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## An interface condition to compute compressible flows in variable cross section ducts



*Conditions d'interface pour le calcul d'écoulements compressibles en conduite à section variable*

Jean-Marc Hérard<sup>a,b</sup>, Jonathan Jung<sup>c,d,e</sup><sup>a</sup> EDF R&D, MFEF, 6, quai Watier, 78400, Chatou, France<sup>b</sup> I2M, UMR CNRS 7353, 39, rue Joliot-Curie, 13453 Marseille, France<sup>c</sup> LMA-IPRA, Université de Pau et des pays de l'Adour, UMR CNRS 5142, avenue de l'Université, 64013 Pau, France<sup>d</sup> INRIA Bordeaux Sud Ouest, Cagire Team, 351, cours de la Libération, 33405 Talence cedex, France<sup>e</sup> IJLL & LRC Manon, 4, place Jussieu, 75005, Paris, France

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### ABSTRACT

We propose an improved interface condition in order to account for head losses in pipe when some discontinuous cross sections occur.

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### R É S U M É

Nous proposons d'améliorer la condition d'interface afin de prendre en compte la perte de charge pour un écoulement en conduite à section variable discontinue.

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## 1. Introduction

We examine in this paper some possible improvement of the well-balanced strategy in order to compute approximations of solutions to Euler-type models in one-dimensional geometries corresponding basically to pipe configurations. From a practical point of view, these methods are very useful in order to build stable and efficient algorithms. If  $\rho, u, p$  respectively denote the density, the velocity and the pressure of the fluid, and noting as usual  $\epsilon(p, \rho)$  the internal energy,  $E = \rho(\epsilon(p, \rho) + u^2/2)$  the total energy, and  $A(x)$  the – steady – cross section of the pipe, the target set of equations is

$$\begin{cases} \partial_t(A\rho) + \partial_x(A\rho u) = 0, \\ \partial_t(A\rho u) + \partial_x(A\rho u^2 + Ap) - p \partial_x A = A(x)S_u^r, \\ \partial_t(AE) + \partial_x(Au(E + p)) = A(x)uS_u^r, \end{cases} \quad (1)$$

E-mail addresses: [jean-marc.herard@edf.fr](mailto:jean-marc.herard@edf.fr) (J.-M. Hérard), [jonathan.jung@univ-pau.fr](mailto:jonathan.jung@univ-pau.fr) (J. Jung).

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and the associated unknown is  $W = (A\rho, A\rho u, AE)^T$ . The terms on the right-hand side  $S_{ij}^r$  stand for the *regular* head losses, which account for viscous effects on the walls of the pipe. This set of equations is meant when the cross-section  $A(x)$  is regular (at least  $C^2$ ). In practice, this information is provided by the user. When convergent or divergent regions occur in the pipe, and when the meshes that are used are rather coarse, the well-balanced strategy naturally arises for those who use Finite Volume methods. In that case, the cross section is assumed to be constant within each cell, and another equation is added to (1), which is  $\partial_t A = 0$ . The idea is very appealing, and many authors have been investigating that particular topic, both at the continuous and discrete levels, among which we may at least cite [2,3,8–10,12]. Though the cross section  $A(x)$  is known, the new unknown is now  $\tilde{W} = (A, A\rho, A\rho u, AE)^T$ , and terms of the form  $\psi(W)\partial_x A$  are now part of the convective subset. Meanwhile, a steady wave associated with  $\lambda = 0$ , corresponding to the “constraint”  $\partial_t A = 0$  arises. The one-dimensional Riemann problem associated with system (1) complemented with  $\partial_t A = 0$  and discontinuous initial conditions may then be investigated. As a consequence, there is the temptation to use this approach not only to deal with regular cross sections, but also to apply it when discontinuities are present in the pipe, which is obviously of major interest for practical computations in an industrial framework. In that case, the classical well-balanced strategy simply assumes that interface conditions that should be enforced at the steady interface separating two cells with distinct cross sections correspond to the Riemann invariants of the steady wave. In the case of Euler equations, this means that quantities  $Q = A\rho u, QH$  where  $H = (E + p)/\rho$  and the entropy  $s(p, \rho)$  should be preserved across the steady interface. Well-Balanced (WB) schemes relying on this approach can then be built, and some of them behave quite well owing to their stability and ability to maintain well-balanced initial conditions, even when the cross section is discontinuous. It even seems that the WB technique is mandatory in order to achieve numerical convergence towards the correct solutions (see for instance [6,11]).

Nonetheless, a major drawback is hidden in the basic formulation. This is due to the fact that for the case of discontinuous cross sections, the physical *singular* head losses have not been accounted for. As a result, the comparison of the “true” multi-dimensional solution and of the one-dimensional well-balanced computational results is rather poor; this is described in [7] for instance. One retrieves the expected results that are the following: the quantities  $Q = A\rho u, QH$  should remain constant across interfaces supporting discontinuous cross sections, but the entropy does not. This is directly linked to the fact that some singular contribution has been omitted in the momentum equation.

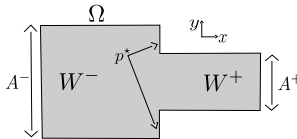
In order to cure that deficiency, at least two strategies may be proposed. A first one simply consists in a modified one-dimensional approach that would rely on an integral formulation of the multi-dimensional set of governing equations. This scheme (called 1D+ in the sequel) has been partially investigated in [1], and this seems to be a rather promising technique in order to handle this difficulty. Another strategy that is appealing in based on similar ideas, and it simply consists in enforcing a new set of interface conditions at the steady interface associated with the cross-section discontinuity. The basic requirement is that these modified interface conditions should account for singular head losses due to pressure effects, at least in a weak way. This is precisely the main objective of the present contribution.

Thus the paper is organized as follows: we firstly introduce the new interface condition, we secondly explain how to compute left and right states satisfying the new interface condition, we thirdly introduce the numerical scheme and we finally do some numerical validations.

## 2. A new formulation for flows in variable cross section ducts

### 2.1. The new interface condition

We omit the regular head losses in System (1), which means that  $S_{ij}^r = 0$  in System (1). Integrating the multi-dimensional Euler equations on the physical domain  $\Omega$  (see Figure below), we may now propose three natural candidates for the *steady* interface conditions



$$\begin{cases} [Q]_{\pm}^{\pm} = 0, \\ [Qu + Ap]_{\pm}^{\pm} - p^* [A]_{\pm}^{\pm} = 0, \\ [QH]_{\pm}^{\pm} = 0, \end{cases} \tag{2}$$

$$\text{with } p^* = \begin{cases} p^-, & \text{if } A^- > A^+, \\ p^+, & \text{otherwise,} \end{cases} \tag{3}$$

where we note by  $\cdot^+$  (resp.  $\cdot^-$ ) the value of the quantity on the right (resp. the left) side of the discontinuous interface. The choice of the wall pressure  $p^*$  is based on the two-dimensional approach. Indeed, if we have a contraction with  $A^- > A^+$  (resp. an expansion with  $A^- < A^+$ ), the pressure on the restricted wall is the left (resp. right) pressure.

### 2.2. Computation of left and right states satisfying the interface condition (2)

We assume that the fluid satisfies a perfect gas equation of states:  $p = (\gamma - 1)\rho\epsilon$ .  $W^+ = W^+(W^-)$ : we assume that  $W^-$  is known and we compute  $W^+$  such that the interface condition (2) is satisfied. For given value of  $W^-$ , we can compute  $Q^- = A^- \rho^- u^-$  and  $H^- = \frac{\gamma}{\gamma-1} \frac{p^-}{\rho^-} + \frac{(u^-)^2}{2}$ . The first two conditions in (2) give

$$u^+ = \frac{Q^-}{A^+\rho^+} \quad \text{and} \quad p^+ = p^- + \frac{Q^-}{\min(A^-, A^+)} \left( u^- - \frac{Q^-}{A^+\rho^+} \right).$$

Injecting these two quantities in the third condition of (2), we deduce that  $\rho^+$  is a root of the second-order polynomial:

$$P_-(X) = H^- X^2 - \frac{\gamma}{\gamma - 1} \left( p^- + \frac{Q^-}{\min(A^-, A^+)} u^- \right) X + \frac{(Q^-)^2}{A^+} \left( \frac{\gamma}{\gamma - 1} \frac{1}{\min(A^-, A^+)} - \frac{1}{2A^+} \right) = 0. \quad (4)$$

$W^- = W^-(W^+)$ : we now assume that  $W^+$  is known and we compute  $W^-$  as a function of  $W^+$ . Given  $W^+$ , we compute  $Q^+ = A^+\rho^+u^+$  and  $H^+ = \frac{\gamma}{\gamma-1} \frac{p^+}{\rho^+} + \frac{(u^+)^2}{2}$ . We get at once:

$$u^- = \frac{Q^+}{A^-\rho^-} \quad \text{and} \quad p^- = p^+ + \frac{Q^+}{\min(A^-, A^+)} \left( u^+ - \frac{Q^+}{A^-\rho^-} \right)$$

where  $\rho^-$  is a root of

$$P_+(X) = H^+ X^2 - \frac{\gamma}{\gamma - 1} \left( p^+ + \frac{Q^+}{\min(A^-, A^+)} u^+ \right) X + \frac{(Q^+)^2}{A^-} \left( \frac{\gamma}{\gamma - 1} \frac{1}{\min(A^-, A^+)} - \frac{1}{2A^-} \right) = 0. \quad (5)$$

If equations (4) and (5) have two distinct real solutions, we choose the one that has the same sonicity as the state on the other side of the interface. If there is only one solution, we choose this unique solution. When there is no real solution, the algorithm is stopped.

### 2.3. Numerical scheme

We present a scheme for approximating solutions to system (1) on  $[x_{\min}, x_{\max}]$ , relying on the arguments in the previous sections. Given an uniform time step  $\Delta t$  and a mesh size  $\Delta x = \frac{x_{\max} - x_{\min}}{N}$ , setting  $x_j = x_{\min} + j \Delta x$ ,  $j = 0 \dots N$ , and  $t_n = n \Delta t$ ,  $n \in \mathbb{N}$ , we denote by  $W_j^n$  the approximation of the values  $W(x_j, t^n)$  of the exact solution to system (1). The *WB-HJ* scheme is defined by

$$W_j^{n+1} = W_j^n - \frac{\Delta t}{\Delta x} \left( F^{\text{Rus}} \left( W_j^n, (W_{j+1}^n)^- \right) - F^{\text{Rus}} \left( (W_{j-1}^n)^+, W_j^n \right) \right) \quad (6)$$

where  $(W_{j+1}^n)^- = W^-(W_{j+1}^n)$ ,  $(W_{j-1}^n)^+ = W^+(W_{j-1}^n)$  are computed in Section 2.2, and the numerical flux  $F^{\text{Rus}}(W_l, W_r)$  is the Rusanov flux

$$F^{\text{Rus}}(W_l, W_r) = \frac{F(W_l) + F(W_r)}{2} - \frac{\max(|u_l| + c_l, |u_r| + c_r)}{2} (W_r - W_l) \quad (7)$$

with  $F(W) = (Q, Qu + Ap, QH)^T$ . The *WB-HJ* scheme is an adaptation of the scheme of Kröner and Thanh [12], called *WB-KT* in the sequel, for which the classical interface condition on  $(Q, s = \frac{p}{\rho^\gamma}, H)$  is replaced by the new interface condition (2).

**Proposition 1.** *The scheme WB-HJ is well-balanced: if for all  $j$ ,  $Q_j^0 = Q^0$ ,  $I_j^0 = I^0$  and  $H_j^0 = H^0$  where  $I := Qu + A(p - p^*)$ , then for all  $j$  and for all  $n$ ,*

$$Q_j^n = Q_j^0 = Q^0, \quad I_j^n = I_j^0 = I^0 \quad \text{and} \quad H_j^n = H_j^0 = H^0. \quad (8)$$

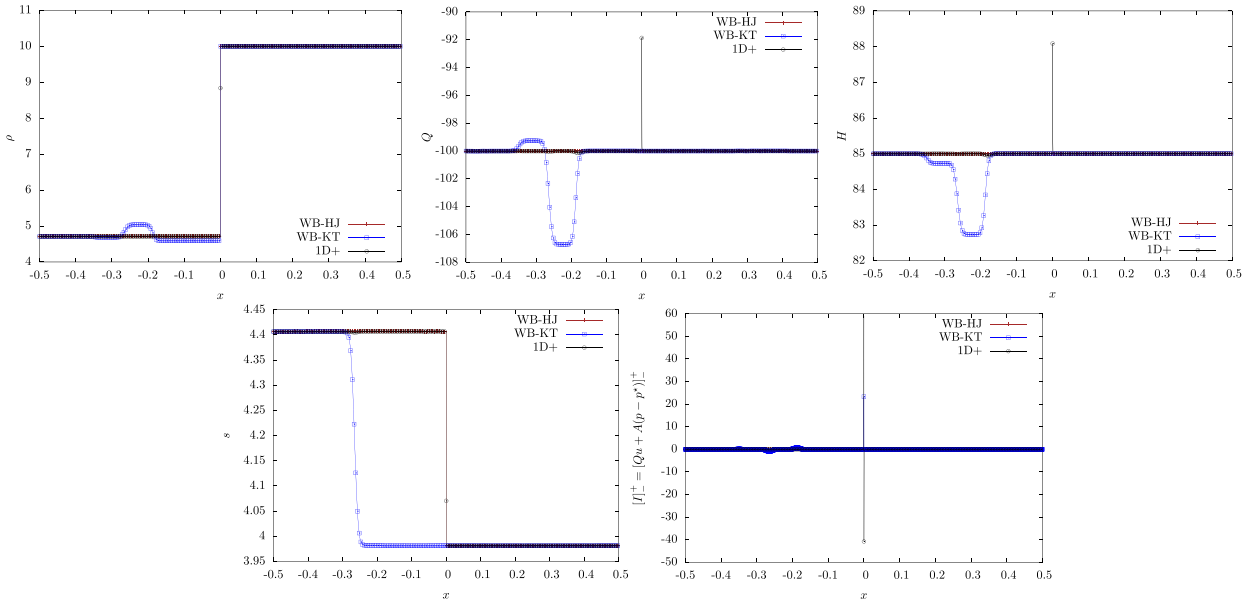
### 3. Numerical results

For all test cases, we choose a perfect gas equation of state:  $p = (\gamma - 1)\rho\epsilon$ , where  $\gamma$  is set to 7/5. The time step  $\Delta t$  satisfies the CFL condition where the CFL constant has been set to 0.4.

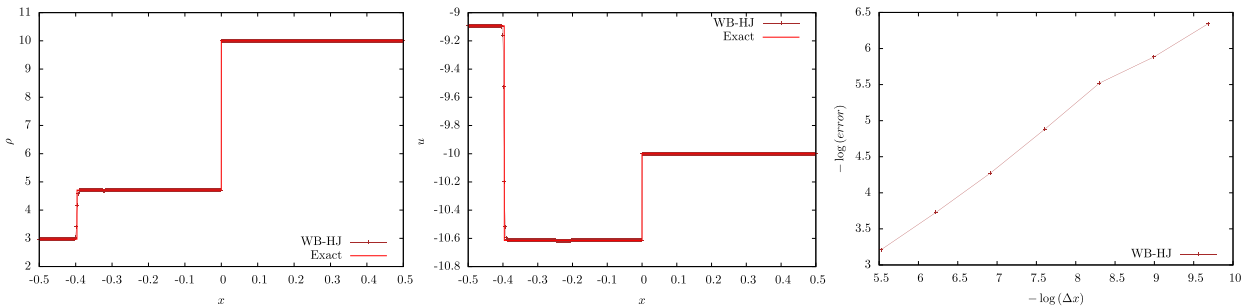
#### 3.1. A stationary flow for the new interface condition (2)

We start with a test case where initial conditions are Well-Balanced in the sense of (8) for the new interface condition (2). In practice, we consider the following values

$$(A, \rho, u, p)(x, t = 0) = \begin{cases} (2, 4.711, -10.614, 38.588), & \text{if } x < 0, \\ (1, 10, -10, 100), & \text{otherwise.} \end{cases} \quad (9)$$



**Fig. 1.** A stationary flow for the new interface condition (2): density  $\rho$  (top left), mean discharge  $Q$  (top middle), total enthalpy  $H$  (top right), physical entropy  $s$  (bottom left) and the variation of the new invariant  $I = Qu + A(p - p^*)$  (bottom right) at time  $t = 0.2$  s for an initial condition satisfying the new interface condition (2).



**Fig. 2.** A classical Riemann problem: density  $\rho$  (left), velocity  $u$  (middle) and  $L^1$  norm of the error for the density (right). The coarser and finer meshes used for the computation of the error contain  $N = 200$  and  $N = 15\,000$  regular cells.

The computational domain is  $[-0.5, 0.5]$ , the mesh contains 4 000 nodes. In Fig. 1, we plot the density, the mean discharge  $Q$ , the total enthalpy  $H$ , the physical entropy  $s$  and the variation of the new invariant  $I = Qu + A(p - p^*)$  at time  $t = 0.025$  s. Across the interface  $x = 0$ ,  $Q$  and  $H$  are preserved by the three schemes:  $1D+$  [1],  $WB-HJ$  and  $WB-KT$ . The difference between these last two schemes is on the invariant  $I$ :  $WB-HJ$  preserves  $I$ , while  $s$  is preserved by the  $WB-KT$  scheme.

### 3.2. A classical Riemann problem

One advantage of the  $WB-HJ$  scheme against the discrete  $1D+$  scheme [1] is that we can compare our result to the exact solution to a classical Riemann problem. The exact solution is computed with the classical approach where we use the three invariants ( $Q, I, H$ ) at the steady contact discontinuity. We consider the Riemann problem involving a shock wave on the left of the interface discontinuity

$$(A, \rho, u, p)(x, t = 0) = \begin{cases} (2, 2.970, -9.093, 20), & \text{if } x < 0, \\ (1, 10, -10, 100), & \text{otherwise.} \end{cases} \tag{10}$$

The final time of computation is  $t = 0.03$  s. In Fig. 2, we plot the density  $\rho$  and the velocity  $u$  obtained with 4 000 nodes. The approximated solution is close to the exact solution. We also perform a convergence study on the density variable. It may be checked that the convergence rate is very close to 0.8 in  $L^1$  norm.

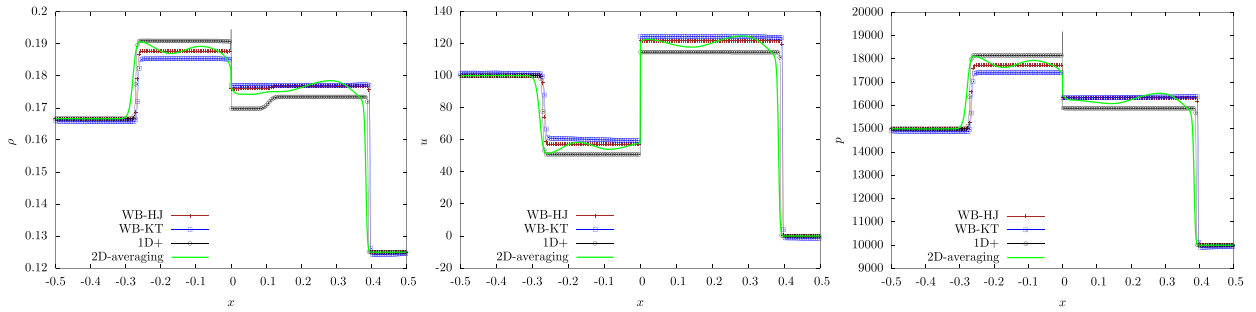


Fig. 3. A shock wave hitting a section contraction: density  $\rho$  (left), velocity  $u$  (middle) and pressure  $p$  (right).

### 3.3. A shock wave hitting a section contraction

We consider the following initial data

$$(A, \rho, u, p)(x, t = 0) = \begin{cases} (0.2, 0.167, 100, 1.5 \times 10^4), & \text{if } x < -0.1, \\ (0.2, 0.125, 0, 10^4), & \text{if } -0.1 \leq x < 0, \\ (0.1, 0.125, 0, 10^4), & \text{otherwise.} \end{cases} \quad (11)$$

A shock wave starting from  $x = -0.1$  travels to the right and hits the section contraction located at  $x = 0$ . Part of the shock wave is reflected and part is transmitted through the right. We compare the results obtained with the three one-dimensional schemes ( $WB-HJ$ ,  $WB-KT$  and  $1D^+$ ) with 4 000 nodes to the results obtained with a two dimensional set of Euler equations on a very fine mesh containing 216 000 cells (1 200 in  $x$ -direction), using Rusanov scheme. The two-dimensional results are averaged in the  $y$ -direction. The computation was done using the toolbox CDMATH [4,5]. In Fig. 3, we plot the density  $\rho$ , the velocity  $u$  and the pressure  $p$  at time  $t = 12 \times 10^{-4}$  s. We take the  $2D$ -averaging solution as the reference solution. The  $WB-HJ$  scheme gives better results than  $WB-KT$  and  $1D^+$  schemes. It remains to study the Riemann problem associated with system (1) together with the new invariant  $(Q, I, H)$  at the steady interface. It is also planned to develop a simple solver [3] associated with the new interface condition (2). This new solver could be more robust than the  $WB-HJ$  scheme. Finally, we emphasize that the present interface condition can be easily extended to other conservative systems.

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