Discrete Mathematics 311 (2011) 211–219

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On the hyperbolicity constant in graphs

José M. Rodríguez [a], José M. Sigarreta [b,*], Jean-Marie Vilaire [a], María Villeta [c]

[a] *Departamento de Matemáticas, Universidad Carlos III de Madrid, Av. de la Universidad 30, 28911 Leganés, Madrid, Spain*

[b] *Facultad de Matemáticas, Universidad Autónoma de Guerrero, Carlos E. Adame 5, Col. La Garita, Acapulco, Guerrero, Mexico*

[c] *Departamento de Estadística e Investigación Operativa III, Universidad Complutense de Madrid, Av. Puerta de Hierro s/n., 28040 Madrid, Spain*

ARTICLE INFO

Article history:
Received 4 November 2009
Received in revised form 29 October 2010
Accepted 8 November 2010
Available online 30 November 2010

Keywords:
Graphs
Connectivity
Geodesics
Gromov Hyperbolicity

ABSTRACT

If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X. The space X is δ*-hyperbolic* (in the Gromov sense) if, for every geodesic triangle T in X, every side of T is contained in a δ-neighborhood of the union of the other two sides. We denote by $\delta(X)$ the sharpest hyperbolicity constant of X, i.e. $\delta(X) := \inf\{\delta \geq 0 : X \text{ is } \delta\text{-hyperbolic}\}$. In this paper, we obtain several tight bounds for the hyperbolicity constant of a graph and precise values of this constant for some important families of graphs. In particular, we investigate the relationship between the hyperbolicity constant of a graph and its number of edges, diameter and cycles. As a consequence of our results, we show that if G is any graph with m edges with lengths $\{l_k\}_{k=1}^m$, then $\delta(G) \leq \sum_{k=1}^m l_k/4$, and $\delta(G) = \sum_{k=1}^m l_k/4$ if and only if G is isomorphic to C_m. Moreover, we prove the inequality $\delta(G) \leq \frac{1}{2} \operatorname{diam} G$ for every graph, and we use this inequality in order to compute the precise value $\delta(G)$ for some common graphs.

1. Introduction

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory; see, for instance [2–5,13–15,22–26,29,31,32,34,37,39,41].

The theory of Gromov hyperbolic spaces was initially applied to study finitely generated groups, field in which it turned out a crucial tool. After that, it was also successfully used in automatic groups (see [33]) and computer science. Recently, new practical uses came up, like its utilization in secure transmission of information through the Internet (see [22–26]), in the spread of viruses through the network (see [23,24]), or in the study of DNA data (see [13]).

In recent years, several investigators have shown their interest in proving that the metrics used in the geometric function theory are Gromov hyperbolic. For instance, the Klein–Hilbert and Kobayashi metrics are Gromov hyperbolic (under particular conditions on the domain of definition, see [8,27,6]), the Gehring–Osgood j-metric is Gromov hyperbolic, and the Vuorinen j-metric is not Gromov hyperbolic except in the punctured space (see [17]). Also, in [28] the hyperbolicity of the conformal modulus metric μ and the related so-called Ferrand metric λ^*, have been studied. Gromov hyperbolicity of the quasihyperbolic and the Poincaré metrics is also the subject of [1,7,10,18–21,30,34–40]. In particular, in [34, Theorem 2.19], [37, Theorem 2.5], [39, Theorem 3.7] and [40, Theorem 4.20] it is proved the equivalence of the hyperbolicity of Riemann surfaces (with their Poincaré metrics) and the hyperbolicity of a simple graph; with all these arguments it seems interesting to have hyperbolicity criteria for graphs.

* Corresponding author.
E-mail addresses: jomaro@math.uc3m.es (J.M. Rodríguez), josemariasigarretaalmira@hotmail.com (J.M. Sigarreta), jvilaire@math.uc3m.es (J.-M. Vilaire), mvilleta@estad.ucm.es (M. Villeta).

0012-365X/$ – see front matter
doi:10.1016/j.disc.2010.11.005

In our study on the hyperbolicity constant in graphs we use the notations of [16]. Now we give the basic facts about Gromov's spaces. If $\gamma : [a, b] \longrightarrow X$ is a continuous curve in a metric space (X, d), we can define the *length* of γ as

$$L(\gamma) := \sup\Big\{\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : a = t_0 < t_1 < \cdots < t_n = b\Big\}.$$

We say that γ is a *geodesic* if it is an isometry, i.e., $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t-s|$ for every $s, t \in [a, b]$. We say that X is a *geodesic metric space* if for every $x, y \in X$, there exists a geodesic joining x and y; we denote by $[xy]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected. If X is a graph, we use the classical notation $[u, v]$ for the edge of a graph joining the vertices u and v.

Throughout the paper we just consider graphs which are connected and locally finite (i.e., each ball contains just a finite number of edges). These conditions guarantee that the graph is a geodesic space (since we consider that every point in any edge of a graph G is a point of G, whether it is a vertex of G or not). We allow loops, multiple edges and edges of arbitrary lengths in our graphs.

If X is a geodesic metric space and $J = \{J_1, J_2, \ldots, J_n\}$, with $J_j \subseteq X$, we say that J is δ-*thin* if for every $x \in J_i$ we have that $d(x, \cup_{j\neq i} J_j) \le \delta$. We denote by $\delta(J)$ the sharpest thin constant of J, i.e. $\delta(J) := \inf\{\delta \ge 0 : J \text{ is } \delta\text{-thin}\}$. If $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$. The space X is δ-*hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e. $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \ge 0$. If X is hyperbolic, then $\delta(X) = \inf\{\delta \ge 0 : X \text{ is } \delta\text{-hyperbolic}\}$. The hyperbolicity constant $\delta(X)$ of a geodesic metric space can be viewed as a measure of how "tree-like" the space is, since those spaces with $\delta(X) = 0$ are precisely the metric trees.

Remark 1. Any bigon, i.e., a triangle with two equal vertices, in a δ-hyperbolic space is obviously δ-thin. Note that any geodesic polygon with $n \ge 3$ sides in a δ-hyperbolic space is $(n-2)\delta$-thin (we just have to decompose the polygon as a union of triangles).

Remark 2. There are several definitions of Gromov hyperbolicity (see e.g. [16]). These different definitions are equivalent in the sense that if X is δ_A-hyperbolic with respect to the definition A, then it is δ_B-hyperbolic with respect to the definition B, and there exist universal constants c_1, c_2 such that $c_1\delta_A \le \delta_B \le c_2\,\delta_A$ (see e.g. [16, p. 41]). However, for a fixed $\delta \ge 0$, the set of δ-hyperbolic graphs with respect to the definition A, is different, in general, from the set of δ-hyperbolic graphs with respect to the definition B. We have chosen this definition since it has a deep geometric meaning (see e.g. [16, Chapter 3]).

Remark 3. Some authors (see e.g. [13]) consider just those geodesic triangles in any graph G that have vertices in $V(G)$; by doing so we obtain a definition which is equivalent (in the sense of Remark 2) to our definition if every edge in G has length 1. However, if we want to deal with graphs with edges of arbitrary length, we must consider geodesic triangles with vertices in G.

We would like to point out that deciding whether or not a space is hyperbolic is usually extraordinarily difficult. Note that, first of all, we have to consider an arbitrary geodesic triangle T, and calculate the minimum distance from an arbitrary point P of T to the union of the other two sides of the triangle to which P does not belong to. And then we have to take supremum over all the possible choices for P and then over all the possible choices for T. It means that if our space is, for instance, an n-dimensional manifold and we select two points P and Q on different sides of a triangle T, the function F that measures the distance between P and Q is a $(3n+2)$-variable function ($3n$ variables describe the three vertices of T and two variables describe the points p and q in the closed curve given by T). In order to prove that our space is hyperbolic we would have to take the minimum of F over the variable that describes Q, and then the supremum over the remaining $3n+1$ variables, or at least prove that it is finite. Without disregarding the difficulty of solving a $(3n+2)$-variable minimax problem, note that the main obstacle is that we do not even know in an approximate way the location of geodesics in the space.

Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \longrightarrow Y$ is said to be an (α, β)-*quasi-isometry*, with $\alpha \ge 1,\ \beta \ge 0$, if for every $x, y \in X$:

$$\alpha^{-1} d_X(x, y) - \beta \le d_Y(f(x), f(y)) \le \alpha\, d_X(x, y) + \beta.$$

We say that f is a *quasi-isometry* if we disregard the constants α and β.

When $\alpha = 1$ and $\beta = 0$, f is said to be an *isometry*.

A quasi-isometry, in general, is not continuous as we can see in the following example.

The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = [x]$ is a $(1, 1)$-quasi-isometry but f is not continuous in $\mathbb{N}$.

Let X be a metric space, Y a non-empty subset of X and ε a real positive number. We define the ε-*neighborhood* of Y in X, as the set $\mathcal{V}_\varepsilon(Y) := \{x \in X : d_X(x, Y) \le \varepsilon\}$.

Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry $f : X \longrightarrow Y$ and a real number $\varepsilon \ge 0$ such that $f(X)$ is ε-*full* in Y, i.e., $\mathcal{V}_\varepsilon(f(X)) = Y$. An (α, β)-*quasigeodesic* of a metric space X is a (α, β)-quasi-isometry $\gamma : I \longrightarrow X$, where I is an interval of $\mathbb{R}$.

If D is a closed subset of X, we always consider in D the *inner metric* obtained by the restriction of the metric in X, that is

$$d_D(z, w) := \inf\big\{L_X(\gamma) : \gamma \subset D \text{ is a continuous curve joining } z \text{ and } w\big\} \geq d_X(z, w).$$

Consequently, $L_D(\gamma) = L_X(\gamma)$ for every curve $\gamma \subset D$.

The following are interesting examples of hyperbolic spaces. $\mathbb{R}$ is 0-hyperbolic: in fact, any point of a geodesic triangle in the real line belongs to two sides of the triangle simultaneously, and therefore we can conclude that $\mathbb{R}$ is 0-hyperbolic. $\mathbb{R}^2$ is not hyperbolic: it is clear that triangles with arbitrarily large diameter can be drawn, and then $\mathbb{R}^2$ is not hyperbolic with the Euclidean metric. This argument can be generalized in a similar way to higher dimensions: a normed vector space E is hyperbolic if and only if $\dim E = 1$. Every metric tree of arbitrary length is 0-hyperbolic. In fact, any point of a geodesic triangle in a tree belongs simultaneously to two sides of the triangle. Every bounded metric space X is $(\operatorname{diam} X)$-hyperbolic. Every simply connected complete Riemannian manifold with sectional curvature verifying $K \leq -k^2$, for some positive constant k, is hyperbolic. We refer to [9,11,12,16] for more background and further results.

2. Hyperbolicity constant in graphs

Since it is not easy to guarantee the hyperbolicity, it is interesting to relate the hyperbolicity constant with other important parameters of a graph or with some properties of the graph.

We start with some results which relate hyperbolicity with local hyperbolicity. We say that a sequence of closed sets $\{K_n\}_n$ in a metric space X is an *exhaustion* of X if $K_n \subseteq K_{n+1}$ for every n and given any compact set $K \subset X$, there exists N with $K \subseteq K_N$.

Theorem 4. *Assume that there exist $\delta \geq 0$ and an exhaustion $\{K_n\}_n$ of a geodesic metric space X such that K_n is δ-hyperbolic for every n. Then X is δ-hyperbolic.*

Proof. Let $T = [xy] \cup [yz] \cup [zx]$ be any geodesic triangle in X, and let $u \in [xy]$. $\mathcal{V}_\delta(T)$ is contained in K_N for some N. Since T is a geodesic triangle in X, it is also a geodesic triangle in K_N. Since K_N is δ-hyperbolic, there exists $v \in [yz] \cup [zx]$ such that $d_X(u, v) \leq d_{K_N}(u, v) \leq \delta$. □

With the same aim, we relate the hyperbolicity of a graph with the hyperbolicity of its subgraphs.

We say that a subgraph Γ of G is *isometric* if $d_\Gamma(x, y) = d_G(x, y)$ for every $x, y \in \Gamma$.

Lemma 5. *If Γ is an isometric subgraph of G, then $\delta(\Gamma) \leq \delta(G)$.*

Proof. Note that, by hypothesis, $d_\Gamma(x, y) = d_G(x, y)$ for every $x, y \in \Gamma$; therefore, every geodesic triangle in Γ is a geodesic triangle in G. Hence, $\delta(\Gamma) \leq \delta(G)$. □

In [16, p. 87] we can find the following result.

Lemma 6 (*Invariance of Hyperbolicity*)**.** *Let $f : X \longrightarrow Y$ be an (α, β)-quasi-isometry between two geodesic metric spaces. If Y is δ-hyperbolic, then X is δ'-hyperbolic, where δ' is a constant which just depends on δ, α and β.*

Besides, if f is ε-full for some $\varepsilon \geq 0$, then X is hyperbolic if and only if Y is hyperbolic. Furthermore, if X is δ'-hyperbolic, then Y is δ-hyperbolic, where δ is a constant which just depends on δ', α, β and ε.

Theorem 7. *Assume that Γ is a subgraph of a graph G such that there exist $\alpha \geq 1$ and $\beta \geq 0$ with $d_\Gamma(x, y) \leq \alpha d_G(x, y) + \beta$, for every $x, y \in \Gamma$. If G is hyperbolic, then Γ is hyperbolic. Moreover, if there exists a constant c such that every connected component E of $G \setminus \Gamma$ satisfies $\operatorname{diam}_G E \leq c$, then G is hyperbolic if and only if Γ is hyperbolic.*

Proof. By Lemma 6, it suffices to note that the inclusion $i : \Gamma \longrightarrow G$ is an (α, β)-quasi-isometry, since $d_G(x, y) \leq d_\Gamma(x, y)$ for every $x, y \in \Gamma$. Furthermore, if every connected component E of $G \setminus \Gamma$ satisfies $\operatorname{diam}_G E \leq c$, then i is c-full. □

The next result relates δ with an important parameter of a graph, the diameter. It is a simple but useful result.

Theorem 8. *In any graph G the inequality $\delta(G) \leq \frac{1}{2} \operatorname{diam} G$ holds, and furthermore, it is sharp.*

Proof. Let us consider a geodesic side γ in any geodesic triangle $T \subset G$. Denote by x, y the endpoints of γ, and by γ_1, γ_2 the other sides of T. For any $p \in \gamma$, it is clear that

$$d(p, \gamma_1 \cup \gamma_2) \leq d(p, \{x, y\}) \leq \frac{1}{2} L(\gamma) \leq \frac{1}{2} \operatorname{diam} G,$$

and consequently, $\delta(G) \leq \frac{1}{2} \operatorname{diam} G$. □

The equality in Theorem 8 is attained by many graphs, as shown in the following theorem.

We will also need the following result (see [37, Lemma 2.1]). As usual, by *cycle* we mean a simple closed curve, i.e. a path with different vertices, unless the last vertex, which is equal to the first one.

Lemma 9. *Let us consider a geodesic metric space X. If every geodesic triangle in X which is a cycle, is δ-thin, then X is δ-hyperbolic.*

This lemma has the following direct consequence.

Corollary 10. *In any geodesic metric space X,*

$$\delta(X) = \sup\Big\{\delta(T) : T \text{ is a geodesic triangle which is a cycle}\Big\}.$$

Theorem 11. *The following graphs with edges of length 1 have these precise values of δ.*

- *The path graphs verify $\delta(P_n) = 0$ for every $n \geq 1$.*
- *The cycle graphs verify $\delta(C_n) = n/4$ for every $n \geq 3$.*
- *The complete graphs verify $\delta(K_1) = \delta(K_2) = 0$, $\delta(K_3) = 3/4$, $\delta(K_n) = 1$ for every $n \geq 4$.*
- *The complete bipartite graphs verify $\delta(K_{1,1}) = \delta(K_{1,2}) = \delta(K_{2,1}) = 0$, $\delta(K_{m,n}) = 1$ for every $m, n \geq 2$.*
- *The Petersen graph P verifies $\delta(P) = 3/2$.*
- *The wheel graph with n vertices W_n verifies $\delta(W_4) = \delta(W_5) = 1$, $\delta(W_n) = 3/2$ for every $7 \leq n \leq 10$, and $\delta(W_n) = 5/4$ for $n = 6$ and for every $n \geq 11$.*

Furthermore, the graphs C_n and K_n for every $n \geq 3$, $K_{m,n}$ for every $m, n \geq 2$, the Petersen graph and W_n for every $4 \leq n \leq 10$, verify $\delta(G) = \frac{1}{2}$ diam G.

Proof. It is clear that $\delta(P_n) = 0$, $\delta(K_1) = \delta(K_2) = 0$ and $\delta(K_{1,1}) = \delta(K_{1,2}) = \delta(K_{2,1}) = 0$, since these graphs are trees.

Since $\operatorname{diam} C_n = n/2$, Theorem 8 gives that $\delta(C_n) \leq n/4$. Let us consider a bigon with two vertices $\{x, y\}$ at a distance $n/2$, with sides γ_1, γ_2 such that $\gamma_1 \cup \gamma_2 = C_n$. The midpoint p of γ_1 satisfies $d(p, \gamma_2) = d(p, \{x, y\}) = n/4$. Consequently, $\delta(C_n) = n/4$. We also have $\delta(K_3) = 3/4$, since $K_3 = C_3$.

If $n \geq 4$, then the diameter of the complete graphs K_n is $\operatorname{diam} K_n = 2$. Therefore, Theorem 8 gives that $\delta(K_n) \leq 1$. Consider a cycle g of length 4 in K_n. Fix a point x in the midpoint of a fixed edge of g; let us consider the point $y \in g$ at a distance 2 from x and the bigon with vertices $\{x, y\}$ and sides γ_1, γ_2 such that $\gamma_1 \cup \gamma_2 = g$. The midpoint p of γ_1 satisfies $d(p, \gamma_2) = d(p, \{x, y\}) = 1$. Hence, $\delta(K_n) = 1$.

The argument for $K_{m,n}$, with $m, n \geq 2$, is similar to this last one.

Let us fix two non-adjacent points x, y in the "exterior" pentagon P_0 of the Petersen graph P and consider the path with length three $g_1 \subset P_0$ joining x and y. Let g_2 be the path with length three not contained in P_0 joining x and y. Let p be the midpoint of g_1. Then we have $\delta(P) \geq d(p, g_2) = d(p, \{x, y\}) = 3/2$.

Note that $\operatorname{diam} V(P) = 2$. Given two points $p_1, p_2 \in P$, let us denote by v_i a vertex with $d(p_i, v_i) \leq 1/2$ for $i = 1, 2$. Then $d(p_1, p_2) \leq d(p_1, v_1) + \operatorname{diam} V(P) + d(p_2, v_2) \leq 1/2 + 2 + 1/2 = 3$, and $\operatorname{diam} P \leq 3$. Hence, Theorem 8 gives that $\frac{3}{2} \leq \delta(P) \leq \frac{1}{2} \operatorname{diam} P \leq \frac{3}{2}$, and we deduce $\delta(P) = \frac{3}{2}$ and $\operatorname{diam} P = 3$.

The wheel graph W_4 is isometric to K_4, and then $\delta(W_4) = 1$. Theorem 8 gives that $\delta(W_n) \leq \frac{1}{2} \operatorname{diam} W_n$. It is not difficult to check that $\operatorname{diam} W_4 = \operatorname{diam} W_5 = 2$, $\operatorname{diam} W_6 = 5/2$ and $\operatorname{diam} W_n = 3$ for every $n \geq 7$. Since W_5 contains a cycle with length 4, then $\delta(W_5) \geq 1$; since $\delta(W_5) \leq \frac{1}{2} \operatorname{diam} W_5 = 1$, we conclude that $\delta(W_5) = 1$.

Let us consider the cycle C in W_n with length $n - 1$ containing every vertex minus the central vertex.

Let x be the midpoint of any edge in C, and consider the points y and z in C at distances $(n - 1)/2$ and $(n - 1)/4$, respectively, from x. Then $T := \{x, y, z\}$ is a geodesic triangle with $[xy] \cup [yz] \cup [zx] = C$ if $n \in \{6, 7\}$ (recall that $\operatorname{diam} W_6 = 5/2$ and $\operatorname{diam} W_7 = 3$). The midpoint p of $[xy]$ verifies $d(p, [yz] \cup [zx]) = d(p, \{x, y\}) = (n - 1)/4$, and consequently, $\delta(W_n) \geq (n - 1)/4$ if $n \in \{6, 7\}$. Therefore, $\delta(W_6) \geq 5/4$ and $\delta(W_7) \geq 3/2$. Since $\operatorname{diam} W_6 = 5/2$ and $\operatorname{diam} W_7 = 3$, we have that $\delta(W_6) = 5/4$ and $\delta(W_7) = 3/2$.

Let x be the midpoint of any edge in C, and consider the points y and z in C at distances 3 and $(n - 4)/2$, respectively, from x in C. Then $T := \{x, y, z\}$ is a geodesic triangle with $[xy] \cup [yz] \cup [zx] = C$ if $n \in \{8, 9, 10\}$ (recall that $\operatorname{diam} W_n = 3$ for every $n \geq 7$). The midpoint p of $[xy]$ verifies $d(p, [yz] \cup [zx]) = d(p, \{x, y\}) = 3/2$, and consequently, $\delta(W_n) \geq 3/2$ if $n \in \{8, 9, 10\}$. Since $\operatorname{diam} W_n = 3$ for every $n \geq 7$, we have that $\delta(W_n) = 3/2$ if $n \in \{8, 9, 10\}$.

If $n \geq 11$, then the cycle C in W_n has length $n - 1 \geq 10$, and it is not a geodesic triangle, since any geodesic γ verifies $L(\gamma) \leq \operatorname{diam} W_n = 3$. Let us consider the cycle C' in W_n with length 9 containing eight consecutive vertices in C and the central vertex v_0 in W_n. Let x be the point in C' at a distance 9/2 from v_0. Consider the points y and z in C' at a distance 3 from v_0. Then $T := \{x, y, z\}$ is a geodesic triangle with $[xy] \cup [yz] \cup [zx] = C'$, since $n \geq 11$. The point q in $[xy]$ with $d(p, x) = 5/4$ verifies $d(p, [yz] \cup [zx]) = d(p, x) = 5/4$, and consequently, $\delta(W_n) \geq \delta(T) \geq 5/4$ if $n \geq 11$. We are proving that this triangle is, in fact, an extremal triangle.

Let us consider any geodesic triangle $T = \{x, y, z\}$ in W_n with $n \geq 11$. By Corollary 10, we can assume that T is also a cycle. Since the cycle T is not C, then it must be a cycle C'' in W_n with length $m \geq 3$ containing $m - 1$ consecutive vertices in C (which we will call $v_1, \ldots, v_{m-1}$) and the central vertex v_0 in W_n. Note that $m \leq 9$, since any geodesic γ verifies $L(\gamma) \leq \operatorname{diam} W_n = 3$.

Assume first that $x = v_0$ is a vertex of T. Since every point $a \in W_n$ verifies $d(a, v_0) \leq 3/2$, then $L([xy]), L([xz]) \leq 3/2$ and hence, $d(p_1, [xz] \cup [yz]) \leq d(p_1, \{x, y\}) \leq 3/4$ for every $p_1 \in [xy]$ and $d(p_2, [xy] \cup [yz]) \leq d(p_2, \{x, z\}) \leq 3/4$ for every $p_2 \in [xz]$. Without loss of generality, we can assume that $d(y, v_1) \leq d(z, v_1)$.

If $d(y, v_0) < 1$, let us denote by y' the point with $y' \in [v_2, v_3]$ and $d(y, y') = 2$. Then $z \in [v_1, v_2] \cup [v_2 y']$, since

$$d(y, y') = d(y, v_1) + d(v_1, v_2) + d(v_2, y') = 2,$$
$$d(v_2, y') = 1 - d(y, v_1),$$
$$d(v_3, y') = 1 - d(v_2, y') = d(y, v_1) = 1 - d(y, v_0),$$
$$d(y, y') = d(y, v_0) + d(v_0, v_3) + d(v_3, y') = 2;$$

therefore, $L([yz]) \le 2$ and $d(p_3, [xy] \cup [xz]) \le d(p_3, \{y, z\}) \le 1$ for every $p_3 \in [yz]$.

If $1 \le d(y, v_0) < 3/2$, then $y \in [v_1, v_2]$; let us denote by y'' the point with $y'' \in [v_3, v_4]$ and $d(y, y'') = 5/2$. Therefore, $z \in [yv_2] \cup [v_2, v_3] \cup [v_3 y'']$, since

$$d(y, y'') = d(y, v_2) + d(v_2, v_3) + d(v_3, y'') = 5/2,$$
$$d(v_3, y'') = 3/2 - d(y, v_2),$$
$$d(v_4, y'') = 1 - d(v_3, y'') = 1 - 3/2 + d(y, v_2) = 1/2 - d(y, v_1),$$
$$d(y, y'') = d(y, v_1) + d(v_1, v_0) + d(v_0, v_4) + d(v_4, y'') = 5/2.$$

Hence, $L([yz]) \le 5/2$ and $d(p_3, [xy] \cup [xz]) \le 5/4$ for every $p_3 \in [yz]$.

If $d(y, v_0) = 3/2$, then y is the midpoint of $[v_1, v_2]$; let us denote by y''' the midpoint of $[v_4, v_5]$. Since $d(y, y''') = 3 = \operatorname{diam} W_n$, we have that $z \in [yv_2] \cup [v_2, v_3] \cup [v_3, v_4] \cup [v_4 y''']$. If $p_3 \in [yz]$ verifies $d(p_3, v_3) \ge 1/4$, then $d(p_3, [xy] \cup [xz]) \le d(p_3, \{y, z\}) \le 3/2 - 1/4 = 5/4$. If $p_3 \in [yz]$ verifies $d(p_3, v_3) \le 1/4$, then $d(p_3, [xy] \cup [xz]) \le d(p_3, v_0) \le d(p_3, v_3) + d(v_3, v_0) \le 1/4 + 1 = 5/4$.

Hence, if v_0 is a vertex of T, we have proved that $\delta(T) \le 5/4$. If v_0 is not a vertex of T, a similar argument gives also $\delta(T) \le 5/4$. Therefore, $\delta(W_n) \le 5/4$ for every $n \ge 11$. Hence, $\delta(W_n) = 5/4$ for every $n \ge 11$.

Finally, it is straightforward that the graphs $C_n, K_n, K_{m,n}$ and W_n verify $\delta(G) = \frac{1}{2} \operatorname{diam} G$ (for the values of n, m appearing in the statement of the theorem), since the hyperbolicity constants of these graphs are known. □

It is interesting to remark the unexpected behavior of $\delta(W_n)$. This illustrates the difficulty of the study of the hyperbolicity constant. The final conclusion of Theorem 11 shows that it is not easy to characterize the graphs verifying $\delta(G) = \frac{1}{2} \operatorname{diam} G$ (even if G has every edge with length 1).

We are interested in other classes of graphs for which we have $\delta(G) = \frac{1}{2} \operatorname{diam} G$.

Theorem 12. *Let $C_{a,b,c}$ be the graph with two vertices and three edges joining them with lengths $a \le b \le c$. Then $\delta(C_{a,b,c}) = (c + \min\{b, 3a\})/4$.*

Proof. Let us denote by x_1, x_2, the vertices of $C_{a,b,c}$, and by A, B, C the edges with lengths a, b, c, respectively.

Assume first that $b \le 3a$. Let x_0 be the point in C with $d(x_0, x_1) = (c + a)/2$ and y_0 be the point in B with $d(y_0, x_1) = (b - a)/2$. Consider the geodesics $[x_0 x_1] \subset C$, $[x_1 y_0] \subset B$ and $[x_0 y_0] = [x_0 x_1] \cup [x_1 y_0]$. Note that $L([x_0 y_0]) = (c + b)/2$. Let p be the point in $[x_0 x_1] \subset C$ with $d(p, x_0) = d(p, y_0) = (c + b)/4$. Then the geodesic bigon $B = \{x_0, y_0\}$ given by the geodesics $[x_0 x_1] \cup [x_1 y_0]$ and $[x_0 x_2] \cup [x_2 y_0]$ has $\delta(B) \ge (c+b)/4$, since $d(p, x_2) = (c+a)/2 - (c+b)/4 + a = (c-b+6a)/4 \ge (c+b)/4$ (since $b \le 3a$), and hence, $d(p, [x_0 x_2] \cup [x_2 y_0]) = d(p, \{x_0, y_0, x_2\}) = (c + b)/4$. Since $\operatorname{diam} C_{a,b,c} = (c + b)/2$, we have that $\delta(C_{a,b,c}) \le (c + b)/4$ by Theorem 8. Therefore, in this case we have $\delta(C_{a,b,c}) = (c + b)/4 = (c + \min\{b, 3a\})/4$.

Now assume that $b > 3a$. Let us consider a geodesic triangle T; in order to compute $\delta(C_{a,b,c})$ without loss of generality, we can assume that T is a cycle, by Corollary 10. If the closed curve given by T is $C \cup A$, then $\delta(T) \le (c + a)/4 < (c + 3a)/4$ and the first inequality is attained by taking T as a geodesic bigon. If the closed curve given by T is $B \cup A$, then $\delta(T) \le (b + a)/4 < (c + 3a)/4$ and the first inequality is attained by taking T as a geodesic bigon.

Let us consider an arbitrary geodesic triangle $T := \{x, y, z\}$ lying in $C \cup B$, and let p be any point in T. Without loss of generality, we can assume that p belongs to the geodesic side $[xy]$ of T.

Assume first that $[xy] \subset B$; then $[xy]$ is contained in the cycle $B \cup A$, which has length $b + a$; since $[xy]$ is a geodesic, then

$$L([xy]) = d(x, y) \le \frac{1}{2} L(B \cup A) = \frac{b + a}{2} < \frac{c + 3a}{2},$$
$$d(p, [xz] \cup [yz]) \le d(p, \{x, y\}) \le \frac{1}{2} L([xy]) < \frac{c + 3a}{4}.$$

In a similar way, if $[xy] \subset C$, then $d(p, [xz] \cup [yz]) \le (c + a)/4 < (c + 3a)/4$. Hence, by symmetry, we can assume that $x_1 \in [xy]$, $x \in C$, $y \in B$ and $p \in [xx_1]$.

Let us define $U := d(x, x_1)$ and $V := d(y, x_1)$. Since $[xx_1]$ is contained in the cycle $C \cup A$, which has length $c + a$, we have $L([xx_1]) = d(x, x_1) \le (c + a)/2$. Then we have $U \in [0, (c + a)/2]$. Since $[x_1 y] \subset B \cup A$ and $[xy] \subset C \cup B$ we deduce, in a similar way, that $V \in [0, (b + a)/2]$ and $U + V \le (c + b)/2$. Let γ_2, γ_3 be the other geodesics in T. We denote by t the distance $d(p, x) =: t$.

Define $U_0 := (c - a)/2$ and $V_0 := (b + a)/2$; we know that $V \le V_0$.

Assume that $U \le U_0$. Since $V \le V_0$, we have that $U + V \le (c+b)/2$. Then $d(p, \gamma_2 \cup \gamma_3) = \min\{t,\ U - t + a,\ U - t + V\}$ and we have (since $U \le U_0$, $V \le V_0$ and $a \le V_0$)

$$\begin{aligned}\max_{p\in[xx_1]} d(p, \gamma_2 \cup \gamma_3) &= \max_{t\in[0,U]} \min\{t,\ U - t + a,\ U - t + V\}\\ &\le \max_{t\in[0,U_0]} \min\{t,\ U_0 - t + a,\ U_0 - t + V_0\}\\ &= \max_{t\in[0,U_0]} \min\{t,\ U_0 - t + a\} = \frac{U_0 + a}{2} = \frac{c+a}{4} < \frac{c+3a}{4}.\end{aligned}$$

Now assume that $(c-a)/2 < U$; recall that $U \le (c+a)/2$. Since we need $U + V \le (c+b)/2$, then

$$V \le \min\left\{\frac{b+a}{2},\ \frac{c+b}{2} - U\right\} = \frac{c+b}{2} - U =: V_1,$$

and

$$\begin{aligned}\max_{p\in[xx_1]} d(p, \gamma_2 \cup \gamma_3) &= \max_{t\in[0,U]} \min\{t,\ U - t + a,\ U - t + V\}\\ &\le \max_{t\in[0,U]} \min\{t,\ U - t + a,\ U - t + V_1\} = \max_{t\in[0,U]} \min\left\{t,\ U - t + a,\ \frac{c+b}{2} - t\right\}.\end{aligned}$$

Since

$$U + a \le \frac{c+a}{2} + a = \frac{c+3a}{2} < \frac{c+b}{2},$$

we deduce that

$$\begin{aligned}\max_{p\in[xx_1]} d(p, \gamma_2 \cup \gamma_3) &\le \max_{t\in[0,U]} \min\left\{t,\ U - t + a,\ \frac{c+b}{2} - t\right\} = \max_{t\in[0,U]} \min\left\{t,\ U - t + a\right\}\\ &\le \max_{t\in[0,U]} \min\left\{t,\ \frac{c+3a}{2} - t\right\} = \frac{c+3a}{4}.\end{aligned}$$

Since every inequality can be attained, we deduce $\max_{p\in[xx_1]} d(p, \gamma_2 \cup \gamma_3) = (c+3a)/4$. Therefore, we have $\delta(C_{a,b,c}) = (c+3a)/4 = (c + \min\{b, 3a\})/4$. □

Proposition 13. *$\delta(C_{a,b,c}) = \frac{1}{2}\operatorname{diam} C_{a,b,c}$ if and only if $b \le 3a$.*

Proof. Using Theorem 12 and $\operatorname{diam} C_{a,b,c} = (c+b)/2$, we have $\delta(C_{a,b,c}) = \frac{1}{2}\operatorname{diam} C_{a,b,c}$ if and only if $\frac{c+b}{4} = \frac{1}{2}\operatorname{diam} C_{a,b,c} = \delta(C_{a,b,c}) = \frac{c+\min\{b,3a\}}{4}$, and this holds if and only if $b \le 3a$. □

In a subsequent work (see [31, Proposition 20]) the authors, using Theorem 12, obtain the following general result. We include a proof for the sake of completeness.

Corollary 14. *Denote by $C_{a_1,a_2,\dots,a_k}$ the graph with two vertices and k edges joining them with lengths $a_1 \le a_2 \le \cdots \le a_k$. Then*

(i) $\delta(C_{a_1,a_2,\dots,a_k}) = \frac{a_k+\min\{a_{k-1},3a_1\}}{4}$.
(ii) $\delta(C_{a_1,a_2,\dots,a_k}) = \frac{1}{2}\operatorname{diam} C_{a_1,a_2,\dots,a_k}$ *if and only if* $a_{k-1} \le 3a_1$.

Proof. Let us denote by x_1, x_2, the vertices of $C_{a_1,a_2,\dots,a_k}$, and by $A_1, A_2, \dots A_k$ the edges with lengths $a_1, a_2, \dots, a_k$, respectively.

Let us consider a geodesic triangle T; in order to compute $\delta(C_{a_1,a_2,\dots,a_k})$ without loss of generality, we can assume that T is a cycle, by Corollary 10. Then the closed curve given by T is $A_i \cup A_j$ with $1 \le i < j \le k$.

If $i = 1$, then $A_1 \cup A_j$ is an isometric subgraph of $C_{a_1,a_2,\dots,a_k}$. If $i > 1$, then $A_1 \cup A_i \cup A_j$ is an isometric subgraph of $C_{a_1,a_2,\dots,a_k}$. Hence, by Lemma 5 and Theorem 12, we have

$$\begin{aligned}\delta(C_{a_1,a_2,\dots,a_k}) &= \max\left\{\max_{1<j\le k} \delta(C_{a_1,a_j}),\ \max_{1<i<j\le k} \delta(C_{a_1,a_i,a_j})\right\}\\ &= \max\left\{\max_{1<j\le k} \frac{a_j + a_1}{4},\ \max_{1<i<j\le k} \frac{a_j + \min\{a_i, 3a_1\}}{4}\right\}\\ &= \max\left\{\frac{a_k + a_1}{4},\ \frac{a_k + \min\{a_{k-1}, 3a_1\}}{4}\right\}\\ &= \frac{a_k + \min\{a_{k-1}, 3a_1\}}{4}. \quad \square\end{aligned}$$

3. Bounds on the hyperbolicity constant in a graph

A path γ between two points in a graph is called a *bridge* if the internal vertices of γ have degree two. In particular, any edge is a bridge, since it has no internal vertices.

Theorem 15. *Assume that γ is a bridge in a graph G and γ' is a geodesic in the closure of $G \setminus \gamma$ joining the same points than γ. Then* $\max\{L(\gamma), L(\gamma')\} \le 4\delta(G)$.

Proof. Let us denote by a and b, the endpoints of γ.

Assume first that γ is a geodesic joining a and b; then $L(\gamma) \le L(\gamma')$. Let c be a point of γ' such that $d_G(a, c) = d_G(b, c) = L(\gamma')/2$; since γ' is a geodesic in the closure of $G \setminus \gamma$, then γ' is the union of two geodesics (in G) $[ac]$ and $[cb]$. Let us consider the geodesic triangle T with sides γ, $[ac]$, $[cb]$. Let u be the midpoint of $[ac]$. Since γ is a bridge and γ' is a geodesic in the closure of $G \setminus \gamma$, we have $d_G(u, \{a, c\}) = d_G(u, \gamma \cup [cb])$. Hence, $\delta(T) \ge d_G(u, \{a, c\}) = L(\gamma')/4$, and we conclude $L(\gamma) \le L(\gamma') \le 4\delta(G)$.

Now assume that γ is not a geodesic; then γ' is a geodesic in G (since γ is a bridge), and $L(\gamma') \le L(\gamma)$. Using the previous argument, changing the role of γ and γ', we also deduce $L(\gamma') \le L(\gamma) \le 4\delta(G)$. □

A curve γ is a *minimal closed geodesic* if γ is a cycle such that for any two points of γ, there exists a geodesic γ' joining them with $\gamma' \subset \gamma$.

Remark 16. Every bridge is contained in a minimal closed geodesic.

Theorem 17. *If G is any graph, then*

$$\delta(G) \ge \frac{1}{4} \sup\{L(\gamma) : \gamma \text{ is a minimal closed geodesic}\}.$$

Proof. Consider any fixed minimal closed geodesic γ. Let $x, y \in \gamma$ such that $d_G(x, y) = L(\gamma)/2$. Then $T = \{x, y\}$ is a bigon, with two geodesics γ_1, γ_2 verifying $\gamma_1 \cup \gamma_2 = \gamma$. Let us consider $u \in \gamma_1$ with $d_G(u, x) = d_G(u, y) = L(\gamma)/4$. Since γ is a minimal closed geodesic, then $d_G(u, \gamma_2) = d_G(u, \{x, y\}) = L(\gamma)/4$, and $\delta(G) \ge \delta(T) \ge L(\gamma)/4$. This gives the result. □

It is interesting to obtain inequalities involving the hyperbolicity constant and other important parameters of a graph. In this sense we obtain the following theorems.

Theorem 18. *Let G be a graph with edges of length 1. If there exist a cycle g in G with length $L(g) \ge 5$ and a vertex $w \in g$ with degree two, then $\delta(G) \ge 5/4$.*

Proof. Let us denote by $u, v \in g$ the two vertices which are the neighbors of w, and by g_1 the subcurve of length 2 joining u and v and containing w. Since the closure h of $g \setminus g_1$ is a curve in G joining u and v with $L(h) \ge 3$ and $h \cap g_1 = \{u, v\}$, the following set M is non-empty

$$M := \left\{\sigma \text{ is a curve in } G \text{ joining } u \text{ and } v \text{ with } L(\sigma) \ge 3 \text{ and } \sigma \cap g_1 = \{u, v\}\right\}.$$

Let us consider a curve g_2 in M verifying $L(g_2) = \min\{L(\sigma) : \sigma \in M\}$; since $g_2 \in M$, we have $L(g_2) \ge 3$.

Let z be the midpoint of g_2; it is clear that the two subarcs of g_2 joining z with u and v are geodesics by the minimizing property of g_2. Since w has degree two and u, v are the neighbors of w, the two subarcs γ_1, γ_2 of $\gamma := g_1 \cup g_2$ joining z with w are geodesics.

Let us consider the bigon $\{w, z\}$ with sides γ_1, γ_2, and the point $p \in \gamma_1$ at a distance 5/4 from w. Since $L(\gamma_1) = L(\gamma_2) = L(\gamma)/2 \ge 5/2$, we deduce $d(p, \{w, z\}) \ge 5/4$. If σ is any curve joining p and $\gamma_2 \setminus \{w, z\}$, then $L(\sigma \cap \gamma_1) \ge 1/4$. Let $q \in V(G)$ be the last point of σ in γ_1; then $d(p, \gamma_2) = L(\sigma \cap \gamma_1) + d(q, \gamma_2) \ge 1/4 + 1 = 5/4$. Then $\delta(G) \ge 5/4$. □

Theorem 19. *Let G be any graph with m edges. Then $\delta(G) \le \sum_{k=1}^{m} l_k/4$, where $l_k = L(e_k)$ for every edge $e_k \in E(G)$. Moreover, $\delta(G) = \sum_{k=1}^{m} l_k/4$ if and only if G is isomorphic to C_m.*

Proof. It is not difficult to check the result for $m = 1$ (then the extremal graph is a vertex with a loop) and for $m = 2$ (in this case the extremal graph has two vertices and a double edge). Now assume that $m \ge 3$.

Let T be any fixed geodesic triangle, $\gamma_1, \gamma_2, \gamma_3$ be the geodesics joining the vertices of the triangle, and $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ be the closed curve given by T. In order to compute $\delta(G)$, by Corollary 10, we can assume that γ is a cycle.

We have $L(\gamma) \le \sum_{k=1}^{m} l_k$, and hence $L(\gamma_j) \le \sum_{k=1}^{m} l_k/2$, for every j. If $x \in \gamma_j =: [yz]$, then $d(x, \{y, z\}) \le L(\gamma_j)/2 \le \sum_{k=1}^{m} l_k/4$ and consequently $\delta(T) \le \sum_{k=1}^{m} l_k/4$. Hence, $\delta(G) \le \sum_{k=1}^{m} l_k/4$.

If $\delta(G) = \sum_{k=1}^{m} l_k/4$, then every inequality in the previous argument must be an equality. In particular, we have that $L(\gamma) = \sum_{k=1}^{m} l_k$ and we deduce $G = \gamma$. Therefore, we conclude that G is a cycle and, consequently, it is isomorphic to C_m. □

We deduce the following result for graphs with edges of length 1.

Corollary 20. *Let G be any graph with m edges. If every edge has length* 1*, then $\delta(G) \leq m/4$. Moreover, $\delta(G) = m/4$ if and only if G is isometric to C_m.*

Given a graph G, we say that a family of subgraphs $\{G_n\}_n$ of G is a *tree-decomposition* of G if $\cup_n G_n = G$, $G_n \cap G_m$ is either a vertex or the empty set for each $n \neq m$, and if the graph R is a tree, where $V(R) = \{v_n\}_n$ and $[v_n, v_m] \in E(R)$ if and only if $G_n \cap G_m \neq \emptyset$.

We will need the following result (see [4, Theorem 5]).

Lemma 21. *Let G be a graph and $\{G_n\}_n$ be a tree-decomposition of G. Then $\delta(G) = \sup_n \delta(G_n)$.*

Furthermore, we have the following result.

Theorem 22. *Let G be any graph with m edges. If every edge has length* 1 *and G is not isometric to C_m, then $\delta(G) \leq (m-1)/4$. Moreover, $\delta(G) = (m-1)/4$ if and only if G is isometric to C_{m-1} with an edge e_0 attached, and we have either that e_0 is a loop or that the other vertex of e_0 has degree* 1 *or e_0 joins two different vertices of C_{m-1} at a distance (in C_{m-1}) less than or equal to* 3*.*

Proof. Let T be a geodesic triangle, $\gamma_1, \gamma_2, \gamma_3$ be the geodesics joining the vertices of the triangle, and $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ be the closed curve given by T. In order to compute $\delta(G)$, by Corollary 10, we can assume that γ is a cycle.

If $L(\gamma) = m$, then $\gamma = G$, and G is isometric to C_m, which is a contradiction. Hence, $L(\gamma) \leq m-1$ and $L(\gamma_j) \leq (m-1)/2$, for every j. If $x \in \gamma_j =: [yz]$, then $d(x, \{y, z\}) \leq L(\gamma_j)/2 \leq (m-1)/4$ and consequently $\delta(T) \leq (m-1)/4$ and $\delta(G) \leq (m-1)/4$.

If $\delta(G) = (m-1)/4$, then every inequality in the previous argument must be an equality. Then we have that $L(\gamma) = m-1$. Since γ is a cycle, we conclude that G is isometric to C_{m-1} with an edge e_0 attached.

A possibility is that e_0 is attached just in some vertex of C_{m-1}. Then we have either that e_0 is a loop or that the other vertex of e_0 has degree 1. Both cases are possible, since $\delta(G) = (m-1)/4$ by Lemma 21 (in both cases, $\{\gamma, e_0\}$ is a tree-decomposition of G).

In other case, e_0 joins two different vertices of C_{m-1}, and G is isometric to some $C_{1,b,c}$, with $b, c \in \mathbb{Z}^+$, $1 + b + c = m$ and $b \leq c$. Theorem 12 gives that $\delta(C_{1,b,c}) = (c + \min\{b, 3\})/4$. Hence, $\delta(G) = (m-1)/4$ if and only if $c + \min\{b, 3\} = m-1$, i.e., $\min\{b, 3\} = b$ or $b \leq 3$. $\square$

Acknowledgements

We would like to thank the referees for a careful reading of the manuscript and for some helpful suggestions.

This research is partially supported by two grants from the Ministerio de Ciencia e Innovación (MTM 2009-07800 and MTM 2008-02829-E), Spain.

References

[1] V. Alvarez, A. Portilla, J.M. Rodríguez, E. Tourís, Gromov hyperbolicity of Denjoy domains, Geom. Dedicata 121 (2006) 221–245.
[2] S. Bermudo, H. Fernau, J.M. Rodríguez, J.M. Sigarreta, Discretization of the hyperbolicity constant. Preprint.
[3] S. Bermudo, J.M. Rodríguez, J.M. Sigarreta, J.-M. Vilaire, Mathematical properties of Gromov hyperbolic graphs, AIP Conf. Proc. 281 (2010) 575–578.
[4] S. Bermudo, J.M. Rodríguez, J.M. Sigarreta, J.-M. Vilaire, Gromov hyperbolic Graphs (submitted for publication).
[5] S. Bermudo, J.M. Rodríguez, J.M. Sigarreta, E. Tourís, Hyperbolicity and complement of graphs (submitted for publication).
[6] Z.M. Balogh, M. Bonk, Gromov hyperbolicity and the Kobayashi metric on strictly pseudoconvex domains, Comment. Math. Helv. 75 (2000) 504–533.
[7] Z.M. Balogh, S.M. Buckley, Geometric characterizations of Gromov hyperbolicity, Invent. Math. 153 (2003) 261–301.
[8] Y. Benoist, Convexes hyperboliques et fonctions quasisymétriques, Publ. Math. Inst. Hautes Études Sci. 97 (2003) 181–237.
[9] M. Bonk, Quasi-geodesics segments and Gromov hyperbolic spaces, Geom. Dedicata 62 (1996) 281–298.
[10] M. Bonk, J. Heinonen, P. Koskela, Uniformizing Gromov hyperbolic spaces, Astérisque 270 (2001).
[11] M. Bonk, O. Schramm, Embeddings of Gromov hyperbolic spaces, Geom. Funct. Anal. 10 (2000) 266–306.
[12] B.H. Bowditch, Notes on Gromov's hyperobolicity criterion for path-metric spaces, in: E. Ghys, A. Haefliger, A. Verjovsky (Eds.), Group Theory from a Geometrical Viewpoint, Trieste, 1990, World Scientific, River Edge, NJ, 1991, pp. 64–167.
[13] G. Brinkmann, J. Koolen, V. Moulton, On the hyperbolicity of chordal graphs, Ann. Comb. 5 (2001) 61–69.
[14] R. Frigerio, A. Sisto, Characterizing hyperbolic spaces and real trees, Geom. Dedicata 142 (2009) 139–149.
[15] G. Gavoille, O. Ly, Distance labeling in hyperbolic graphs, Lect. Notes Comput. Sci. 3827 (2005) 1071–1079.
[16] E. Ghys, P. de la Harpe, Sur les groupes hyperboliques d'après Mikhael Gromov, in: Progress in Mathematics 83, Birkhäuser Boston Inc, Boston, MA, 1990.
[17] P.A. Hästö, Gromov hyperbolicity of the j_G and $\tilde{j}_G$ metrics, Proc. Amer. Math. Soc. 134 (2006) 1137–1142.
[18] P.A. Hästö, H. Lindén, A. Portilla, J.M. Rodríguez, E. Tourís, Gromov hyperbolicity of Denjoy domains with hyperbolic and quasihyperbolic metrics, J. Math. Soc. Japan (in press).
[19] P.A. Hästö, A. Portilla, J.M. Rodríguez, E. Tourís, Gromov hyperbolic equivalence of the hyperbolic and quasihyperbolic metrics in Denjoy domains, Bull. London Math. Soc. 42 (2010) 282–294.
[20] P.A. Hästö, A. Portilla, J.M. Rodríguez, E. Tourís, Comparative Gromov hyperbolicity results for the hyperbolic and quasihyperbolic metrics, Complex Var. Elliptic Equ. 55 (2010) 127–135.
[21] P.A. Hästö, A. Portilla, J.M. Rodríguez, E. Tourís, Uniformly separated sets and Gromov hyperbolicity of domains with the quasihyperbolic metric. Medit. J. Math. in press, (doi:10.1007/s00009-010-0059-7).
[22] E. Jonckheere, P. Lohsoonthorn, A hyperbolic geometry approach to multipath routing, in: Proceedings of the 10th Mediterranean Conference on Control and Automation, MED 2002, Lisbon, Portugal, July, 2002. FA5-1.
[23] E.A. Jonckheere, Contrôle du trafic sur les réseaux à géometrie hyperbolique–Une approche mathématique a la sécurité de l'acheminement de l'information, J. Européen de Systèmes Automatisés 37 (2) (2003) 145–159.
[24] E.A. Jonckheere, P. Lohsoonthorn, Geometry of network security, in: American Control Conference, ACC, 2004, pp. 111–151.
[25] E.A. Jonckheere, P. Lohsoonthorn, F. Ariaesi, Upper bound on scaled Gromov-hyperbolic delta, Appl. Math. Comput. 192 (2007) 191–204.

[26] E.A. Jonckheere, P. Lohsoonthorn, F. Bonahon, Scaled Gromov hyperbolic graphs, J. Graph Theory 2 (2007) 157–180.
[27] A. Karlsson, G.A. Noskov, The Hilbert metric and Gromov hyperbolicity, Enseign. Math. 48 (2002) 73–89.
[28] H. Lindén, Gromov hyperbolicity of certain conformal invariant metrics, Ann. Acad. Sci. Fenn. Math. 32 (1) (2007) 279–288.
[29] J.H. Koolen, V. Moulton, Hyperbolic bridged graphs, European J. Combin. 23 (2002) 683–699.
[30] K. Matsuzaki, J.M. Rodríguez, Twists and Gromov hyperbolicity of Riemann surfaces, Acta Math. Sinica (in press).
[31] J. Michel, J.M. Rodríguez, J.M. Sigarreta, M. Villeta, Hyperbolicity and parameters of graphs, Ars Combin. (in press).
[32] J. Michel, J.M. Rodríguez, J.M. Sigarreta, M. Villeta, Gromov hyperbolicity in cartesian product graphs, Proc. Indian Acad. Sci. Math. Sci. 120 (5) (2010) 1–17.
[33] K. Oshika, Discrete Groups, AMS Bookstore, 2002.
[34] A. Portilla, J.M. Rodríguez, E. Tourís, Gromov hyperbolicity through decomposition of metric spaces II, J. Geom. Anal. 14 (2004) 123–149.
[35] A. Portilla, J.M. Rodríguez, E. Tourís, Stability of Gromov hyperbolicity, J. Adv. Math Stud. 2 (2009) 1–20.
[36] A. Portilla, E. Tourís, A characterization of Gromov hyperbolicity of surfaces with variable negative curvature, Publ. Mat. 53 (2009) 83–110.
[37] J.M. Rodríguez, E. Tourís, Gromov hyperbolicity through decomposition of metric spaces, Acta Math. Hung. 103 (2004) 53–84.
[38] J.M. Rodríguez, E. Tourís, A new characterization of Gromov hyperbolicity for Riemann surfaces, Publ. Mat. 50 (2006) 249–278.
[39] J.M. Rodríguez, E. Tourís, Gromov hyperbolicity of Riemann surfaces, Acta Math. Sinica 23 (2007) 209–228.
[40] E. Tourís, Graphs and Gromov hyperbolicity of non-constant negatively curved surfaces, J. Math. Anal. Appl. (in press).
[41] Y. Wu, C. Zhang, Chordality and hyperbolicity of a graph (submitted for publication).