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Continuous Dependence of the Solution of a Stefan-like Problem Describing the Hydration of Tricalcium Silicate

FOUAD A. MOHAMED

Department of Mathematics, Texas Tech University, Lubbock, Texas 79409

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1. INTRODUCTION

Recently Ranhoeft [7] has established the existence of a unique classical solution for a one-phase, one-dimensional Stefan-like problem which is characterized by two free boundaries. In physical terms, the problem describes conceptually the principal phenomenon of hydration of tricalcium silicate (C_3S) as the major constituent of Portland cement [5, 6]. The mathematical description of the process of chemical reaction and diffusion of chemical reactants (water) through an ever-thickening spherical hydrate layer around the C_3S-particles gives rise to the following system:

$$u_t(x, t) - k(t)\, u_{xx}(x, t) = 0 \qquad \text{in } Q_T \tag{1.1}$$

$$u(x, 0) = \phi(x), \qquad a \leqslant x \leqslant b \tag{1.2}$$

$$u(r(t), t) = 0, \qquad 0 < t < T \tag{1.3}$$

$$u(s(t), t) = Cs(t), \qquad 0 < t < T \tag{1.4}$$

$$k(t)\, u_x(r(t), t) = -\lambda \frac{d}{dt}[r(t)]^2, \qquad 0 < t < T \tag{1.5}$$

$$\frac{d}{dt}[s(t)]^3 = -\mu \frac{d}{dt}[r(t)]^3, \qquad 0 < t < T \tag{1.6}$$

$$r(t) > 0, \qquad s(t) < L, \; 0 < t < T \tag{1.7}$$

together with

$$r(0) = a, \qquad s(0) = b, \qquad \phi(a) = 0, \qquad \phi(b) = Cb. \tag{1.8}$$

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In these equations, subscripts denote differentiation with respect to the indicated variables, C, λ, μ, and T are positive constants, $\phi(x)$ is a continuously differentiable, non-negative function, $k(t)$ is a continuous function satisfying

$$0 < k_* \leqslant k(t) \leqslant k^* < \infty \qquad \text{for} \quad t \geqslant 0,$$

where k_* and k^* are suitable positive constants, and

$$Q_T = \{(x, t): r(t) < x < s(t), 0 < t < T\}.$$

By a *solution* $(r(t), s(t), u(x, t))$ of (1.1)–(1.8) in some time interval $[0, T]$, we mean

(i) $r(t)$ and $s(t)$ are continuously differentiable in $(0, T)$ and continuous in $[0, T]$ with $0 < r(t) < s(t) < L$;

(ii) $u(x, t)$ is continuous in $\bar{Q}_T$ except for a finite number of discontinuities at the boundaries $x = 0$, $t = 0$, $x = L$ where both $\liminf u(x, t)$ and $\limsup u(x, t)$ are bounded;

(iii) $u_x(x, t)$ is continuous in $\bar{Q}_T$, $u_{xx}(x, t)$ and $u_t(x, t)$ are continuous in Q_T.

For conciseness, one may summarize the results of Ranhoeft [7] as follows:

THEOREM 1.1. *There exists a time value T so that Problem* (1.1)–(1.8) *possesses a unique solution $(r(t), s(t), u(x, t))$ for $t \in [0, T]$. Moreover, $r(t)$ is monotonically decreasing and $s(t)$ is monotonically increasing in* $[0, T]$.

The proof of Theorem 1 follows by utilizing potential theoretic arguments and the maximum principle for parabolic equations.

Remark 1.1. In the above theorem, T represents the supremum of the width of time intervals in which the triple (r, s, u) constitutes a solution of Problem (1.1)–(1.8) and either $T = +\infty$, or one of the following cases occurs: $\lim_{t \to T} r(t) = 0$ or $\lim_{t \to T} s(t) = L$.

The object of this note is to prove the stability of the free boundaries in (1.1)–(1.8). To state the pertinent continuous dependence theorem consider two solutions (r_1, s_1, u_1) and (r_2, s_2, u_2) of (1.1)–(1.8) corresponding to the data functions ϕ_1 and ϕ_2 and the coefficients k_1 and k_2 in some time intervals $(0, T_1)$ and $(0, T_2)$, respectively. Moreover, set

$$T=\min[T_1, T_2]$$
$$a_0=\min[\min_t r_1(t), \min_t r_2(t)], \qquad 0\leqslant t\leqslant T$$
$$b_0=\min[\min_t s_1(t), \min_t s_2(t)], \qquad 0\leqslant t\leqslant T \tag{1.9}$$
$$a_1=\max[\max_t r_1(t), \max_t r_2(t)], \qquad 0\leqslant t\leqslant T$$
$$b_1=\max[\max_t s_1(t), \max_t s_2(t)], \qquad 0\leqslant t\leqslant T.$$

In the next section we prove the following:

THEOREM 1.2. *Under the assumptions prescribed on the data and coefficients of the given problem, constants M and T^* ($\leqslant T$) can be found a priori such that*

$$\|(r_1, s_1)-(r_2, s_2)\|_{C^1(0,T^*)} \leqslant M\{|a_1-a_2|+|b_1-b_2|+\|\phi_1-\phi_2\|_{C^1(a,b)}+\|k_1-k_2\|_{C^0[0,T^*]}\}. \tag{1.10}$$

In (1.10), by $\|\psi\|_{C^1(I)}$ for a continuously differentiable function $\psi(x)$ defined on the interval I, we mean

$$\|\psi\|_{C^1(I)}=\|\psi\|_{C^0(I)}+\|d\psi/dx\|_{C^0(I)},$$

where $\|\psi\|_{C^0(I)}=\sup_{x\in I}|\psi(x)|$. The norm $\|\psi\|_{C^N(I)}$ is sometimes denoted by $\|\psi\|_N$ for $N=0, 1, 2, \ldots$.

2. PROOF OF THEOREM 2

It is convenient to perform the transformations

$$y=\frac{x-r(t)}{s(t)-r(t)}, \qquad v(y,t)=u(ys(t)+(1-y)\,r(t),t) \tag{2.1}$$

so that (1.1)–(1.8) convert into

$$v_t=k(t)[s(t)-r(t)]^{-2}\,v_{yy}+[\dot s(t)-\dot r(t)]^{-1}\,[ys(t)+(1-y)\,r(t)]\,v_y, \quad (y,t)\in D_T\equiv(0,1)\times(0,T) \tag{2.2}$$

$$v(y,0)=\phi(by+a(1-y))\equiv f(y), \qquad 0<Y<1 \tag{2.3}$$

$$v(0,t)=0, \qquad 0<t<T \tag{2.4}$$

$$v(1,t)=Cs(t), \qquad 0<t<T \tag{2.5}$$

$$\frac{dr(t)}{dt} = -k(t)[2\lambda r(t)(s(t)-r(t))]^{-1} v_y(0, t), \qquad 0 < t < T \tag{2.6}$$

$$\frac{ds(t)}{dt} = -\mu r^2(t)\, s^{-2}(t) \frac{dr(t)}{dt}, \qquad 0 < t < T. \tag{2.7}$$

For purposes of reference, let the symbol Λ signify the set of pairs of real valued function $z(t) = (r(t), s(t))$ defined for $0 \leqslant t \leqslant T$ and continuously differentiable for $0 < t < T$ with $z(0) = (a, b)$ and $0 < a < b < L$ such that r and s satisfy

$$|\dot{r}(t)| + |\dot{s}(t)| \leqslant R, \qquad t \in (0, T), \tag{2.8}$$

where R is a finite positive constant.

From now on, let M denote a constant that depends on L, R, T, a_0, a_1, b_0, b_1 along with the bounds $\|\phi_j\|_1$ and $\|k_j\|_0$ $(j = 1, 2)$.

We now formulate the following

LEMMA 2.1. *Under the assumptions given on the problem data and coefficients, the solution of* (2.2)–(2.5) *satisfies the condition*

$$v_{yy}(y, t)| < M_0 + Mt^{\nu/2}, \qquad (y, t) \in D_T, \tag{2.9}$$

where M_0 is a constant that depends on the quantities L, a_0, a_1, b_0, b_1, $\|\phi_j\|_1$, and $\|k_j\|_0$ $(j = 1, 2)$ and $\nu \in (0, 1)$ depends on the same quantities that M does.

For the proof of (2.9), it suffices to apply the arguments and techniques of Appendix 3 in [2] to v as the solution of (2.2)–(2.5).

Now, on replacing the function r, s, v, k, and f in (2.2)–(2.7) by respective ones v_j, s_j, v_j, k_j, and f_j $(j = 1, 2)$, then one can easily verify that the differences

$$\begin{aligned} &w(y, t) = v_1(y, t) - v_2(y, t) \\ &\delta(t) = r_1(t) - r_2(t), \qquad \delta^*(t) = s_1(t) - s_2(t) \\ &\tilde{f}(y) = f_1(y) - f_2(y), \qquad \tilde{k}(t) = k_1(t) - k_2(t), \qquad \Delta(t) = (\delta(t), \delta^*(t)) \end{aligned} \tag{2.10}$$

satisfy the conditions

$$w_t = A(y, t)\, w_{yy} + H(y, t) \qquad \text{in } D_T \tag{2.11}$$

$$w(y, 0) = \tilde{f}(y), \qquad 0 < y < 1 \tag{2.12}$$

$$w(0, t) = 0, \qquad 0 < t < T \tag{2.13}$$

$$w(1, t) = C\delta^*(t), \qquad 0 < t < T \tag{2.14}$$

$$\frac{d}{dt}\delta(t) = -E(t)\, w_y(0, t) + F(0, t), \qquad 0 < t < T \tag{2.15}$$

$$\frac{d}{dt}\delta^*(t) = -\mu[(r_1/s_1)^2\, \dot{\delta} + (\dot{r}_2/s_2^2)\{\delta^*(s_2 + s_2)(r_1/s_1)^2 + \delta(r_1 + r_2)\}], \tag{2.16}$$

where

$$A(y, t) = k_1/(s_1 - r_1)^2, \tag{2.17}$$

$$B(y, t) = [y\dot{s}_1 + (1 - y)\, \dot{r}_1]/(s_1 - r_1), \tag{2.18}$$

$$E(t) = k_1/[2\lambda r_1 (s_1 - r_1)], \tag{2.19}$$

$$\begin{aligned} H(y, t) = (\delta - \delta^*)\{&(s_1 + s_2 - r_1 - r_2)\, k_1 v_{2,yy}/[(s_1 - r_1)(s_2 - r_2)]^2 \\ &+ [y\dot{s}_1 + (1 - y)\, \dot{r}_1]\, v_{2,y}/[(s_1 - r_1)(s_2 - r_2)]\} \\ &+ \tilde{k} v_{2,yy}/(s_2 - r_2)^2 + [y\dot{\delta}^* + (1 - y)\, \dot{\delta}]\, v_{2,y}/(s_2 - r_2), \end{aligned} \tag{2.20}$$

$$\begin{aligned} F(0, t) = -v_{2,y}(0, t)&[r_1(s_1 - r_1)\, \tilde{k} + k_1\{(r_1 + r_2 - s_1)\, \delta - r_2\delta^*\}]/ \\ &[2\lambda r_1 r_2 (s_1 - r_1)(s_2 - r_2)]. \end{aligned} \tag{2.21}$$

Next, utilizing (2.8) and (2.9) yields

$$\max_{y \in [0, 1]} |H(y, t)| \leqslant M(\|\dot{\delta}\|_t + \|\dot{\delta}^*\|_t + \|\tilde{D}\|_t). \tag{2.22}$$

(Here, e.g., $\|\delta\|_t$ denotes the sup in $(0, t)$ of $\delta(\tau)$.)
Let $w(y, t)$ be decomposed into the sum

$$w(y, t) = W(y, t) + yC\delta^*(t). \tag{2.23}$$

Then $W(y, t)$ solves the problem

$$W_t = A(y, t)\, W_{yy} + H^*(y, t), \qquad (y, t) \in D_T \tag{2.24}$$

$$W(y, 0) = \tilde{f}(y) - yC\delta^*(0), \qquad y \in (0, 1) \tag{2.25}$$

$$W(0, t) = 0, \qquad t \in (0, T) \tag{2.26}$$

$$W(1, t) = 0, \qquad t \in (0, T), \tag{2.27}$$

where $H^*(y, t) = B(y, t)[W_y + C\delta^*] - yC\dot{\delta}^* + H(y, t)$.

Let $G(y, t; \zeta, \tau)$ denote the Green's function for the operator $\partial/\partial t - A(y, t)\, \partial^2/\partial x^2$ in D_T. Then the solution of (2.24)–(2.27) is represented by the integral

$$\begin{aligned} W_y(y, t) = &\int_0^t \int_0^1 G_y(y, t; \zeta, \tau)\, H^*(\zeta, \tau)\, d\zeta\, d\tau \\ &+ \int_0^1 G_y(y, t; \zeta, 0)\, W(\zeta, 0)\, d\zeta, \end{aligned} \tag{2.28}$$

Applying the analysis and techniques of Appendix 1 in [2] implies that the second integral in (2.28) is bounded by $M\|\tilde{f}\|_1$.

On the other hand, recalling well-known estimates on Green's function (see, e.g., [1; 3, p. 413; 4]), we deduce that the first integral in the right side of (2.28) is bounded by

$$M\left\{\int_0^t\int_0^1 (t-\tau)^{-1}\exp\left[\frac{-\gamma(y-\zeta)^2}{t-\tau}\right]\right.$$
$$\cdot \max_{\zeta\in[0,1]} [B(\zeta,\tau)(W_y(\zeta,\tau)+\zeta C\|\delta^*\|_\tau)]\,d\zeta\,d\tau$$
$$+\int_0^t\int_0^1 (t-\tau)^{-1}\exp\left[\frac{-\gamma(y-\zeta)^2}{t-\tau}\right]$$
$$\left.\cdot \max_{\zeta\in[0,1]} [\zeta C\|\delta^*\|_\tau + H(\zeta,\tau)]\,d\zeta\,d\tau\right\}.$$

Employing the definition of B and the inequality (2.22), then the last expression becomes bounded by

$$M\left\{\int_0^t (t-\tau)^{-1/2}\max_{y\in[0,1]}|W_y(y,\tau)|\,d\tau\right.$$
$$\left.+\int_0^t (t-\tau)^{-1/2}\|\dot{\Delta}\|_\tau\,d\tau+\|\tilde{k}\|_t\right\}, \tag{2.29}$$

where

$$\|\dot{\Delta}\|_t \equiv \|\dot{\delta}\|_t + \|\dot{\delta}^*\|_t. \tag{2.30}$$

From the above results, we deduce that

$$\max_{y\in[0,1]}|W_y(y,t)| \leqslant M\left\{\int_0^t (t-\tau)^{-1/2}\max_{y\in[0,1]}|W_y(y,\tau)|\,d\tau\right.$$
$$\left.+\int_0^t (t-\tau)^{-1/2}\|\dot{\Delta}\|_t\,d\tau+\|\tilde{f}\|_1+\|\tilde{k}\|_t\right\}. \tag{2.31}$$

At this stage, an application of Gronwall's lemma yields

$$\max_{y\in[0,1]}|W_y(y,t)| \leqslant M\left\{\int_0^t (t-\tau)^{-1/2}\|\dot{\Delta}\|_\tau\,d\tau+\|\tilde{f}\|_1+\|\tilde{k}\|_t\right\}. \tag{2.32}$$

In view of (2.23) and (2.32), we get

$$|w_y(0,t)| \leqslant M\left\{\int_0^t (t-\tau)^{-1/2}\|\dot{\Delta}\|_\tau\,d\tau+\|\tilde{f}\|_1+\|\tilde{k}\|_t\right\}. \tag{2.33}$$

Finally, the utility of (2.14), (2.15), (2 .33) and the definitions of $E(t)$ and $F(0, t)$, lead to

$$\|\dot{\delta}\|_t, \|\dot{\delta}^*\|_t \leqslant M\left\{\int_0^t (t-\tau)^{-1/2} \|\dot{\Delta}\|_\tau \, d\tau + \|\tilde{f}\|_1 + \|\tilde{k}\|_t\right\}. \quad (2.34)$$

Hence by virtue of (2.30) and Gronwall's lemma, we obtain

$$\|\dot{\Delta}\|_t \leqslant M(\|\tilde{f}\|_1 + \|\tilde{k}\|_t), \quad (2.35)$$

from which (1.10) follows, concluding the proof of Theorem 2.

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