is to use perturbation analysis from the family Γ_2^h. This approach works for h bounded away from 0. When $h \to 0$ the curves Γ_2^h become unbounded, and as in the case of the Melnikov computation [16, Section 3.6] it is convenient to carry out a significant part of the analysis in K_1. In order to understand the flow in chart K_2 near infinity we use chart K_1 and analyze perturbations of the curves Γ_1^h. The analysis for h close to 1/4 based on the Hopf bifurcation theorem is given in the proof of Theorem 3.1. Now we consider $h \leqslant h_0$, where $h_0 \in (0, 1/4)$.

Let $\gamma_2^h(t) = (x_2^h(t), y_2^h(t))$ be a solution corresponding to Γ_2^h such that $x_2^h(0) = 0$ and $y_2^h(0) > 0$. For each $(r_2, \lambda_2) \in [0, \sqrt{\varepsilon_0}) \times (-\mu, \mu)$ let $\gamma_{r_2, \lambda_2}^h$ and $\hat{\gamma}_{r_2, \lambda_2}^h$ be the forward solution and the backward solution of (4.2) satisfying

$$\gamma_{r_2, \lambda_2}^h(0) = \hat{\gamma}_{r_2, \lambda_2}^h(0) = \gamma_2^h(0).$$

Let $(0, y_{r_2, \lambda_2}^h)$ and $(0, \hat{y}_{r_2, \lambda_2}^h)$ be the points of intersection of $\gamma_{r_2, \lambda_2}^h$ and $\hat{\gamma}_{r_2, \lambda_2}^h$, respectively, with the negative part of the y_2-axis. Equation (4.2) has a periodic orbit passing through $\gamma_2^h(0)$ if and only if $y_{r_2, \lambda_2}^h = \hat{y}_{r_2, \lambda_2}^h$. We define the function

$$\mathscr{D}_s(h, r_2, \lambda_2) = H(0, y_{r_2, \lambda_2}^h) - H(0, \hat{y}_{r_2, \lambda_2}^h).$$

Since $\frac{\partial H}{\partial y}(0, y) \neq 0$ it follows that periodic orbits of (4.2) correspond to solutions of

$$\mathscr{D}_s(h, r_2, \lambda_2) = 0. \tag{4.8}$$

The subscript in $\mathscr{D}_s$ denotes "small" cycles including "small" canard cycles, i.e. all cycles which stay in V.

Our goal is to solve (4.8) for $\lambda_2 = \lambda_2(h, r_2)$. Since we attempt to find solutions of (3.1) restricted to the neighborhood V we assume that $|r_2^2 y_{r_2, \lambda_2}^h(t)| \leqslant \rho^2$ along the solution. Since we expect the maximum of $y_{r_2, \lambda_2}^h(t)$ to occur close to $y_2^h(0)$ this gives us an approximate bound on r_2, depending on h. This bound is $r_2 \leqslant \rho(y_2^h(0))^{-1/2}$. Let $\sigma(h) = (y_2^h(0))^{-1}$ which is the minimal value of ε_1 along Γ_1^h. In order to apply the implicit function theorem on the set

$$U_0 = (0, h_0) \times [0, \rho\sqrt{\sigma(h)}) \times (-\mu, \mu), \tag{4.9}$$

we will prove that the following properties hold:

(i) $\lambda_2(h, r_2)$ is defined for $(h, r_2) \in (0, h_0) \times [0, \rho\sqrt{\sigma(h)})$,

(ii) the solution is unique for the flow restricted to V, that is for every (h, r_2) there exists a unique $\lambda_2 = \lambda_2(h, r_2)$ solving (4.8).

(iii) there exists a curve $\lambda_{sc,2}(r_2)$ such that for every $r_2 \in (0, \sqrt{\varepsilon_0})$ and for every $\lambda_2 \in (0, \lambda_{sc,2}(r_2))$ there exists a unique h such that $\lambda_2 = \lambda_2(h, r_2)$.

The reason why items (i)–(iii) require some care is that $\frac{\partial}{\partial h}\mathscr{D}_s(h, r_2, \lambda_2)$ blows up as $h \to 0$. One way of dealing with this issue is to restrict attention to the open interval $(0, h_0)$, apply the implicit function theorem for $h \in (0, h_0)$ and show existence and uniqueness by means of suitable estimates on $\mathscr{D}_s$. To prove (iii) we need to estimate $\frac{\partial}{\partial h}\mathscr{D}_s(h, r_2, \lambda_2)$ as $h \to 0$. We now state a result leading to the proof of (i), (ii) and (iii) and subsequently make some additional remarks. Let T^h be the half period of Γ_2^h, i.e. $\gamma_2^h(T^h) = (\gamma_2^h(-T^h)$. We set

$$d_{r_2}^h = \int_{-T^h}^{T^h} grad\, H(\gamma_2^h(t)) \cdot G(\gamma_2^h(t))\, dt$$

$$d_{\lambda_2}^h = \int_{-T^h}^{T^h} grad\, H(\gamma_2^h(t)) \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} dt.$$

PROPOSITION 4.1. *Let the constant ρ used in the definition of V_1 be sufficiently small. Then, for $r_2 \leqslant \rho\sqrt{\sigma(h)}$, $\lambda_2 \in (-\mu, \mu)$, and $h \in (0, h_0)$ the function $\mathscr{D}_s$ has the expansion*

$$\mathscr{D}_s(h, r_2, \lambda_2) = r_2 d_{r_2}^h + \lambda_2 d_{\lambda_2}^h + Q(h, r_2, \lambda_2), \tag{4.10}$$

with Q satisfying the following estimates for some constant $K > 0$:

$$|Q(h, r_2, \lambda_2)| \leqslant K(r_2 + |\lambda_2|)^2, \tag{4.11a}$$

$$\left|\frac{\partial}{\partial h} Q(h, r_2, \lambda_2)\right| \leqslant K\rho(r_2 + |\lambda_2|)\, \sigma(h)^{-3/2}. \tag{4.11b}$$

Moreover, the partial derivatives $|(\partial/\partial r_2)\, \mathscr{D}_s(h, r_2, \lambda_2)|$, $|(\partial/\partial \lambda_2)\, \mathscr{D}_s(h, r_2, \lambda_2)|$, $|(\partial^2/\partial r_2^2)\, \mathscr{D}_s(h, r_2, \lambda_2)|$, $|(\partial^2/\partial \lambda_2^2)\, \mathscr{D}_s(h, r_2, \lambda_2)|$ and $|(\partial^2/\partial r_2\, \partial \lambda_2)\, \mathscr{D}_s(h, r_2, \lambda_2)|$ are uniformly bounded.

As noted above we will apply the implicit function theorem on the open interval $(0, h_0)$. Due to the lack of compactness we need to make sure that domain of definition of the solution of (4.8) $\lambda_2(h, r_2)$ does not excessively shrink as $h \to 0$. This can be achieved by showing that

$$\frac{\partial}{\partial \lambda_2} \mathscr{D}_s(h, r_2, \lambda_2) \neq 0 \tag{4.12}$$

for $(h, r_2, \lambda_2) \in U_0$. As will be clear in the sequel $d^h_{\lambda_2} \neq 0$ for $h \in (0, h_0)$. Consequently, property (4.12) follows from boundedness of the second partial derivatives, as asserted in Proposition 4.1.

The approach traditionally taken in solving a problem of the type (4.8) is to define a new bifurcation function

$$\tilde{\mathscr{D}}_s(h, r_2, \lambda_2) = r_2 P(h) + \lambda_2 + \frac{Q(h, r_2, \lambda_2)}{d^h_{\lambda_2}}, \tag{4.13}$$

where $P(h) = d^h_{r_2}/d^h_{\lambda_2}$, and consider the equivalent problem

$$\tilde{\mathscr{D}}_s(h, r_2, \lambda_2) = 0. \tag{4.14}$$

We will show later (Theorem. 4.3(iii)) that

$$\frac{d}{dh} P(h) \geqslant \text{const.}\ \sigma(h)^{-3/2}.$$

Consequently, estimate (4.11b) is sufficient for proving condition (iii).

The proof of Proposition 4.1 will be based on the dynamics in chart K_1. Inadvertently some transition between K_1 and K_2 will be necessary and one may be tempted to carry out the entire proof in K_2. However, we feel that chart K_1 is in a very natural way related to the original coordinates—all that the blow-up transformation in K_1 does is to replace y by $r_1 = \sqrt{y}$ and to blow up x, i.e. the distance between the two branches of S to $O(1)$. In particular, in K_1 the condition $r_2 \leqslant \rho \sqrt{\sigma(h)}$ corresponds to $r_1(0) \leqslant \rho$, where $r_1(0)$ is the initial condition of the periodic orbit. This clearly means that the family of periodic orbits under consideration corresponds to the transition from small periodic orbits born in the Hopf bifurcation to canard cycles of size $O(\rho^2)$ in y-direction.

We now further investigate the dynamics in chart K_1. We define

$$\Sigma = \{p = (0, r_1, \varepsilon_1, \lambda_1) \in V_1\}. \tag{4.15}$$

Consider the subset $\Sigma_{\varepsilon_1} \subset \Sigma$ defined by fixing ε_1. If $p \in \Sigma_{\varepsilon_1}$ then $\tilde{H}(p) = \frac{1}{4}(1 + \frac{2}{\varepsilon_1})\, e^{-2/\varepsilon_1}$. Note that $\tilde{H}(p) = h$ for each $p \in \Sigma_{\sigma(h)}$. Let $p(h) = (0, 0, \sigma(h), 0)$. For $p \in \Sigma$ let γ_p be the solution of (4.5) with $\gamma_p(0) = p$. Let

$\gamma_1^h = \gamma_{p(h)}$. We will measure the separation between γ_p and γ_1^h and $\Sigma_{l,1}^{out}$. Let T_p be such that $\gamma_p(T_p) \in \Sigma_{l,1}^{out}$. Define

$$\Pi(p) = \tilde{H}(\gamma_p(T_p)) - \tilde{H}(\gamma_1^h(T_{p(h)}))$$

for all $p \in \Sigma_{\sigma(h)}$. Since $\tilde{H}(\gamma_1^h(T_{p(h)})) = h = \tilde{H}(p)$ it follows that

$$\Pi(p) = \tilde{H}(\gamma_p(T_p)) - \tilde{H}(p) = \int_0^{T_p} \frac{d}{dt} \tilde{H}(\gamma_p(t))\, dt.$$

From (4.7) we derive:

$$\begin{aligned} \frac{\partial \tilde{H}}{\partial x_1} &= -e^{-2/\varepsilon_1} \frac{x_1}{\varepsilon_1}, \\ \frac{\partial \tilde{H}}{\partial \varepsilon_1} &= -e^{-2/\varepsilon_1} \frac{1}{\varepsilon_1^2} \left(\frac{1}{2} x_1^2 - \frac{x_1^2}{\varepsilon_1} + \frac{1}{\varepsilon_1} \right). \end{aligned} \tag{4.16}$$

Then, using (4.16), we derive the following formula for $\frac{d\tilde{H}}{dt}$, evaluated along γ_p.

$$\begin{aligned} \frac{d\tilde{H}}{dt} = -e^{-2/\varepsilon_1} \varepsilon_1^{-3/2} \Big\{ & r_2 x_1 (a_1 \varepsilon_1 x_1 - a_2 x_1 + a_3 x_1^3) \\ & + (1 - x_1^2)(r_2(a_4 x_1^2 + a_5) + \lambda_2 \varepsilon_1) + \frac{1}{\sqrt{\varepsilon_1}} O(r_2(r_1 + |\lambda_2|)) \Big\}. \end{aligned} \tag{4.17}$$

For simplicity we now set $\lambda_2 = 0$ and compute $\Pi(p)$ as a function of r_2. The estimates of terms involving λ_2 are similar. Let $p(r_2) = (0, r_2/\sqrt{\sigma(h)}, \sigma(h), 0)$ and let

$$\begin{aligned} \tilde{\eta}(x_1, \varepsilon_1) = e^{-2/\varepsilon_1} \varepsilon_1^{-3/2} \{ & x_1 (a_1 \varepsilon_1 x_1 - a_2 x_1 + a_3 x_1^3) \\ & + (1 - x_1^2)(a_4 x_1^2 + a_5) \}. \end{aligned}$$

It follows that

$$\Pi(p(r_2)) = r_2 \int_0^{T_p} \tilde{\eta}(\gamma_{p(r_2)}(t))\, dt + O(r_2^2).$$

Let

$$\tilde{Q}(h, r_2) = \int_0^{T_p} \tilde{\eta}(\gamma_{p(r_2)}(t))\, dt - \int_0^{T_{p(h)}} \tilde{\eta}(\gamma_1^h(t))\, dt.$$

The main technical result leading to a proof of Proposition 4.1 is the following.

PROPOSITION 4.2. *The remainder term $\tilde{Q}$ and its derivative satisfy the following estimates,*

$$|\tilde{Q}(h, r_2)| \leqslant K r_2, \tag{4.18a}$$

$$\left|\frac{\partial}{\partial h}\tilde{Q}(h, r_2)\right| \leqslant K\rho(\sigma(h))^{-3/2}, \tag{4.18b}$$

for some constant K.

Let $(x^h_{1,r_2}, \varepsilon^h_{1,r_2})$, respectively (x^h_1, ε^h_1), be the x_1 and the ε_1 components of $\gamma_{p(r_2)}$ and γ^h_1. In the proof of Proposition 4.2 it would be most convenient to parametrize x^h_{1,r_2} by ε_1. In particular, it would then suffice to estimate the distance between x^h_{1,r_2} and x^h_1, respectively between $\frac{\partial}{\partial h}x^h_{1,r_2}$ and $\frac{\partial}{\partial h}x^h_1$. However, parametrizing x^h_{1,r_2} by ε_1 may not be possible, since the right hand side of (4.5) may vanish when $x^h_{1,r_2}(t)$ is small. This problem could be circumvented by letting the initial condition depend on (r_2, λ_2) so that $(\varepsilon^h_{1,r_2})'(0) = 0$. Instead we break-up the analysis in two parts. We first consider parametrization of $(x^h_{1,r_2}, \varepsilon^h_{1,r_2})$ by t, for $t \in [0, \hat{T}]$, $\hat{T} > 0$. More precisely, let $\beta \in (0, 1)$ and let T^* and $T^*_{r_2}$ be defined by $x^h_1(T^*) = x^h_{1,r_2}(T^*_{r_2}) = -1 + \beta$. Further let $\varepsilon^*_1 = \varepsilon^h_1(T^*)$ and $\varepsilon^*_{1,r_2} = \varepsilon^h_{1,r_2}(T^*_{r_2})$. In Lemma 4.1 we obtain a result on the distance between ε^*_{1,r_2} and ε^*_1. In Lemma 4.2 we give estimates on $\frac{\partial}{\partial h}(x^h_{1,r_2}(t), \varepsilon^h_{1,r_2}(t))$ and $|\frac{\partial}{\partial h}(x^h_{1,r_2}(t) - x^h_1, \varepsilon^h_{1,r_2}(t)) - \varepsilon^h_1)|$ for $t \in [0, T^*_{r_2}]$. For $\varepsilon_1 \geqslant \varepsilon^*_{1,r_2}$ we parametrize x^h_{1,r_2} by ε_1. In Lemma 4.3 we estimate the distance between x^h_{1,r_2} and x^h_1, and between $\frac{\partial}{\partial h}x^h_{1,r_2}$ and $\frac{\partial}{\partial h}x^h_1$.

LEMMA 4.1. *The following estimate holds:*

$$\left|\frac{1}{\varepsilon^*_{1,r_2}} - \frac{1}{\varepsilon^*_1}\right| = O\left(\frac{r_2}{\sqrt{\varepsilon^*_1}}\right). \tag{4.19}$$

Proof. Write (4.5c) in the form

$$\left(\frac{1}{\varepsilon_1}\right)' = F(x_1, r_1, \varepsilon_1, \lambda_1). \tag{4.20}$$

It follows that $\varepsilon^*_1/\sigma(h)$ and $\varepsilon^*_{1,r_2}/\sigma(h)$ are bounded uniformly in h. Note that

$$|x^h_{1,r_2}(t) - x^h_1(t)| = O(r_2/\sqrt{\sigma(h)}), \qquad t \in [0, T^*].$$

We conclude that

$$|x^h_{1,r_2}(t) - x^h_1(t)| = O(r_2/\sqrt{\varepsilon^h_{1,r_2}(t)}), \qquad t \in [0, T^*]. \tag{4.21}$$

The result now follows from (4.5c), (4.21) and the fact that $F(x_1, r_1, \varepsilon_1, 0) = x_1 + O(r_1)$. ∎

The following result will be necessary for estimating the derivative of $\mathscr{D}_s$ with respect to h.

LEMMA 4.2. *Fix $\hat{T} > 0$. Then, there exists a constant K such that, for any $t \in [0, \hat{T}]$,*

$$\left|\frac{\partial}{\partial h} x^h_{1,r_2}(t)\right| \leqslant K\sigma(h)^2 \, e^{2/\sigma(h)}, \tag{4.22a}$$

$$\left|\frac{\partial}{\partial h} \varepsilon^h_{1,r_2}(t)\right| \leqslant K\sigma(h)^3 \, e^{2/\sigma(h)}, \tag{4.22b}$$

$$\left|\frac{\partial}{\partial h} x^h_{1,r_2}(t) - \frac{\partial}{\partial h} x^h_1(t)\right| \leqslant K r_2 \sigma(h)^{3/2} \, e^{2/\sigma(h)}, \tag{4.22c}$$

$$\left|\frac{\partial}{\partial h} \varepsilon^h_{1,r_2}(t) - \frac{\partial}{\partial h} \varepsilon^h_1(t)\right| \leqslant K r_2 \sigma(h)^3 \, e^{2/\sigma(h)}. \tag{4.22d}$$

Proof. Follows from smooth dependence of solutions on initial conditions and from the identity

$$\frac{d}{dh} \sigma(h) = \sigma(h)^3 \, e^{2/\sigma(h)}. \quad ∎$$

For $t \geqslant T^*_{r_2}$ the right hand side of (4.5c) is positive. Hence, for $\varepsilon_1 \in [\varepsilon^*_{1,r_2}, \delta]$, we can parametrize x^h_{1,r_2} by ε_1. From (4.5) we conclude that $x^h_{1,r_2}(\varepsilon_1)$ satisfies the equation

$$\frac{dx_1}{d\varepsilon_1} = \frac{1}{\varepsilon_1^2} \left(-\frac{-1 + x_1^2}{x_1} + \frac{1}{2} \varepsilon_1 x_1 + O(r_1) \right). \tag{4.23}$$

Now write

$$x^h_{1,r_2}(\varepsilon_1) = x^h_1(\varepsilon_1) + r_1(\varepsilon_1) \, z(\varepsilon_1, h), \tag{4.24}$$

where $r_1(\varepsilon_1) = r_2/\sqrt{\varepsilon_1}$. We have the following result.

LEMMA 4.3. *There exists a constant K such that, for* $\varepsilon_1 \in [\varepsilon^*_{1,r_2}, \delta]$,

$$|z(\varepsilon_1)| \leqslant K, \tag{4.25a}$$

$$\left|\frac{\partial z}{\partial h} z(\varepsilon_1)\right| \leqslant K\varepsilon_1 e^{2/\varepsilon_1}. \tag{4.25b}$$

Proof. Using (4.23) we obtain the following equation for z

$$\frac{dz}{d\varepsilon_1} = \frac{1}{\varepsilon_1^2}((-c(\varepsilon_1) + O(r_1))\, z + \psi(x_1^h, \varepsilon_1) + O(r_1 z^2)),$$

where

$$c(\varepsilon_1) = 2 + \frac{-1 + x_1^h(\varepsilon_1)^2}{x_1^h(\varepsilon_1)^2} - \frac{1}{2}\varepsilon_1,$$

and ψ is a smooth function. To complete the proof of (4.25a) it remains to show that $|z(\varepsilon^*_{1,r_2})|$ is uniformly bounded (as a function of h). By the estimate (4.19) we have

$$|T^*_{r_2} - T^*| = O(r_2/\sqrt{\varepsilon_1^*}). \tag{4.26}$$

Hence

$$|x^h_{1,r_2}(\varepsilon_1^*) - x_1^h(\varepsilon_1^*)| = O(r_2/\sqrt{\varepsilon_1^*}).$$

The estimate (4.25a) follows. We now prove (4.25b). Let $\zeta(\varepsilon_1, h) = \frac{\partial}{\partial h} z(\varepsilon_1, h)$. Then ζ satisfies the equation

$$\frac{d\zeta}{d\varepsilon_1} = \frac{1}{\varepsilon_1^2}((-c(\varepsilon_1) + O(r_1))\, \zeta + O(\varepsilon_1 e^{2/\varepsilon_1})). \tag{4.27}$$

By definition $x^h_{1,r_2}(t) = x^h_{1,r_2}(\varepsilon^h_{1,r_2}(t))$. It follows that

$$\frac{\partial x^h_{1,r_2}}{\partial h}(\varepsilon^h_{1,r_2}(t)) = \frac{\partial}{\partial h} x^h_{1,r_2}(t) - \frac{\partial}{\partial \varepsilon_1} x^h_{1,r_2}(\varepsilon^h_{1,r_2}(t)) \cdot \frac{\partial}{\partial h}\varepsilon^h_{1,r_2}(t).$$

For ε_1 sufficiently close to ε^*_{1,r_2} Lemma 4.2 still applies. The term $(\partial/\partial\varepsilon_1)\, x^h_{1,r_2}(\varepsilon^h_{1,r_2}(t))$ can be estimated using (4.23). From estimates (4.22) and (4.26) we conclude that $|\zeta(\varepsilon^*_{1,r_2})|$ is uniformly bounded. The result follows. ∎

Proof of Proposition 4.2. Let $\eta(x_1, \varepsilon_1)$ be defined by

$$\eta(x_1, \varepsilon_1) = e^{-2/\varepsilon_1}\varepsilon_1^{-7/2}\left(a_1 x_1 \varepsilon_1 - a_2 x_1 + a_3 x_1^3 + \frac{1 - x_1^2}{x_1}(a_4 x_1^2 + a_5)\right). \tag{4.28}$$

Write $\tilde{Q}=\tilde{Q}_1+\tilde{Q}_2+\tilde{Q}_3$, with

$$\tilde{Q}_1=\int_0^{T^*_{r_2}}\tilde{\eta}(\gamma_{p(r_2)}(t))\,dt-\int_0^{T^*}\tilde{\eta}(\gamma_1^h(t))\,dt,$$

$$\tilde{Q}_2=\int_{\varepsilon_1^*}^{\varepsilon^*_{1,r_2}}\eta(x_1^h(\varepsilon_1),\varepsilon_1)\,d\varepsilon_1, \tag{4.29}$$

$$\tilde{Q}_3=\int_{\varepsilon^*_{1,r_2}}^{\delta}\eta(x^h_{1,r_2}(\varepsilon_1),\varepsilon_1)\,d\varepsilon_1-\int_{\varepsilon^*_{1,r_2}}^{\delta}\eta(x_1^h(\varepsilon_1),\varepsilon_1)\,d\varepsilon_1.$$

Estimates (4.22a), (4.22b), (4.19) and (4.25a) imply that

$$|\tilde{Q}|\leqslant|\tilde{Q}_1|+|\tilde{Q}_2|+|\tilde{Q}_3|=O(r_2).$$

Estimate (4.18a) follows. We now prove (4.18b). First note that the estimate

$$\left|\frac{\partial}{\partial h}\tilde{Q}_1\right|=O(1) \tag{4.30}$$

follows directly from (4.22c), (4.22d) and Lemma 4.1. Next we use the implicit function theorem and implicit differentiation, as well as (4.22b), to obtain

$$\left|\frac{\partial}{\partial h}\varepsilon^*_{1,r_2}\right|=O(\sigma(h)^3\,e^{2/\sigma(h)})\quad\text{and}\quad\left|\frac{d}{dh}\varepsilon_1^*\right|=O(\sigma(h)^3\,e^{2/\sigma(h)}). \tag{4.31}$$

Now, using the fact that $\frac{\partial}{\partial h}x_1^h(\varepsilon_1)=\varepsilon_1 e^{2/\varepsilon_1}$, we obtain

$$\left|\frac{\partial}{\partial h}\tilde{Q}_2\right|\leqslant K\left|\int_{\varepsilon_1^*}^{\varepsilon^*_{1,r_2}}\varepsilon_1^{-5/2}\,d\varepsilon_1\right|\leqslant\rho K\sigma(h)^{-3/2}, \tag{4.32}$$

where K is some constant. To obtain an estimate on $|\frac{\partial}{\partial h}\tilde{Q}_3|$ we write $x^h_{1,r_2}(\varepsilon_1)$ in the form (4.24). Note that $\partial\eta/\partial x_1$ is Lipschitz and that $r_1(t)\leqslant\rho$. Using (4.25) and (4.31) we obtain

$$\left|\frac{\partial}{\partial h}\tilde{Q}_3\right|\leqslant K\rho\int_{\varepsilon^*_{1,r_2}}^{\delta}\varepsilon_1^{-5/2}\,d\varepsilon_1\leqslant\rho K(h)^{-3/2}, \tag{4.33}$$

where K is some constant. The result follows. ∎

Remark 4.2. Due to the presence of the term $\varepsilon_1^{-7/2}$ in the function η the quantity $|\frac{\partial}{\partial h}\tilde{Q}|$ is genuinely of the order $O(\sigma(h)^{-3/2})$. Due to the presence of a similar term in $\frac{d}{dh}P(h)$ (see the proof of Theorem 4.3) the two expressions are of the same order in $\sigma(h)^{-1}$. However, the coefficient of $\sigma(h)^{-3/2}$ in (4.18b) has the factor ρ, due to the fact that $\tilde{Q}$ is of higher order in r_1, see (4.24).

Proof of Proposition 4.1. We first prove (4.11a). Let $\gamma^h_{r_2} = \gamma^h_{r_2, 0}$ and $y^h_{r_2} = y^h_{r_2, 0}$. Let $(x^h_{r_2}, \delta^{-1})$ be the point of intersection of $\gamma^h_{r_2}$ with $\{y_2 = \delta^{-1}\}$. Recall the function Π defined prior to the statement of Proposition 4.2. We have

$$H(0, y^h_{r_2}) - h = H(0, y^h_{r_2}) - H(x^h_{r_2}, \delta^{-1}) + \Pi(p(r_2)).$$

By standard methods [6]

$$H(0, y^h_{r_2}) - H(x^h_{r_2}, \delta^{-1}) = r_2 \int_{T^{h,\delta}}^{T^h} grad\, H(\gamma^h_2(t)) \cdot G(\gamma^h_2(t))\, dt + O(r_2^2),$$

where $T^{h,\delta}$ is defined by $\gamma^h_2(T^{h,\delta}) \in \{y_2 = \delta^{-1}\}$. Proposition 4.2 now implies that

$$H(0, y^h_{r_2}) - h = r_2 \int_0^{T^h} grad\, H(\gamma^h_2(t)) \cdot G(\gamma^h_2(t))\, dt + O(r_2^2).$$

In a similar fashion one derives

$$H(0, \hat{y}^h_{r_2}) - h = -r_2 \int_{-T^h}^{0} grad\, H(\gamma^h_2(t)) \cdot G(\gamma^h_2(t))\, dt + O(r_2^2).$$

Hence

$$\mathscr{D}_s(h, r_2, 0) = r_2 \int_{-T^h}^{T^h} grad\, H(\gamma^h_2(t)) \cdot G(\gamma^h_2(t))\, dt + O(r_2^2).$$

By similar methods an analogous formula holds for the separation with respect to λ_2. Estimate (4.11a) follows.

We now prove (4.11b). We analyze the case of $\gamma^h_{r_2, \lambda_2}$. The analysis for $\hat{\gamma}^h_{r_2, \lambda_2}$ is similar. The estimate

$$\frac{\partial}{\partial h}\left(\Pi(p(r_2)) - r_2 \int_0^{T_p} \tilde{h}(\gamma_{p(r_2)}(t))\, dt\right) = r_2 \rho O(\sigma(h)^{-3/2})$$

can be obtained similarly as the estimate of $\frac{\partial}{\partial h}\tilde{Q}$, through an estimate analogous to (4.30) and an estimate analogous to (4.33). The estimate

$$\left|\frac{\partial}{\partial h}(H(0, y^h_{r_2}) - H(x^h_{r_2}, \delta^{-1}))\right| = O(1)$$

can obtain by standard methods. It follows from Proposition 4.2 that

$$\left|\frac{\partial}{\partial h} Q(h, r_2, 0)\right| = r_2 \rho O(\sigma(h)^{-3/2}).$$

Estimates of terms involving λ_2 can be obtained analogously. Estimate (4.11b) follows.

Finally, the proof of the boundedness of first and second order partial derivatives is a simplified version of the proof of (4.11). Its main part consists of estimating integrals of total derivatives of the right hand side of (4.17). The relevant integrands all have a factor e^{-2/ε_1}, involve negative powers of $\varepsilon_1^{1/2}$ and appropriate λ_2 and r_2 derivatives of $x^h_{1,p}(t)$ and $\varepsilon^h_{1,p}(t)$. Due to the presence of the factor e^{-2/ε_1} it suffices to obtain estimates of derivatives of $x^h_{1,p}(t)$ in terms of negative powers $\varepsilon^h_{1,p}(t)$. Such estimates are obtained in a similar way as (4.25b). ∎

We define

$$I_1(h) = \int_{-T^h}^{T^h} (x_2^h(t)^2 \, e^{-2y_2^h(t)}) \, dt,$$

$$I_2(h) = \int_{-T^h}^{T^h} (x_2^h(t)^4 \, e^{-2y_2^h(t)}) \, dt,$$

$$P(h) = \frac{I_2(h)}{I_1(h)}.$$

These definitions are valid for $h \in [0, \frac{1}{4})$ with $T^0 = \infty$.

LEMMA 4.4. *The coefficients $d^h_{r_2}$ and $d^h_{\lambda_2}$ from Proposition* 4.1 *are given by*

$$d^h_{r_2} = -(a_1 + a_5)\, I_1(h) - \frac{A}{3}\, I_2(h), \qquad d^h_{\lambda_2} = -2 I_1(h),$$

with A defined by Eq. (3.13).

Proof. The proof is a computation based on repeated integration by parts. ∎

It follows from Proposition 4.1 that, for $r_2 \leqslant \rho \sqrt{\sigma(h)}$, bifurcation Eq. (4.8) is equivalent to

$$\lambda_2 = r_2 \left(-\frac{a_1 + a_5}{2} - \frac{1}{6} AP(h) \right) + O((|\lambda_2| + r_2)^2). \tag{4.34}$$

THEOREM 4.3. *The function* $P(h)$ *has the following properties*:

(i) $P(h)>0$, *for* $h\in[0, 1/4)$ *and* $\lim_{h\to 1/4-} P(h)=0$,

(ii) $P'(h)<0$, *for* $h\in(0, 1/4)$,

(iii) $\lim_{h\to 0}\sigma(h)^{3/2}P'(h)<0$.

Proof. Parts (i) and (ii) are proved in [8, Appendix]. Here we prove part (iii). In the course of the proof we simplify the notation, writing x_2 for x_2^h and y_2 for y_2^h. Let $\xi(h)$ be the positive solution of $H(0, y_2)=h$ and note that

$$I_1(h)=-2\int_{\xi(h)}^{\sigma(h)^{-1}} x_2 e^{-2y_2}\,dy_2,$$

$$I_2(h)=-2\int_{\xi(h)}^{\sigma(h)^{-1}} x_2^3 e^{2y_2}\,dy_2,$$

where $x_2=x_2(h, y_2)$ is the solution of

$$\tfrac{1}{2}e^{-2y_2}(y_2-x_2^2+\tfrac{1}{2})=h.$$

By implicit differentiation $\partial x_2/\partial h=-e^{2y_2}(1/x_2)$. It follows that

$$\frac{dI_1(h)}{dh}=-2\int_{\xi(h)}^{\sigma(h)^{-1}}\frac{1}{x_2}\,dy_2,$$

$$\frac{dI_2(y)}{dh}=-6\int_{\xi(h)}^{\sigma(h)^{-1}} x_2\,dy_2.$$

We first consider, for h sufficiently small, the integrals

$$J_1=-\int_{\delta^{-1}}^{\sigma(h)^{-1}}\frac{1}{x_2}\,dy_2,$$

$$J_2=-\int_{\delta^{-1}}^{\sigma(h)^{-1}} x_2\,dy_2.$$

Transforming to K_1 we get

$$J_1=\int_{\sigma(h)}^{\delta}\frac{1}{\varepsilon_1^{3/2}x_1}\,d\varepsilon_1,$$

$$J_2=\int_{\sigma(h)}^{\delta}\frac{x_1}{\varepsilon_1^{5/2}}\,d\varepsilon_1.$$

It follows from the fact that $\varepsilon_1^*/\sigma(h)$ is uniformly bounded in h (see proof of Lemma 4.1) that

$$|J_2| \geqslant K\sigma(h)^{-3/2},$$

for some $K>0$. We now show that

$$|J_1| = O(\sqrt{\sigma(h)^{-1}}). \tag{4.35}$$

It is clear that

$$\left| \int_{\varepsilon_1^*}^{\delta} \frac{1}{\varepsilon_1^{3/2} x_1} \, d\varepsilon_1 \right| = O(\sqrt{\sigma(h)^{-1}}).$$

Further, using (4.6), we get

$$\left| \int_{\sigma(h)}^{\varepsilon_1^*} \frac{1}{\varepsilon_1^{3/2} x_1} \, d\varepsilon_1 \right| = \left| \int_0^{T^*} \sqrt{\varepsilon_1(t)} \, dt \right| = O(\sqrt{\sigma(h)}). \tag{4.36}$$

The estimate (4.35) follows. Finally we have

$$\left| \int_{\xi(h)}^{\delta^{-1}} \frac{1}{x_2} \, dy_2 \right| = \left| \int_{T^{h,\,\delta}}^{T^h} dt \right| = O(1),$$

$$\left| \int_{\xi(h)}^{\delta^{-1}} x_2 \, dy_2 \right| = \left| \int_{T^{h,\,\delta}}^{T^h} x_2(t)^2 \, dt \right| = O(1).$$

Hence

$$|P'(h)| = \left| \frac{I_2'(h) \, I_1(h) - I_1'(h) \, I_2(h)}{I_1(h)^2} \right| \geqslant K\sigma(h)^{-3/2}$$

for some $K>0$. ∎

Proof of Theorem 4.1. (i) If V, ε_0 and λ_0 are sufficiently small then V contains no equilibria other than p_e. Using a chart K_5 corresponding to $\bar{\lambda} = -1$ in the blow-up transformation (3.4) one shows that for $\lambda \in (-\lambda_0, \lambda_H(\sqrt{\varepsilon}))$ the neighborhood V contains no periodic orbits. Hence the assertion follows from the Poincaré–Bendixson theorem.

(ii) Let $h_0 \in (0, 1/4)$. A standard argument based on Theorems 3.1 and 4.3 [6] shows that there exists a curve $\lambda_2(r_2, h_0)$ such that, for any $r_2 \in (0, \rho)$ and $\lambda_H(r_2) < \lambda < \lambda_2(r_2, h_0)$ there exists a unique attracting periodic orbit for the flow of (4.2) restricted to a large disk centered at the origin. Using the information on the relative position of $S_{l,\,\varepsilon}$ and $S_{m,\,\varepsilon}$ [16] we conclude that there is a unique periodic orbit in V and the phase portrait is as shown in Fig. 9, region II.

Further on we consider $h \in (0, h_0)$. Recall the neighborhood U_0 defined by (4.9). Since the implicit function theorem at $h = 0$ cannot be applied, a more delicate argument is necessary. Note that $(h, 0, 0)$ is a solution of (4.34) for all $h \in (0, h_0)$. By Proposition 4.1 the second partial derivatives of $\mathcal{D}_s$ with respect to λ_2 and r_2 are uniformly bounded and $\partial \mathcal{D}_s / \partial \lambda_2 (h, 0, 0) = d^h_{\lambda_2} \neq 0$. It follows that

$$\frac{\partial \mathcal{D}_s}{\partial \lambda_2}(h, r_2, \lambda_2) \neq 0 \qquad \text{for} \quad (h, r_2, \lambda_2) \in U_0. \tag{4.37}$$

Hence a solution of (4.34) in the form $\lambda_2(h, r_2)$ can be obtained by repeated application of the implicit function theorem. To prove uniqueness of $\lambda_2(h, r_2)$ consider a solution $(h^*, r_2^*, \lambda_2^*)$ of (4.34) and apply the implicit function theorem at $(h^*, r_2^*, \lambda_2^*)$, getting a solution $\tilde{\lambda}_2(h, r_2)$. By (4.11a) and (4.37) the curve $\tilde{\lambda}_2$ can be extended up to $r_2 = 0$ and $\tilde{\lambda}_2(h, 0) = 0 = \lambda_2(h, 0)$, $h \in (0, h_0)$. It follows that the curves $\lambda_2(h, r_2)$ and $\tilde{\lambda}_2(h, r_2)$ coincide, implying $\lambda_2^* = \lambda_2(h^*, r_2^*)$. Finally, by (4.11b) and Theorem 4.3 (iii),

$$\frac{\partial \lambda_2}{\partial h}(h, r_2) < 0 \qquad \text{for} \quad (h, r_2) \in (0, h_0) \times (0, \sqrt{\sigma(h)}). \tag{4.38}$$

We now define the curve $\lambda_{sc}(r_2)$. For a trajectory of (4.5) starting at $(0, r_2 \sigma^{-1/2}, \sigma, \lambda_2 \sigma^{1/2})$ let $\varepsilon_{1,m}$ be the minimal value of ε_1 along this trajectory. Using similar arguments as in the proof of Lemma 4.1 we obtain

$$\frac{\partial}{\partial \sigma} \varepsilon_{1,m}(r_2, \sigma, \lambda_2) = 1 + O(\rho, \mu).$$

It follows that

$$\frac{d}{dh} \varepsilon_{1,m}(r_2, \sigma(h), \lambda_2(h, r_2)) = \frac{d}{dh} \sigma(h)(1 + O(\rho)).$$

Now we slightly decrease ρ and define $h_{sc}(r_2)$ by the requirement:

$$\varepsilon_{1,m}(r_2, \sigma(h_{sc}), \lambda_2(h_{sc}(r_2), r_2)) = \rho.$$

The curve λ_{sc} is given by

$$\lambda_{sc}(r_2) = r_2 \lambda_2(h_{sc}(r_2), r_2).$$

It now follows that for fixed $\varepsilon = r_2^2$ and $\lambda \in (\lambda_H(r_2), \lambda_{sc}(r_2))$ there is a unique, stable periodic orbit in V. The periodic orbits existing for

$\lambda_2(h, r_2) > \lambda_{sc}(r_2)$ must, by construction, leave the neighborhood V. Note that

$$r_2 \sigma(h_{sc}(r_2))^{-1/2} = \rho(1 + O(\rho)). \tag{4.39}$$

It follows that the orbit $\Gamma_{1, sc}$ (in chart K_1), corresponding to $\lambda_2(h_{sc}(r_2), r_2)$, passes $O(e^{-K/\varepsilon})$ close to both $M_{l,1}$ and $M_{m,1}$, where $K > 0$ is some constant. Since $\Gamma_{2, sc}$ is a closed orbit it follows that the separation between $M_{l,2}$ and $M_{m,2}$ must be of the order $O(e^{-K/\varepsilon})$. It follows that

$$\mathscr{D}_c(r_2, \lambda_2(h_{sc}, (r_2))) = O(e^{-K/\varepsilon}),$$

where $\mathscr{D}_c(r_2, \lambda_2)$ is the function describing the separation between $M_{l,2}$ and $M_{m,2}$ (see [16, Proof of Theorem 3.1]). Hence, since $(\partial/\partial\lambda_2)\, D_c(0, 0) \neq 0$,

$$|\lambda_2(h_{sc}(r_2) - \lambda_{c,2}(r_2))| = O(e^{-K/\varepsilon}).$$

This concludes the proof of (ii), except for the issue of stability, which will be discussed in the proof Proposition 4.3.

(iii) It follows from the proof of (ii), in particular from the construction of λ_{sc}, that for $Lr_2 \geqslant \lambda > \lambda_{sc}(r_2)$, where $L > 0$ is a large constant, that all orbits entering V, except for p_e, must leave V. The case of $\lambda > Lr_2$ is proved in chart K_4 defined by $\bar{\lambda} = 1$. ∎

Proof of Theorem 4.2. Similar to the proof of Theorem 4.1. ∎

We end this section with a result which is a straightforward consequence of the above analysis, but will be important for the description of canard explosion.

PROPOSITION 4.3. *Assume that $A \neq 0$ and that ρ and ε_0 are sufficiently small. Then, for $\varepsilon \in (0, \varepsilon_0)$, there exists a continuous family of periodic orbits*

$$s \to (\lambda(s, \sqrt{\varepsilon}), \Gamma(s, \sqrt{\varepsilon})), \qquad s \in (0, \rho^2), \qquad \Gamma(s, \sqrt{\varepsilon}) \subset V,$$

where $\lambda(s, \sqrt{\varepsilon})$ is C^k in $(s, \sqrt{\varepsilon})$ and $\Gamma(s, \sqrt{\varepsilon})$ passes through the point $(0, s)$. If $A < 0$ then $\frac{\partial}{\partial s}\lambda(s, \sqrt{\varepsilon}) > 0$ and the periodic orbit is stable. If $A > 0$ then $\frac{\partial}{\partial s}\lambda(s, \sqrt{\varepsilon}) < 0$ and the periodic orbit is unstable. Any periodic orbit in V is a member of the family $\Gamma(s, \sqrt{\varepsilon})$.

Proof. It follows from the proof of Theorem 4.1 that there exists a family of periodic orbits

$$h \to (\lambda(h, r_2), \Gamma(h, r_2)), \qquad r_2 \leqslant \rho \sqrt{\sigma(h)}.$$

We change the parametrization, letting $s=\varepsilon/\sigma(h)$. It follows from the definition of σ and from a straightforward computation that $\frac{ds}{dh}(h)<0$ for all $h\in(0,\frac{1}{4})$. The statements on the sign of $\frac{\partial}{\partial s}\lambda(s,\sqrt{\varepsilon})$ and on uniqueness of periodic orbits in V now follow from the proof of Theorem 4.1. To prove stability we consider the section

$$\Delta_s=\{(0,s,\varepsilon,\lambda): s\in(0,s_0), \varepsilon\in(0,\varepsilon_0), \lambda\in(-\lambda_0,\lambda_0)\},$$

and study the return map of the flow of (3.1) from Δ_s to itself. Let π_s: $\Delta_s\to\Delta_s$ be this return map. Suppose $A<0$ (the other case is similar). Differentiating the identity

$$\pi_s(s,\lambda(s,r_2),r_2)=s$$

with respect to s we get

$$1-\frac{\partial\pi_c}{\partial s}=\frac{\partial\pi_c}{\partial\lambda}(s,\lambda(s,r_2),r_2)\frac{\partial\lambda}{\partial s}(s,r_2). \tag{4.40}$$

Since the right hand side is positive and $\partial\pi_c/\partial s>0$ by the orientation preserving property of the flow, it follows that $\partial\pi_c/\partial s<1$, i.e. the periodic orbit is stable. ∎

5. GLOBAL ASPECTS OF THE FLOW

5.1. *Construction of Canard Cycles*

In this section we prove Theorems 3.3 and 3.5. W begin by constructing the family of limit cycles $\Gamma(s,\sqrt{\varepsilon})$. We will distinguish five types of limit cycles, depending on the construction used in obtaining them. These are:

1. Hopf cycles and small canard cycles, corresponding to perturbations of $\Gamma(s)$, $s\in[0,\rho^2)$,

2. canard cycles (without head) corresponding to perturbations of $\Gamma(s)$ for $s\in(s_0, y_M-s_0)$, s_0 small,

3. canard cycles passing close to the fold point (x_M, y_M) corresponding to perturbations of $\Gamma(s)$ for $s\in(y_M-\tilde{s}_0, y_M+\tilde{s}_0)$, $\tilde{s}_0>s_0$ small,

4. canard cycles (with head) corresponding to perturbations of $\Gamma(s)$ for $s\in(y_M+s_0, 2y_M-s_0)$,

5. canard cycles and relaxation oscillations corresponding to perturbations of $\Gamma(s)$ for $s\geqslant 2y_M-\tilde{s}_0$, $\tilde{s}_0>s_0$.

Our strategy can be described as follows. We blow-up the canard point and the fold point by the appropriate blow-up transformations (see [16] for the fold point). The dynamics in neighborhoods of the canard point and the fold point is now described by the corresponding blown-up vector fields. Thus, we obtain a blown-up family $\bar{\Gamma}$ of singular cycles. The situation corresponding to $\lambda_2 = 0$ and $\bar{r} = 0$ (in both blow-ups) is shown schematically in Fig. 13, where typical singular cycles labeled as 1–15 are shown. Roughly speaking the five types of cycles described above will be obtained as perturbations of singular cycles as follows: Type 1 from orbits 1–4, type 2 from orbits 5, type 3 from orbits 6–12, type 4 from orbits 13, and type 5 from orbits 14 and 15. Actually, not all cycles of type 5 can be obtained by perturbing from $\lambda_2 = 0$, the connection to relaxation cycles must be done in chart K_4.

Note the "overlap" between the various types of orbits, i.e. orbit 4 can be considered as type 1 but also type 2. Similarly, orbits 6, 12, and 14 can be considered as belonging to two adjacent types.

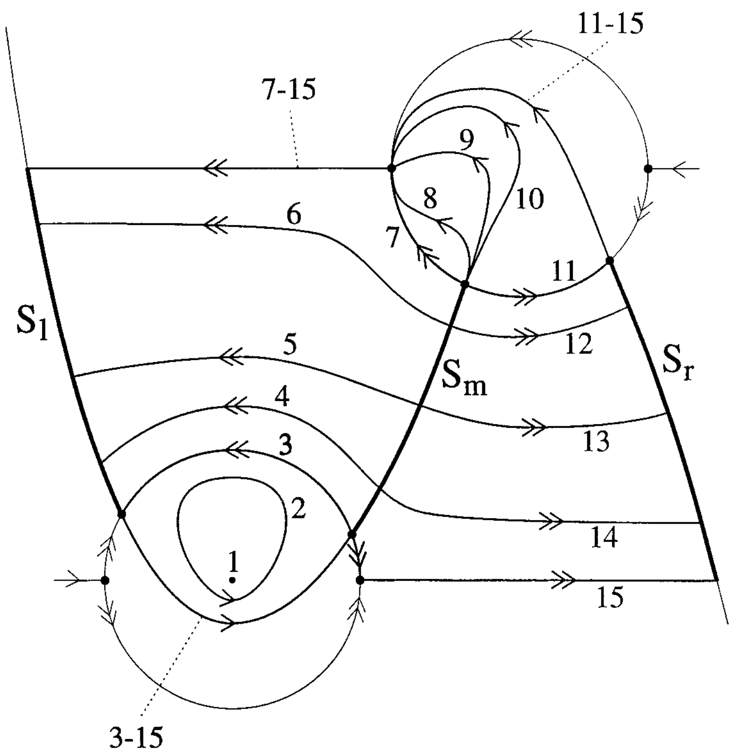

FIG. 13. Blown-up singular cycles for $\lambda_2 = 0$.

Cycles of type 1 are those whose existence is guaranteed by Proposition 4.3. Consequently we begin by constructing cycles of type 2. Let

$$\Delta = \{(0, s, \varepsilon, \lambda) : s \in (s_0, y_M - s_0), \varepsilon \in (0, \varepsilon_0), \lambda \in (-\lambda_0, \lambda_0)\}.$$

We consider forward and backward orbits of points in Δ under the flow of (3.1). Let $p \in \Delta$ and let $\gamma_p(t)$ be the forward trajectory of p. For any p we have $\gamma_p(t) \in V$ for t in some interval $I_p \subset [0, \infty)$. for the relevant choices of p the trajectory γ_p gives rise to a trajectory $\bar{\gamma}_p$ of $\bar{X}$ and to associated trajectories $\gamma_{p,j}$, $j = 1, 2$, in charts K_1 and K_2. Let $(0, y_{p,2})$ be the point where $\gamma_{p,2}$ crosses the x_2-axis. Let $(0, \hat{y}_{p,2})$ be analogously defined for the backward trajectory $\hat{\gamma}_p$. Let $\gamma_{l,2}(t) = (x_{l,2}(t), y_{l,2}(t))$ and $\gamma_{m,2} = (x_{m,2}(t), y_{m,2}(t))$ be solutions contained in $M_{l,2}$ and $M_{m,2}$, respectively, with $x_{l,2}(0) = x_{m,2}(0) = 0$. These solutions are analogous to $\gamma_{a,2}$ and $\gamma_{r,2}$ defined in the proof of [16, Proposition 3.4]. We have the following result.

LEMMA 5.1. *There exists $K > 0$ such that, for any $p \in \Delta$,*

$$\begin{aligned} |y_{p,2} - y_{l,2}| &= O(e^{-K/\varepsilon}), \\ |\hat{y}_{p,2} - y_{m,2}| &= O(e^{-K/\varepsilon}), \end{aligned} \tag{5.41}$$

where $\varepsilon = r_2^2$. Analogous estimates hold for the partial derivatives of $y_{p,2} - y_{l,2}$ and $\hat{y}_{p,2} - y_{m,2}$ with respect to r_2 and λ_2.

Proof. The trajectories $\gamma_{p,1}$ and $\hat{\gamma}_{p,1}$ pass through $\Sigma^{out}_{l,1}$ and $\Sigma^{in}_{m,1}$, respectively. Consequently the estimate follows from Fenichel theory and [16, Proposition 3.1]. ∎

The following result asserts the existence of periodic orbits obtained when $y_{p,2} = \hat{y}_{p,2}$.

PROPOSITION 5.1. *Consider $s \in (s_0, y_M - s_0)$, and $\varepsilon \in (0, \varepsilon_0]$. There exists a C^k smooth function $\lambda(s, \sqrt{\varepsilon})$ such that the orbit of (4.2) passing through $(0, s)$ is periodic if and only if $\lambda = \lambda(s, \sqrt{\varepsilon})$.*

Proof. Let $r_2 = \sqrt{\varepsilon}$ and $\lambda = r_2 \lambda_2$. Let

$$\mathscr{D}(s, r_2, \lambda_2) = y_{p,2} - \hat{y}_{p,2}.$$

Let $\mathscr{D}_c(r_2, \lambda_2)$ be a function measuring the separation between $\gamma_{l,2}$ and $\gamma_{m,2}$ defined analogously as in [16, Section 3.8]. Let d_{r_2} and d_{λ_2} be the coefficients of first order separation between $M_{l,2}$ and $M_{m,2}$, respectively. These coefficients are defined in [16] in formula (3.29). Lemma 5.1 implies that the expression

$$\mathscr{D}(s, r_2, \lambda_2) - \mathscr{D}_c(r_2, \lambda_2)$$

and its partial derivatives with respect to r_2 and λ_2 are $O(e^{-K/\varepsilon})$ small. Hence, the equation

$$\mathscr{D}(s, r_2, \lambda_2) = 0$$

can be solved by the implicit function theorem for λ_2 as a function of s and r_2, with the solution satisfying the following estimate:

$$\lambda_2 = -\frac{d_{r_2}}{d_{\lambda_2}} r_2 + o(r_2).$$

The result follows. ∎

Proposition 5.1 guarantees the existence of the family

$$(\lambda(s, \sqrt{\varepsilon}), \Gamma(s, \sqrt{\varepsilon})), \qquad s \in (s_0, y_M - s_0),$$

with s_0 fixed, but arbitrarily small. Choosing $s_0 < \rho^2$ we see that the families obtained from Propositions 4.3 and 5.1, i.e. cycles of type 1 and type 2 have overlapping domains of definition. Note that each of the families has a uniqueness property, which can be described as follows: for every $s \in (s_0, y_M - s_0)$, respectively $s \in (0, \rho^2)$, there exists a unique λ and a corresponding periodic orbit, unique with respect to the property that it passes through the point $(0, s)$. It follows that the two families can be combined to one family

$$(\lambda(s, \sqrt{\varepsilon}), \Gamma(s, \sqrt{\varepsilon})), \qquad s \in (0, y_M - s_0).$$

The construction of the family $(\lambda(s, \sqrt{\varepsilon}), \Gamma(s, \sqrt{\varepsilon}))$, $s \in (y_M + s_0, 2y_M - s_0)$, i.e. cycles of type 4 is completely analogous. We outline it below, leaving the details to the reader. Let

$$\tilde{\Delta} = \{(x_M, 2y_M - s, \varepsilon, \lambda) : s \in (y_M + s_0, 2y_M - s_0),$$
$$\varepsilon \in (0, \varepsilon_0), \lambda \in (-\lambda_0, \lambda_0)\}.$$

For $p \in \tilde{\Delta}$ consider the backward and the forward trajectories of p, denoted by γ_p and $\hat{\gamma}_p$, with $\gamma_{p,j}$ and $\hat{\gamma}_{p,j}$, $j = 1, 2$, corresponding to the associated trajectories in K_1 and K_2. The forward solution $\hat{\gamma}$ is exponentially attracted by $S_{r,\varepsilon}$, then passes the fold point, and is finally exponentially attracted by $S_{l,\varepsilon}$.

Choose ρ sufficiently small so that for $p \in \tilde{\Delta}$ the unstable fiber through p does not cross V, i.e. the backward solution γ_p is exponentially close to $S_{m,\varepsilon}$ when it enters the neighborhood V.

Thus, both solutions can be ultimately described in chart K_2. Let $(0, y_{p,2})$ and $(0, \hat{y}_{p,2})$ be the points where $\gamma_{p,2}$ and $\hat{\gamma}_{p,2}$ cross the x_2-axis. Define

$$\mathscr{D}(s, r_2, \lambda_2) = y_{p,2} - \hat{y}_{p,2}.$$

For $p \in \tilde{\Delta}$ the function $\mathscr{D}$ satisfies the C^1 estimate

$$\mathscr{D}(s, r_2, \lambda_2) = \mathscr{D}_c(r_2, \lambda_2) + O(e^{-K/\varepsilon}).$$

The existence of the family

$$(\lambda(s, \sqrt{\varepsilon}), \Gamma(s, \sqrt{\varepsilon})), s \in (y_M + s_0, 2y_M - s_0)$$

follows by the same arguments as in the proof of Proposition 5.1.

In the next two sections we construct cycles of types 3 and 5. The difficulty in applying the method described above to cycles of type 5 is that the backward trajectory of $q = 2y_M - s$ will not be uniformly close to $S_{m,\varepsilon}$, or its continuation in K_1, as $s \to 0$. Consequently we need to find periodic orbits as fixed points of a return map rather than by matching the forward and the backward trajectory. In the case of cycles of type 3 the construction is done by matching the forward and the backward trajectory, only the initial condition has to be chosen in the phase space of the blown up vector field at the fold point (x_M, y_M). We discuss both cases below, cycles of type 5 in more detail. We sketch the proof for cycles of type 3, leaving the details to the reader.

Canard cycles and relaxation oscillations corresponding to perturbations of $\Gamma(s)$ for $s \geqslant 2y_M - s_0$. To study cycles close to a relaxation oscillation we need to consider orbits that leave the vicinity of the canard point close to the critical fiber. Thus we need to study chart K_3 of the blow-up at the canard point. In chart K_3, corresponding to $\bar{x} = 1$, the blow-up transformation is given by:

$$x = r_3, \qquad y = r_3^2 y_3, \qquad \varepsilon = r_E^2 \varepsilon_3, \qquad \lambda = r_3 \lambda_3. \tag{5.42}$$

In order to understand cycles existing for larger values of λ we need to consider chart K_4 corresponding to $\bar{\lambda} = 1$. The corresponding blow-up transformation is:

$$x = r_4 x_4, \qquad y = r_4^2 y_4, \qquad \varepsilon = r_4^2 \varepsilon_4, \qquad \lambda = r_4. \tag{5.43}$$

Note that chart K_4 is a rescaling, i.e. r_4 and λ_4 are time independent parameters.

Our strategy for proving the existence of a family of cycles of type 5 is as follows. For s sufficiently large $R(s) < 0$ holds. Hence, the existence of cycles close to relaxation oscillations can be proved by the contraction mapping theorem as in the proof of Theorem 2.1. To prove Theorem 3.3 we need to show that the corresponding initial conditions in $\tilde{\Delta}$ vary monotonically in λ. To this end we study the passage of orbits from K_1 to K_3. Depending on the relative sizes of λ and ε these orbits can take three different routes:

1. from K_1 through K_2 back to K_1 and on to K_3,
2. from K_1 via K_2 to K_3,
3. from K_1 via K_4 to K_3.

These three cases are treated below.

The chart transformation κ_{23} is given by

$$r_3 = x_2 r_2, \qquad y_3 = x_2^{-2} y_2, \qquad \varepsilon_3 = x_2^{-2}, \qquad \lambda_3 = x_2^{-1} \lambda_2 \tag{5.44}$$

and the chart transformation κ_{13} is given by

$$r_3 = x_1 r_1, \qquad y_3 = x_1^{-1}, \qquad \varepsilon_3 = x_1^{-2} \varepsilon_1, \qquad \lambda_3 = x_1^{-1} \lambda_1. \tag{5.45}$$

The equations in K_3 have the form:

$$\begin{aligned} r_3' &= r_3 F(r_3, y_3, \varepsilon_3, \lambda_3), \\ y_3' &= -2 y_3 F(r_3, y_3, \varepsilon_3, \lambda_3) + \varepsilon_3 (1 - \lambda_3 + O(r_3)), \\ \varepsilon_3' &= -2 \varepsilon_3 F(r_3, y_3, \varepsilon_3, \lambda_3), \\ \lambda_3' &= -\lambda_3 F(r_3, y_3, \varepsilon_3, \lambda_3). \end{aligned} \tag{5.46}$$

The exit point we are interested in is located at the origin. Recall that $r_2 = \varepsilon^{1/2}$. We begin by defining some sections of the flow of the blown up vector field. In K_2 we consider the section Σ_{x_2} in the form

$$\Sigma_{x_2} = \{(x_2, y_2, r_2, \lambda_2) \colon x_2 = \delta^{-1/2}\}.$$

In K_1 we define

$$\Sigma_{x_1} = \left\{(x_1, r_1, \varepsilon_1, \lambda_1) \colon x_1 = \beta, \varepsilon_1 < \frac{\delta}{2}, 0 < r_1 < \rho_1\right\},$$

for $\beta > 1$ and some ρ_1 sufficiently small. In K_3 we define

$$\Sigma^{out} = \{(r_3, y_3, \varepsilon_3, \lambda_3) \colon r_3 = \rho\}.$$

It follows that points in $\kappa_{23}(\Sigma_{x_2})$ are of the form $(\delta^{-1/2} r_2, \delta y_2, \delta, \delta^{1/2} \lambda_2)$ and points in $\kappa_{13}(\Sigma_{x_1})$ are of the form $(\gamma^{-1/2} r_1, \gamma, \gamma \varepsilon_1, \gamma^{1/2} \lambda_1)$.

Let $\Pi_{23}\colon \Sigma_{x_2} \to \Sigma^{out}$ be the composition of κ_{23} with the transition map from $\kappa_{23}(\Sigma_{x_2})$ to Σ^{out} for the flow of (5.46). Similarly, let $\Pi_{13}\colon \Sigma_{x_1} \to \Sigma^{out}$ be the composition of κ_{13} with the transition map from $\kappa_{13}(\Sigma_{x_1})$ to Σ^{out} for the flow of (5.46). Using the analogous approach as in [16, Section 2.6] one shows that Π_{23} is given by

$$\Pi_{23}(\delta^{-1/2}, y_2, r_2\lambda_2) = \left(\rho, \frac{\varepsilon}{\delta\rho^2}\,\theta(r_2, y_2, \lambda_2), \frac{\varepsilon}{\rho^2}, \frac{\lambda}{\rho}\right), \tag{5.47}$$

with

$$\theta(r_2, y_2, \lambda_2) = \delta\left(y_2 - \ln\left(\frac{r_2}{\delta^{1/2}\rho}\right)\right) + O(r_2 \ln(r_2)),$$

$$\frac{\partial\theta}{\partial\lambda_2} = O(r_2 \ln(r_2)),$$

$$\frac{\partial\theta}{\partial y_2} = \delta + O(r_2 \ln(r_2)).$$

Similarly, the transition map Π_{13} is given by

$$\Pi_{13}(r_1, \beta, \varepsilon_1, \lambda_1) = \left(\rho, \frac{r_1^2}{\rho^2} - \ln\left(\frac{r_1\beta^{1/2}}{\rho}\right)\frac{\varepsilon}{\beta} + O(r_1^3 \ln(r_1)), \frac{\varepsilon}{\rho^2}, \frac{\lambda}{\rho}\right). \tag{5.48}$$

It follows that Π_{13} is a diffeomorphism with bounded derivative and Π_{13}^{-1} has at most algebraic growth. It follows also that there exist positive constants L_1 and L_2 such that

$$J_1 \stackrel{\text{def}}{=} \left\{\left(\rho, y_3, \frac{\varepsilon}{\rho^2}, \frac{\lambda}{\rho}\right) : -L_1\varepsilon\ln(\varepsilon) < y_3 < L_2\right\} \subset \Pi_{31}(\Sigma_{x_1}).$$

We have the following result.

Proposition 5.2. *Fix $q \in (0, s_0)$. Then there exists $\varepsilon(q) > 0$ and a branch of periodic solutions*

$$(\lambda(q, \sqrt{\varepsilon}), \Gamma(q, \sqrt{\varepsilon})), \qquad \varepsilon \in (0, \varepsilon(q))$$

with initial condition (x_M, q). Moreover

$$\frac{\partial\lambda(q, \sqrt{\varepsilon})}{\partial q} < 0. \tag{5.49}$$

Proof. We define $\varepsilon(q)$ so that the backward trajectory $\hat{\gamma}_q(t)$ of (x_M, q) transformed to K_3 passes through J_1. It is clear that $\varepsilon(q)>0$. For $\varepsilon \in (0, \varepsilon(q))$ the trajectory $\hat{\gamma}_{q,1}(t)$ (i.e. the image of $\hat{\gamma}$ in K_1, as it passes through $\Sigma^{in}_{m,1}$, must be $C^1 - O(e^{-\delta/\varepsilon})$ close to $M_{m,1}$. The existence of the branch of periodic solutions follows analogously as for cycles of type 4. The estimate (5.49) follows from the fact that $R(s)<0$ for s sufficiently close to $2y_M$. ∎

This construction gives cycles which pass from K_1 through K_2, back to K_1, and on to K_3. Note that $\varepsilon(q)$ may shrink to 0 as $q \to 0$. Below we construct cycles for ε in an interval containing $(\varepsilon(q), \varepsilon_0)$ using an approach similar as for the case of relaxation oscillations.

In the following let Γ_i, $i=1, ..., 4$ be the trajectory corresponding to $S_{l,\varepsilon}$ in chart K_i. Let p_i be the intersection of Γ_i with Σ_{x_i}, $i=2, 4$, and p_3 be the intersection of Γ_3 with Σ^{out}. Let η_i be the y_i coordinate of p_i. Let $p(\varepsilon, \lambda)$ be the intersection of $S_{l,\varepsilon}$ with $\tilde{\Delta}$, and let $\eta(\varepsilon, \lambda)$ be the y-coordinate of p. We have the following result concerning the passage of the extension of $S_{l,\varepsilon}$ from K_1 via K_2 to K_3.

LEMMA 5.2. *Let η denote the y-coordinate of p. Then there exists $L>0$ such that*

$$\frac{\partial \eta}{\partial \lambda}<0$$

for all $\lambda = r_2 \lambda_2$ such that Γ_2 intersects Σ_{x_2} and $\lambda_2 \leqslant L$.

Proof. Let $L>0$ be a constant. We claim that

$$\frac{\partial \eta_2}{\partial \lambda_2}(r_2, \lambda_2)<0 \tag{5.50}$$

for all $\lambda_2 \leqslant L$, $r_2 < \sqrt{\varepsilon_0}$ such that p_2 is defined. To see that (5.50) holds we introduce new coordinates

$$\bar{x}_2 = x_2 - \lambda_2, \qquad \bar{y}_2 = y_2 - \lambda_2^2.$$

Transformed to the coordinates $(\bar{x}_2, \bar{y}_2)$ the vector field (4.2) has the rotational property, which holds for all $|\lambda_2| < L$, $r_2 < \sqrt{\varepsilon_0}$. The claim follows.

Let $(\varepsilon_o, \lambda_o) = (\varepsilon/\rho^2, \lambda/\rho)$. Since $p_3(\varepsilon_o, \lambda_o)$ is the function p_2 transformed to Σ^{out} by the map Π_{23} a straightforward computation shows that

$$\eta_3(\varepsilon_o, \lambda_o) = \frac{\varepsilon_o}{\delta}\, \theta(\delta \eta_2(\rho \varepsilon_o^{1/2}, \lambda_o \varepsilon_o^{-1/2})).$$

It follows that

$$\frac{\partial \eta_3}{\partial \lambda_o} = \varepsilon_o^{1/2} \left(\frac{\partial \eta_2}{\partial \lambda_2} (\rho \varepsilon_o^{1/2}, \lambda_o \varepsilon_o^{-1/2}) + O(r_2 \ln(r_2)) \right).$$

The result follows, since p_3 is $C^1 - O(\varepsilon)$ close to p. ▮

In order to understand cycles existing for larger values of λ we need to consider chart K_4, which is just a rescaling, i.e. r_4 and λ_4 are time independent parameters. After substitution and desingularisation (3.3) transforms to the following equations:

$$\begin{aligned} x_4' &= -y_4 + x_4^2 + O(r_4) \\ y_4' &= \varepsilon_4 (x_4 - 1 + O(r_4)). \end{aligned} \tag{5.51}$$

Note that for ε_4 small this is exactly the problem of a singularly perturbed fold point, studied in [16]. In the following we often use $r_4 = \lambda$ and $\varepsilon_4 = \varepsilon / \lambda^2$.

Our objective is to follow the extension of the slow manifold $S_{l,\varepsilon}$ as it passes from K_1 through K_4 and then on to K_3, finally arriving at Σ^{out}. To this end we define the section Σ_{λ_1} in chart K_1 as follows

$$\Sigma_{\lambda_1} = \{(x_1, r_1, \varepsilon_1, \lambda_1) \colon \lambda_1 = \delta\}.$$

Note that for $\lambda \geqslant L r_2$, with $L > 0$ a constant, the extension of $S_{l,\varepsilon}$ to K_1 must pass through Σ_{λ_1}. The coordinate transformation κ_{14} is given by

$$x_4 = \lambda_1^{-1} x_1, \qquad y_4 = \lambda_1^{-2}, \qquad \varepsilon_4 = \lambda_1^{-2} \varepsilon_1, \qquad r_4 = \lambda_1 r_1.$$

It follows that points in $\kappa_{14}(\Sigma_{\lambda_1})$ are of the form $(\delta^{-1} x_1, \delta^{-2}, \lambda^{-2}\varepsilon, \lambda)$.

We now consider the transition map induced by the flow of (5.51) from $\kappa_{14}(\Sigma_{\lambda_1})$ to the section Σ_{x_4} which is defined by

$$\Sigma_{x_4} = \{(x_4, y_4, \varepsilon_4, r_4) \colon x_4 = \delta^{-1}\}.$$

Let p_4 denote the intersection of the extension of $S_{l,\varepsilon}$ with Σ_{x_4}. Note that if $\lambda \geqslant L r_2$ then the ε_4 coordinate in $\kappa_{14}(\Sigma_{\lambda_1})$, given by $\lambda^{-2}\varepsilon$, is bounded by $1/L^2$. By [16, Theorem 2.1 and Remark 2.14], provided that L is sufficiently large, p_4 is given by

$$p_4 = (\delta^{-1}, -\Omega_0 \lambda^{-4/3} \varepsilon^{2/3} + o(\lambda^{-4/3} \varepsilon^{2/3}), \lambda^{-2}\varepsilon, \lambda).$$

The chart transformation κ_{43} is given by

$$r_3 = r_4 x_4, \qquad y_3 = y_4 x_4^{-2}, \qquad \varepsilon_3 = x_4^{-2} \varepsilon_4, \qquad \lambda_3 = x_4^{-1}.$$

It follows that points in $\kappa_{43}(\Sigma_{x_4})$ are of the form $(\delta^{-1}r_4, \delta^2 y_4, \delta^2\varepsilon_4, \delta)$. The map $\Pi_{43}\colon \Sigma_{x_4} \to \Sigma^{out}$ obtained by composing κ_{43} with the transition map for the flow of (5.46) is given by

$$\Pi_{43}(\delta, y_4, \varepsilon_4, r_4) = \left(\rho, \frac{\lambda^2}{\rho^2}\,\theta(y_4, \varepsilon_4, r_4), \frac{\varepsilon}{\rho^2}, \frac{\lambda}{\rho}\right),$$

where

$$\theta(y_4, \varepsilon_4, r_4) = \delta^2\left(y_4 - \varepsilon_4 \ln\left(\frac{r_4}{\delta\rho}\right)\right) + O(r_4 \ln(r_4)).$$

It follows that

$$\begin{aligned} p_3 &= \Pi_{43}(p_4) \\ &= \left(\rho, \frac{1}{\rho^2}\left(-\Omega_0 \lambda^{2/3}\varepsilon^{2/3} - \varepsilon \ln\left(\frac{\lambda}{\delta\rho}\right)\right) + o(\lambda^{2/3}\varepsilon^{2/3}) \right. \\ &\quad \left. + O(-\lambda^3 \ln(\lambda)), \frac{\varepsilon}{\rho^2}, \frac{\lambda}{\rho}\right). \end{aligned} \tag{5.52}$$

Eq. (5.52) leads to the following result.

LEMMA 5.3. *Suppose $\lambda \geqslant L\varepsilon^{1/2}$. Then there exists $\mu > 0$ such that*

$$\frac{\partial \eta_3}{\partial \lambda} < -\mu\varepsilon^{2/3}. \tag{5.53}$$

Proof. Follows directly from differentiation of (5.52) and from the estimate $\lambda \geqslant L\varepsilon^{1/2}$. ∎

Remark 5.1. It is interesting to note that the relative size of the leading terms in (5.52) changes as λ varies compared to ε. As the term $-\Omega_0\lambda^{2/3}\varepsilon^{2/3}$ becomes dominant one can speak of the corresponding periodic orbit as a relaxation oscillation. Since the λ-derivatives of both terms are negative the overall sign of the λ-derivative does not change.

Now we conclude the construction of cycles of type 5. Let

$$\tilde{\Delta}(s_0) = \{(x_M, y), : -\mu < y < s_0\},$$

where $\mu > 0$ is small, and consider the return map $\pi\colon \tilde{\Delta}(s_0) \to \tilde{\Delta}(s_0)$ for the flow of (3.1). Let $q_r(\varepsilon, \lambda)$ be the intersection of the continuation of $S_{r,\varepsilon}$ with

$\tilde{\Delta}(s_0)$. We choose s_0 sufficiently small, so that $R(s)<0$ for $s\in[2y_M-s_0, 2y_M]$. It follows that

$$\pi(y)=\eta(\varepsilon, \lambda)+P(y, \varepsilon\,\lambda), \tag{5.54}$$

with $P(y, \varepsilon, \lambda)$ and $\frac{\partial}{\partial\lambda}P(y, \varepsilon, \lambda)$, and $\frac{\partial}{\partial y}P(y, \varepsilon, \lambda)$ being $O(e^{-K/\varepsilon})$ small. Hence the equation

$$\pi(y)=y$$

can be solved by the implicit function theorem for a unique fixed point $(x_M, q(\varepsilon, \lambda))$ for every value of (ε, λ) such that $\pi(\tilde{\Delta}(s_0)\subset\tilde{\Delta}(s_0)$. We can now strengthen Proposition 5.2 in the following way.

PROPOSITION 5.3. *Suppose ε_0 and s_0 are sufficiently small. Then there exists a smooth family of periodic solutions*

$$(\lambda(q, \sqrt{\varepsilon}), \Gamma(q, \sqrt{\varepsilon})), \qquad \varepsilon\in(0, \varepsilon_0), \quad q\in[0, s_0)$$

with initial condition (x_M, q). Moreover

$$\frac{\partial\lambda(q, \sqrt{\varepsilon})}{\partial q}<0. \tag{5.55}$$

Proof. We first show that the family of periodic orbits with initial condition $(x_M, q(\varepsilon, \lambda))$ can be parametrized by q, i.e. that the function, $\lambda\to q(\varepsilon, \lambda)$ can be inverted. We claim that $\frac{\partial}{\partial\lambda}q(\varepsilon, \lambda)<0$. This follows from the form of (5.54), from Lemma 5.2 and from Lemma 5.3. Hence there exists a family $(\tilde{\lambda}(q, \varepsilon), \tilde{\Gamma}(q, \varepsilon)$ corresponding to the fixed points of π. We now show that domains of definition of this family and the family whose existence was proved in Proposition 5.2 overlap, giving rise to the family whose existence is asserted in Proposition 5.3.

Let $q^o=(\rho, y_3, \varepsilon^2/\rho^2, \lambda/\rho)\subset\Sigma^{out}$ be such that the forward trajectory of (3.1) passes through q. The backward trajectory of the blown-up vector field starting at q^o passes either through Σ_{x_1}, Σ_{x_2}, or through Σ_{x_4}. Hence, for each $q\in(0, s_0)$, $\varepsilon\in(0, \varepsilon_0)$ there exists a $\lambda\in\{\lambda(q, \varepsilon), \tilde{\lambda}(q, \varepsilon)\}$ for which a cycle exists. Uniqueness and the estimate (5.55) follow from the fact that both families of periodic orbits were constructed using the implicit function theorem and from the fact that all periodic orbits are stable, which is due to the estimate $R(s)<0$ for $s\in[2y_M-s_0, 2y_M]$. ∎

Canard cycles passing close to the fold point (x_M, y_M). Consider local coordinates $(\tilde{x}, \tilde{y})$ given by $(x, y)=(x_M+\tilde{x}, y_M+\tilde{y})$. We subsequently drop the ~ for the duration of this section and apply the blow-up transformation as in [16]. Note that the local coordinates (x, y) used here are obtained from the canonical coordinates in [16] by applying the reflection $(x, y) \to (-x, -y)$. This means that the directional blow-ups as well as pictures must be interpreted with this transformation in mind. We consider three sections of the flow. One in chart K_1, given by

$$\Sigma_1 = \{(0, r_1, \varepsilon_1): \rho \geqslant r_1 \geqslant 0, \delta \geqslant \varepsilon_1 \geqslant 0\}$$

and two sections in chart K_3, namely

$$\Sigma_2 = \{(r_3, y_3, \delta): \rho \geqslant r_3 \geqslant 0,\ y_3 \in (-y_{3,0}, y_{3,0})\},$$

$$\Sigma_3 = \{(r_3, \tfrac{1}{2}, \varepsilon_1): \rho \geqslant r_3 \geqslant 0, \tilde{\delta} \geqslant \varepsilon_3 \geqslant 0\},$$

where $y_{3,0}$ is a positive constant. Consider the section $\Sigma^{in}_{m,1}$ in K_1 as defined Section 4.2. It follows that there is a choice of $y_{3,0} \in (0, 1)$ and $\tilde{\delta} > \delta$ so that any trajectory passing through $\Sigma^{in}_{m,1}$ goes through either Σ_1, Σ_2 or Σ_3. In Fig. 13 the trajectories 6–12 pass close to the fold point, with 6 and 7 passing through Σ_3, 8–10 through Σ_2, and 11 and 12 through Σ_1. We now consider points in Σ_j, $j = 1, 2, 3$ and match their forward and backward trajectories in chart K_2 of the blown-up vector field at the canard point. This construction is analogous to the construction of cycles of types 2 and 4. As shown in [16, Sections 2.6–2.7] all trajectories passing near the fold point must arrive to a section of the flow Σ^{out}_3 in chart K_3. Moreover, the derivatives of transition maps Π_j from Σ_j to Σ^{out}_3 can be bounded by $e^{\alpha/\varepsilon}$, with $\alpha > 0$ arbitrarily small. This growth is compensated by the exponential contraction along S_l, respectively S_m, in forward time, respectively in backward time. It follows that an equivalent of Lemma 5.1 holds. Thus we obtain three families of canard cycles corresponding to initial conditions in Σ_j, $j = 1, 2, 3$. By construction, the regions of existence of the canard cycles belonging to those families overlap. Since each family was constructed using the implicit function theorem the families coincide on the overlap region. By conveniently reparametrizing the three families one obtains one family of canard cycles indexed by $s \in J$, where J is an interval containing $[y_M - s_0, y_M + s_0]$, with s_0 chosen sufficiently small. This concludes the proof of the existence of the family $\Gamma(s, \sqrt{\varepsilon})$.

We now prove properties (i), (ii), (iii) and (v). Property (iv) is a consequence of Lemma 5.4 below. Properties (i) and (ii) follow directly from the construction of $\Gamma(s, \sqrt{\varepsilon})$. Property (iii) follows from the fact that for $s \in [\varepsilon^\nu, 2y_M - \varepsilon^\nu]$ the orbit $\Gamma(s, \sqrt{\varepsilon})$, as it passes through chart K_1 of the blown-up vector field near the canard point, must remain for the time of at least $O(1/\varepsilon^{1-\nu})$ near $M_{m,1}$ and $M_{l,1}$ (by the time variable we mean the independent variable of (4.5)). Note that it is necessary to take $\nu \in (0, 1)$, otherwise the estimate (3.17) is meaningless. Property (v) follows from the construction of $\Gamma(s, \sqrt{\varepsilon})$ (implicit function theorem) and from Theorem 2.1. Property (iv) is a consequence of the following lemma.

LEMMA 5.4. *For $s > \varepsilon^\nu$ the orbit $\Gamma(s, \sqrt{\varepsilon})$ is $O(\varepsilon^\alpha)$ close to $\Gamma(s)$, where*

$$0 < \alpha \leqslant \max\{1/2, 2(1-\nu)\}.$$

Proof. We consider $s \in (\varepsilon^\nu, y_M - s_0]$. The proof in the other cases is similar. Let $\gamma_s(t)$ be the forward trajectory starting at $(0, s)$. We first show that $\gamma_s(t)$ remains close to the union of $\{(x, s), x \in (0, x_l(s))\}$ and $S_{l,\varepsilon}$. For $(0, s) \notin V$ this follows directly from Fenichel theory. If $(0, s) \in V$ then let $p = (0, \sqrt{s}, \varepsilon_1, \lambda_1)$ be the corresponding point in K_1 and $\gamma_{s,1}(t) = (x_{s,1}(t), r_{s,1}(t), \varepsilon_{s,1}(t), \lambda_{s,1}(t))$ be the forward trajectory of (3.1) starting at this point. Note that $\varepsilon_1 \leqslant \varepsilon^{1-\nu}$. It follows that the transition time to $\Sigma_{l,1}^{out}$ is of the order $T = O(\varepsilon_1^{-(1-\nu)})$. We now estimate $\Delta\varepsilon_1 = \varepsilon_{s,1}(t) - \varepsilon_1$ for $t \in [0, O(-\ln(\varepsilon))]$. We have

$$\frac{1}{\varepsilon_{1,s}(t)} - \frac{1}{\varepsilon_1} = O(-\ln(\varepsilon)).$$

It follows that

$$\varepsilon_{s,1}(t) = \frac{\varepsilon_1}{1 + \varepsilon_1 O(-\ln(\varepsilon))}$$

and

$$\varepsilon_{s,1}(t) - \varepsilon_1 = O(-\varepsilon_1^2 \ln(\varepsilon)).$$

From (4.5) it follows that $r_{s,1}(t) = \sqrt{s}(1 + O(-\varepsilon_1^2 \ln(\varepsilon))$. Since $\varepsilon_1 \leqslant \varepsilon^{1-\nu}$ it follows that $y_s(t) = r_{s,1}(t)^2 = s(1 + O(\varepsilon^{2(1-\nu)} \ln(\varepsilon)))$. The claim follows, since for $t \geqslant O(-\ln(\varepsilon))$ the trajectory $\gamma_{s,1}(t)$ remains $O(\varepsilon)$ close to $M_{l,1}$.

Finally we show that $S_{l,\varepsilon}$ remains $O(\varepsilon^{1/2})$ close to S_l until it enters a neighborhood of the origin of size $O(\varepsilon)$. To see this note that points of the form $(x_1, r_1, \varepsilon_1, \lambda_1)$ are $O(\varepsilon_1)$ away from $M_{l,1} \cap \{\varepsilon_1 = 0\}$ and the blow-down of $\Sigma_{l,1}^{out}$ is $O(\varepsilon^{1/2})$ away from the origin.

An analogous argument holds for the backward trajectory $\hat{\gamma}_s(t)$. The result follows. ∎

This concludes the proof of Theorem 3.3. The proof of Theorem 3.5 is similar.

5.2. *Details of Canard Explosion*

Let $r_2 = \sqrt{\varepsilon}$. Let $P(s, r_2)$ be the Floquet exponent of $\Gamma(s, r_2)$. We have the following result.

PROPOSITION 5.4. *Fix s_0 sufficiently small. There exists a function $\theta(s, r_2)$, C^k in s and C^1 in r_2, such that*

$$P(s, r_2) = \frac{1}{\varepsilon}\{R(s) + \theta(s, r_2)\}, \qquad s \in (s_0, 2y_M), \tag{5.56}$$

with $\theta(s, r_2)$ and $\frac{\partial\theta}{\partial s}(s, r_2)$ converging to 0 uniformly in s as $r_2 \to 0$.

Proof. We carry out the proof for $s \in (s_0, y_M - s_0)$. The proof for $s > y_M + s_0$ is completely analogous. For $s \approx y_M$ the argument is similar, yet some care is necessary, since the parametrization by s is not so natural. We make some remarks pertaining to this issue at the end of this section.

Let $\Gamma = \Gamma(s, r_2)$ and let $\gamma(t) = (x(t), y(t))$, $t \in [0, T]$ be the solution of (3.1) corresponding to the orbit Γ with $\gamma(0) = \gamma(T) \in \Delta$. It is known that

$$P(s, r_2) = \int_0^T \operatorname{div} X(\gamma(t))\, dt,$$

where T is the period of Γ. We will break Γ into a number of different pieces and we will use blow-up charts as well as Fenichel theory to estimate the portion of $P(s, r_2)$ corresponding to each of the pieces.

By Fenichel theory we can choose $t_1 = O(-\ln(\varepsilon))$ (depending on ε), so that $\gamma(t_1)$ is C^1 $O(\varepsilon)$ close to $S_{l,\varepsilon}$. Let t_2 be such that $\Phi_1^{-1}(\gamma(t_2)) \in \Sigma_{l,1}^{in}$. It follows from Fenichel theory that $\gamma(t)$ is $O(\varepsilon)$ close to $S_{l,\varepsilon}$ for $t \in [t_1, t_2]$. In particular $g(\gamma(t)) \neq 0$ and $\{\gamma(t)\colon t \in [t_1, t_2]\}$ can be written in the form

$$\{(x(y), y)\colon y \in [y_1, y_2]\}, \qquad y_j = y(t_j), \quad j = 1, 2.$$

The function $x(y)$ depends also on s and r_2, but we suppress this dependence to simplify the notation.

Let $\gamma_{l,\varepsilon}$ be the solution of (3.1) contained in $S_{l,\varepsilon}$ with the initial condition chosen sot hat $(0, s)$ is in the stable fiber of $\gamma_{l,\varepsilon}(0)$. Then the points $(x(y(t)), y(t))$ are in the stable fibers at $\gamma_{l,\varepsilon}(t)$. It follows that

$$\begin{aligned} y &= s + O(-\varepsilon \ln(\varepsilon)), \\ \frac{dy_1}{ds} &= 1 + O(-\varepsilon \ln(\varepsilon)). \end{aligned} \tag{5.57}$$

Sine the right hand side of (4.5b) does not vanish we conclude that the solution in K_1 corresponding to $\gamma(t)$ can be written in the form $\{(x_1(r_1), \varepsilon_1, r_1, \lambda_1)\}$. Let t_3 be such that $\Phi_1^{-1}(\gamma(t_3)) \in \Sigma_{l,1}^{out}$. It follows that $y(t_3) = \frac{\varepsilon}{\delta}$ and $\{\gamma(t)\colon t \in [t_1, t_3]\}$ can be written in the form $\{(x(y), y)\colon y \in [y_1, \frac{\varepsilon}{\delta}]\}$. Hence

$$\int_{t_1}^{t_3} \operatorname{div} X(\gamma(t))\, dt$$

$$= \int_{y_1}^{y_3} \frac{1}{\varepsilon} \left\{ \frac{\frac{\partial f}{\partial x}(x(y), y, \varepsilon, \lambda)}{g(x(y), y, \varepsilon, \lambda)} + \varepsilon \frac{\frac{\partial g}{\partial y}(x(y), y, \varepsilon, \lambda)}{g(x(y), y, \varepsilon, \lambda)} \right\} dy. \tag{5.58}$$

Since $g(x(y), y, \varepsilon, \lambda) = -\sqrt{y} + O(\varepsilon, y)$ it follows that

$$\theta_1(s, r_2) \stackrel{\text{def}}{=} \int_{y_1}^{y_3} \varepsilon \frac{\frac{\partial g}{\partial y}(x(y), y, \varepsilon, \lambda)}{g(x(y), y, \varepsilon, \lambda)}\, dy = O(r_2). \tag{5.59}$$

Estimate (5.57) implies that $\frac{\partial \theta_1}{\partial s} = O(\varepsilon)$. We claim that $\frac{\partial x(y)}{\partial r_2}$ is bounded. For $y \leqslant y_2$ this follows from Fenichel theory. For $y > y_2$ recall that $M_{l,1}$ is a graph of a function $x_1(r_1, \varepsilon_1)$ and note that $x(y, r_2) = \sqrt{y}\, x_1(\sqrt{y}, r_2/\sqrt{y})$. The claim follows.

Let t_4 be such that $\Phi_1^{-1}(\gamma(t_4)) \in \Sigma_{m,1}^{in}$. For $t \in [t_3, t_4]$ the solution $\gamma(t)$ can be written in the form

$$\gamma(t + t_3) = (r_2 x_2(t/r_2), r_2^2 y_2(t + r_2)),$$

where $(x_2(\tau), y_2(\tau))$ is the solution of (4.2) joining $\Phi_2^{-1}(\gamma(t_3))$ to $\Phi_2^{-1}(\gamma(t_4))$. Let

$$\theta_2(s, r_2) = \varepsilon \int_{t_3}^{t_4} \operatorname{div} X(\gamma(t))\, dt.$$

It follows that

$$\begin{aligned} \theta_2(s, r_2) &= \int_{t_3/r_2}^{t_4/r_2} \left(r_2 \frac{\partial f}{\partial x}(r_2 x_2(s), r_2^2 y_2(s)) + r_2^3 \frac{\partial g}{\partial y}(r_2 x_2(s), r_2^2 y_2(s)) \right) ds \\ &= O(r_2), \end{aligned}$$

since $\frac{\partial f}{\partial x}(r_2 x_2(s), r_2^2 y_2(s)) = O(r_2)$ and $t_j = O(r_2)$, $j = 3, 4$. Similarly $\frac{\partial \theta_2}{\partial s}$ is uniformly $O(r_2)$.

Let t_5 and t_6 be defined analogously as t_1 and t_2. Let $y_6 = y(t_6)$. For $t \in [t_4, t_6]$ we get an expression analogous to (5.58) namely

$$\int_{t_4}^{t_6} \operatorname{div} X(\gamma(t))\, dt$$

$$= \int_{y_4}^{y_6} \frac{1}{\varepsilon} \left\{ \frac{\frac{\partial f}{\partial x}(\tilde{x}(y), y, \varepsilon, \lambda)}{g(\tilde{x}(y), y, \varepsilon, \lambda)} + \varepsilon \frac{\frac{\partial g}{\partial y}(\tilde{x}(y), y, \varepsilon\, \lambda)}{g(\tilde{x}(y), y, \varepsilon, \lambda)} \right\} dy, \qquad (5.60)$$

where $(\tilde{x}(y), y)$ is an alternative parametrization of $\gamma(t)$, $t \in [t_4, t_6]$. Let

$$\theta_3(s, r_2) \stackrel{\text{def}}{=} \int_{y_4}^{y_6} \varepsilon \frac{\frac{\partial g}{\partial y}(\tilde{x}(y), y, \varepsilon, \lambda)}{g(\tilde{x}(y), y, \varepsilon, \lambda)}\, dy. \qquad (5.61)$$

It follows that $\theta_3(s, r_2)$ and $\partial\theta_3/\partial s(s, r_2)$ are uniformly $O(r_2)$.

Let

$$\theta_4(s, r_2) \stackrel{\text{def}}{=} \varepsilon \left\{ \int_{t_6}^{T} \operatorname{div} X(\gamma(t))\, dt + \int_0^{t_1} \operatorname{div} X(\gamma(t))\, dt \right\}. \qquad (5.62)$$

It is clear that $\theta_4(s, r_2)$ and $\partial\theta_4/\partial s(s, r_2)$ are of order $O(r_2)$. It also follows from Fenichel theory that $\partial\theta_4/\partial r_2(s, r_2)$ is continuous and equal to 0 as $r_2 = 0$.

Finally let

$$\theta_0(s, r_2) \stackrel{\text{def}}{=} R(s)$$

$$= \int_{y_1}^{y_3} \frac{\frac{\partial f}{\partial x}(x(y), y, \varepsilon, \lambda)}{g(x(y), y, \varepsilon, \lambda)}\, dy - \int_{y_4}^{y_6} \frac{\frac{\partial f}{\partial x}(\tilde{x}(y), y, \varepsilon, \lambda)}{g(\tilde{x}(y), y, \varepsilon, \lambda)}\, dy. \qquad (5.63)$$

Note that $\{(x(y), y)\}$ and $\{(\tilde{x}(y), y)\}$ are $O(r_2)$ close to S_l and S_m, respectively. Moreover $y_3 = y_4 = \frac{\varepsilon}{\delta}$ and y_6 satisfies an estimate analogous to (5.57). Hence $\theta_0(s, r_2)$ and $\partial\theta_0/\partial s$ are uniformly $O(r_2)$. We define θ by

$$\theta(s, r_2) = \sum_{j=0}^{4} \theta_j(s, r_2).$$

The result follows.

To obtain the proof for $s \approx y_M$ we need to use the parametrizations in the sections Σ_j, $j=1, 2, 3$. It is necessary to refine the splitting of the orbits $\Gamma(s, r_2)$ into additional pieces, according to the construction, and carry out the estimates for each piece similarly as in the case of $s \in (s_0, y_M - s_0)$. ∎

Proof of Theorem 3.4. It follows from Proposition 4.3 that $P(s, r_2) < 0$, $s \in (0, s_0)$. We assume that $s \in (0, y_M - s_0)$. For the other cases the proof is similar. Let π: $\Delta \to \Delta$ (respectively π: $\tilde{\Delta} \to \tilde{\Delta}$) be the return map. Differentiating the identity

$$\pi(s, \lambda(s, r_2), r_2) = s$$

with respect to s we get

$$1 - e^{P(s, r_2)} = \frac{\partial \pi}{\partial \lambda}(s, \lambda(s, r_2), r_2) \frac{\partial \lambda}{\partial s}(s, r_2). \tag{5.64}$$

It follows from the Melnikov computation in [16, Section 3.8] that

$$\frac{\partial \pi}{\partial \lambda}(s, \lambda(s, r_2), r_2) > 0. \tag{5.65}$$

It follows that $\frac{\partial \lambda}{\partial s}(s, r_2) > 0$. ∎

Proof of Theorem 3.6. Consider the equation

$$P(s, r_2) = 0. \tag{5.66}$$

It follows from Proposition 5.4 and the implicit function theorem that there exists a C^1 curve $s_{lp}(r_2)$ such that $P(s, r_2) = 0$ for $s = s_{lp}(r_2)$. Moreover $P(s, r_2) > 0$ for $s < s_{lp}(r_2)$ and $P(s, r_2) < 0$ for $s > s_{lp}(r_2)$. The result now follows from (5.64) and (5.65). ∎

6. EXAMPLE-VAN DER POL EQUATION

The van der Pol equation

$$\begin{aligned} x' &= y - \frac{x^2}{2} - \frac{x^3}{3} \\ y' &= \varepsilon(\lambda - x) \end{aligned} \tag{6.67}$$

provides the best known example of a canard explosion [4, 8, 11]. We change coordinates, letting

$$x \to -x, \qquad \lambda \to -\lambda.$$

The transformed equation has the form

$$\begin{aligned} x' &= -y + \frac{x^2}{2} - \frac{x^3}{3}, \\ y' &= \varepsilon(x - \lambda). \end{aligned} \tag{6.68}$$

For (6.68) the assumptions Theorem 3.4 can be easily checked. Here we check that $R'(s) < 0$ for $h \in (0, y_M]$. We leave the verification of the other assumptions to the reader. To prove that $R'(s) < 0$ for $h \in (0, y_M]$ we differentiate the relation

$$\frac{x^2}{2} - \frac{x^3}{3} = s$$

with respect to s, obtaining

$$x(s)(1 - x(s)) \frac{dx}{ds}(s) = 1.$$

It follows that

$$R'(s) = \left[x(s)(1 - x(s))^2 \frac{dx}{dh}(s) \right]_{x_l(s)}^{x_m(s)} = x_l(s) - x_m(s).$$

The estimate follows since $x_m(s) > 0$ and $x_l(s) < 0$ and both $x_r(s)$ and $x_l(s)$ are increasing as a function of s.

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