Abstract

We consider the stabilization of the nonnegative solutions of linear parabolic equation by controls localized on a curve. The main results of the article give a necessary and sufficient condition for positive stabilizability in terms of the principal eigenvalue of a certain elliptic operator. In case of positive stabilizability, some feedback stabilizing controls are indicated.

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1. Introduction

This paper deals with stabilization problems for parabolic equations, in dimension $N = 2$, when the control is localized on a curve $\gamma$. More precisely consider the equation

\[
\frac{\partial y}{\partial t} - \Delta y + a(x)y = \delta_{\gamma} u \quad \text{in } Q_T, \\
y = 0 \quad \text{on } \Sigma_T, \quad y(0) = y_0 \quad \text{in } \Omega,
\]

where $\Omega$ is a bounded domain in $\mathbb{R}^2$ with a boundary $\Gamma$ of class $C^2$, $Q_T = \Omega \times (0, T)$, $\Sigma_T = \Gamma \times (0, T)$, $\gamma \subset \Omega$ is a one-dimensional Lipschitz manifold given by $\gamma = \{\alpha(\tau) \mid \tau \in [0, 1]\}$, $\alpha$ is Lipschitz from $[0, 1]$ into $\mathbb{R}^2$ and of class $C^1$ except at a finite set of points.

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and the coefficient $a$ belongs to $L^\infty(\Omega)$. Here, for any $v \in L^1(\gamma)$, $\delta_y^v$ denotes the measure defined by

$$\delta_y^v(\varphi) = \int_{\gamma} v \varphi \, d\sigma,$$

for any $\varphi \in C(\Omega)$. The terminal time $T$ may be finite or infinite, and if $T := \infty$, we set $Q_\infty = \Omega \times (0, \infty)$, $\Sigma_\infty = \gamma \times (0, \infty)$.

We suppose that $y_0 \in L^2(\Omega)$, $y_0 \geq 0$, and we are interested in the following question:

**Definition 1.** Equation (1) is said to be positively stabilizable in $L^2(\Omega)$ if for every nonnegative function $y_0 \in L^2(\Omega)$, there exists a control $u \in L^{p}_{loc}([0, \infty); L^p(\gamma))$ such that (2) holds for some $1 \leq p \leq \infty$.

Equation (1) is said to be positively stabilizable in $H^1_0(\Omega \setminus \gamma)$ if for every nonnegative function $y_0 \in H^1_0(\Omega \setminus \gamma)$, there exists a control $u \in L^p_{loc}([0, \infty); L^p(\gamma))$ such that (2) holds for some $1 \leq p \leq \infty$.

For internal controls the notion of positive stabilizability is introduced in [2].

Recall that $H^1_0(\Omega \setminus \gamma)$ is the closure of $C_0^\infty(\Omega \setminus \gamma)$ in the norm $$(u)(\Omega \setminus \gamma) = \left( \int_{\Omega \setminus \gamma} (|\nabla y|^2 + y^2) \, dx \right)^{1/2}.$$ Even if $\Omega \setminus \gamma$ does not possess the $H^1$-extension property, for every $f \in L^2(\Omega \setminus \gamma)$, the existence of a unique solution $y_f$ to the problem $y \in H^1_0(\Omega \setminus \gamma)$, $-\Delta y = f$ in $\Omega \setminus \gamma$, follows from the Lax–Milgram theorem. The only difference with the case of a regular bounded domain is that $y_f \in D(-\Delta) = \{ y \in H^1_0(\Omega \setminus \gamma) \mid -\Delta y \in L^2(\Omega \setminus \gamma) \}$, but $D(-\Delta) \not\subset H^2(\Omega)$. (See [3, Remark 2.3] for a capacity method approach for this kind of problem.)

The operator $(-\Delta, D(-\Delta))$ is a positive selfadjoint operator on $L^2(\Omega \setminus \gamma)$. Its inverse is a bounded operator from $L^2(\Omega \setminus \gamma)$ into $H^1_0(\Omega \setminus \gamma)$. Since the imbedding from $H^1_0(\Omega \setminus \gamma)$ into $L^2(\Omega \setminus \gamma)$ is compact [1, Theorem 6.2, Part 4], the inverse of $-\Delta$ with Dirichlet boundary conditions on $\Gamma \cup \gamma$ is a compact operator on $L^2(\Omega \setminus \gamma)$. From classical results it follows that the spectrum of $-\Delta + a$ consists of a countable set of eigenvalues $\{\lambda_k(\Omega \setminus \gamma) \mid k \in \mathbb{N} \}$ such that $\lim_{k \to \infty} \lambda_k(\Omega \setminus \gamma) = \infty$. (See also [4,5] for more precise results in the case of polygonal domains.)

Positive stabilization is closely related to the sign of the first eigenvalue $\lambda_1(\Omega \setminus \gamma)$ of the elliptic operator $-\Delta + a$ in $\Omega \setminus \gamma$ with Dirichlet boundary conditions. Recall that there exists $\phi_1 \geq 0$, $\phi_1 > 0$ a.e. in $\Omega$, such that $-\Delta \phi_1 + a(x)\phi_1 = \lambda_1 \phi_1$ in $\Omega \setminus \gamma$, $\phi_1 = 0$ on $\partial(\Omega \setminus \gamma)$. 


In the next sections we will characterize the positive stabilizability in $H^1_0(\Omega \setminus \gamma)$ and $L^2(\Omega)$, respectively, in terms of the sign of $\lambda_1(\Omega \setminus \gamma)$. In Section 4 a lower bound for this eigenvalue is found. The last section gives some hints on possible extensions of the main results.

2. Necessary and sufficient condition for positive stabilization in $H^1_0(\Omega \setminus \gamma)$

**Theorem 1.** Equation (1) is positively stabilizable in $H^1_0(\Omega \setminus \gamma)$ if and only if $\lambda_1(\Omega \setminus \gamma) > 0$.

Before proving this theorem let us state some existence and regularity results for parabolic equations associated with Eq. (1).

**Definition 2.** Let $T \in (0, +\infty)$. A function $y \in L^1(0, T; W^{1,1}_0(\Omega))$ is a weak solution to Eq. (1) if
\[
\int_{\Omega_T} (-y\phi_t + \nabla y \nabla \phi + a(x)y\phi) \, dx \, dt = \int_{\gamma \times (0,T)} u\phi \, d\sigma \, dt + \int_{\Omega} y_0(x)\phi(x,0) \, dx,
\]
for every $\phi \in C^1(\bar{\Omega_T})$, such that $\phi(T) = 0$, and $\phi = 0$ on $\Sigma_T$.

We say that $y \in L^1_{\text{loc}}([0, T]; W^{1,1}_0(\Omega))$ is a weak solution to (1) for $T := +\infty$, if $y$ is a weak solution to (1) for all $T \in (0, +\infty)$.

**Lemma 2.** Suppose that $T \in (0, +\infty)$ and $3/2 < p$. Then, for every $u \in L^p(0, T; L^p(\gamma))$, Eq. (1) admits a unique weak solution $y$ in $L^2(0, T; H^1_0(\Omega))$. Moreover, $y$ belongs to $C_4([0, T]; L^2(\Omega))$.

**Proof.** The proof can be established as in [7, Proposition 2.3].

Consider the equation
\[
\frac{\partial z}{\partial t} - \Delta z + a(x)z = 0 \quad \text{in} \quad (\Omega \setminus \gamma) \times (0, T),
\]
\[
z = 0 \quad \text{on} \quad \Sigma_T \cup (\gamma \times (0, T)), \quad z(0) = y_0 \quad \text{in} \quad \Omega.
\]
In the following, we denote by $n^+$ and $n^-$ the two unit normals to $\gamma$.

**Lemma 3.** Assume that $T \in (0, +\infty)$ and $y_0 \in H^1_0(\Omega \setminus \gamma)$. Let $z \in W(0, T; H^1_0(\Omega \setminus \gamma)$, $H^{-1}(\Omega \setminus \gamma))$ be the weak solution to Eq. (4), and denote by $\tilde{z}$ the function in $W(0, T; H^1_0(\Omega), H^{-1}(\Omega))$ defined by
\[
\begin{cases}
\tilde{z} = z & \text{in} \quad \Omega \setminus \gamma, \\
\tilde{z} = 0 & \text{on} \quad \gamma.
\end{cases}
\]
Then $\tilde{z}$ is also the weak solution to the equation
\[
\frac{\partial h}{\partial t} - \Delta h + a(x)h = \delta_{\gamma} \quad \text{in } QT,
\]
\[h = 0 \quad \text{on } \Sigma_T, \quad h(0) = y_0 \quad \text{in } \Omega,
\]
with
\[u := \frac{\partial z}{\partial n^+} + \frac{\partial z}{\partial n^-} \quad \text{on } \gamma \times (0, T).
\]

Proof. First observe that, due to [5, Theorem 2.4], the solution $z$ to Eq. (4) belongs to
\[H^{1/4-\varepsilon}(0, T; L^2(\Omega \setminus \gamma)) \cap L^2(0, T; H^{3/2-\varepsilon}(\Omega \setminus \gamma)) \quad \text{for all } 0 < \varepsilon < 1/4.
\]
This is not sufficient to define $\partial z/\partial n^+$ and $\partial z/\partial n^-$ (if we take into account only that $z(t) \in H^{3/2-\varepsilon}(\Omega \setminus \gamma)$). However, due to the particular form of the singularity of the solution (see [5]), it is possible to define the normal derivatives $\partial z/\partial n^+$ and $\partial z/\partial n^-$ in $L^2(0, T; L^p(\gamma))$ for all $p < 2$. With a Green formula, we have
\[
\int_{(\Omega \setminus \gamma) \times (0, T)} (-\tilde{z}\phi_t + \nabla \tilde{z} \nabla \phi + a(x)\tilde{z}\phi) \, dx \, dt
\]
\[= \int_{(\Omega \setminus \gamma) \times (0, T)} (-z\phi_t + \nabla z \nabla \phi + a(x)z\phi) \, dx \, dt
\]
\[= \int_{\gamma \times (0, T)} \left( \frac{\partial z}{\partial n^+} + \frac{\partial z}{\partial n^-} \right) \phi \, d\sigma \, dt + \int_{\Omega} y_0\phi(0) \, dx,
\]
for every $\phi \in C^1(\overline{Q_T})$, such that $\phi(T) = 0$, and $\phi = 0$ on $\Gamma \times (0, T)$. The proof is complete. \(\square\)

Lemma 4. Let $z$ be the weak solution of Eq. (4), and $y$ be the weak solution to Eq. (1) associated with some $u \in L^1_{\text{loc}}([0, \infty); L^1(\gamma))$. If
\[y|_{\gamma \times (0, T)} \geq 0,
\]
then
\[y \geq z \quad \text{in } (\Omega \setminus \gamma) \times (0, T).
\]

Proof. The function $w = y - z$ is the solution of the equation
\[
\frac{\partial w}{\partial t} - \Delta w + a(x)w = 0 \quad \text{in } (\Omega \setminus \gamma) \times (0, T),
\]
\[w = 0 \quad \text{on } \Sigma_T, \quad w = g \quad \text{on } \gamma \times (0, T), \quad w(0) = 0 \quad \text{in } \Omega.
\]
Since $g = y|_{\gamma \times (0, T)} \geq 0$, from the maximum principle [6], it follows that $w \geq 0$ and the proof is complete. \(\square\)

Proof of Theorem 1. Let us prove first that if Eq. (1) is positively stabilizable then $\lambda_1(\Omega \setminus \gamma) > 0$. We argue by contradiction.
Suppose that $\lambda_1(\Omega \setminus \gamma) \leq 0$. Let $\phi_1$ be the nonnegative eigenfunction associated with $\lambda_1(\Omega \setminus \gamma)$. It is well known that

$$\phi_1 > 0 \quad \text{in} \ \Omega \setminus \gamma.$$ 

Let $z_1$ be the solution to Eq. (4) for $y_0 := \phi_1$. Then

$$z_1 = \phi_1 \quad \text{if} \ \lambda_1(\Omega \setminus \gamma) = 0, \quad \text{and} \quad z_1 \geq \phi_1 \quad \text{if} \ \lambda_1(\Omega \setminus \gamma) < 0.$$ 

This last inequality can be deduced from a comparison principle.

Let $u_1$ be a control for which the solution $y_1$ of Eq. (1) corresponding to $u_1$ and $y_0 := \phi_1$ satisfies the stabilizability condition (2). By Lemma 4 it follows that:

$$y_1 \geq z_1 \geq \phi_1 > 0 \quad \text{in} \quad (\Omega \setminus \gamma) \times (0, \infty).$$

This is clearly in contradiction with the fact that $y_1$ satisfies the stabilizability condition (2).

Conversely, suppose that $\lambda_1(\Omega \setminus \gamma) > 0$, and let us prove that Eq. (1) is positively stabilizable in $H^1_0(\Omega \setminus \gamma)$. Let $y_0 \in H^1_0(\Omega \setminus \gamma)$ be nonnegative. Then the solution $z$ to Eq. (4) is nonnegative. Moreover, if $\lambda_1(\Omega \setminus \gamma) > 0$ then

$$\lim_{t \to \infty} \|z(t)\|_{L^2(\Omega)} \leq \lim_{t \to \infty} M e^{-\lambda_1 t} \|y_0\|_{L^2(\Omega)} = 0,$$

where $M > 0$ is a constant. Set

$$u := \frac{\partial z}{\partial n^+} + \frac{\partial z}{\partial n^-} \quad \text{on} \ \gamma \times (0, \infty).$$

By Lemma 3 we deduce that the weak solution $y^u$ of Eq. (1) satisfies

$$y^u = z \quad \text{in} \quad (\Omega \setminus \gamma) \times (0, \infty).$$

The proof is complete. \□

Remark. We have actually proved that Eq. (1) with the feedback control

$$u := \frac{\partial y}{\partial n^+} + \frac{\partial y}{\partial n^-} \quad \text{on} \ \gamma \times (0, \infty),$$

has at least a weak solution when $y_0$ belongs to $H^1_0(\Omega \setminus \gamma)$. The uniqueness of the weak solution follows by Green’s formula and using appropriate test functions $\phi$ in (3).

Thus if $\lambda_1(\Omega \setminus \gamma) > 0$, we have found a feedback control which positively stabilizes Eq. (1).

3. Necessary and sufficient condition for positive stabilization in $L^2(\Omega)$

**Theorem 5.** Equation (1) is positively stabilizable in $L^2(\Omega)$ if and only if $\lambda_1(\Omega \setminus \gamma) > 0$.

Before proving the theorem we establish a preliminary result. For any real number $\mu \geq 0$ we consider the eigenvalue problem

$$-\Delta \phi + a(x)\phi = -\mu \delta_{\gamma} \phi + \lambda \phi \quad \text{in} \ \Omega, \quad \phi = 0 \quad \text{on} \ \Gamma, \quad (5)$$

and we denote by $\lambda_1^\mu$ its principal eigenvalue.
Lemma 6.
\[ \lim_{\mu \to +\infty} \lambda_1^\mu = \lambda_1(\Omega \setminus \gamma). \]

Proof. Using the Rayleigh’s principle we infer that
\[ \lambda_1^\mu = \min_{\phi \in H_0^1(\Omega), \|\phi\|_{L^2(\Omega)} = 1} \int_\Omega \left( |\nabla \phi|^2 + a(x) |\phi|^2 \right) \, dx + \mu \int_\gamma |\phi|^2 \, d\sigma \leq \lambda_1(\Omega \setminus \gamma). \]

On the other hand, we have that
\[ \lambda_1^{\mu_1} \leq \lambda_1^{\mu_2} \quad \text{if} \quad 0 \leq \mu_1 \leq \mu_2. \]

Thus
\[ \lim_{\mu \to +\infty} \lambda_1^\mu = \bar{\lambda} \leq \lambda_1(\Omega \setminus \gamma). \]

Let us prove the equality. For all \( \mu > 0 \), consider \( \phi^\mu \in H_0^1(\Omega) \) such that \( \|\phi^\mu\|_{L^2(\Omega)} = 1 \), and
\[ \lambda_1^\mu = \int_\Omega \left( |\nabla \phi^\mu|^2 + a(x) |\phi^\mu|^2 \right) \, dx + \mu \int_\gamma |\phi^\mu|^2 \, d\sigma. \]  

(6)

We may infer that
\[ \int_\Omega |\nabla \phi^\mu|^2 \, dx \leq M, \quad \forall \mu > 0, \]
and
\[ \mu \int_\gamma |\phi^\mu|^2 \, d\sigma \leq M, \quad \forall \mu > 0. \]

As a consequence there exists a subsequence (still denoted by \( (\phi^\mu)_{\mu} \)) such that
\[ \phi^\mu \to \phi^* \quad \text{in} \quad L^2(\Omega), \]
\[ \phi^\mu \rightharpoonup \phi^* \quad \text{in} \quad H_0^1(\Omega), \]
and
\[ \phi^\mu \to 0 \quad \text{in} \quad L^2(\gamma), \]
as \( \mu \to +\infty \). We deduce that \( \phi^* = 0 \) on \( \gamma \). By (6) we deduce that
\[ \lim_{\mu \to +\infty} \lambda_1^\mu \geq \int_\Omega \left( |\nabla \phi^*|^2 + a(x) |\phi^*|^2 \right) \, dx \geq \lambda_1(\Omega \setminus \gamma). \]

This implies that
\[ \lambda_1(\Omega \setminus \gamma) \geq \bar{\lambda} = \lim_{\mu \to +\infty} \lambda_1^\mu \geq \lambda_1(\Omega \setminus \gamma), \]
and we finally get the conclusion. \( \square \)
Proof of Theorem 5. If Eq. (1) is positively stabilizable then it follows as in Theorem 1 that \( \lambda_1(\Omega \setminus \gamma) > 0 \). \( \square \)

Conversely, suppose that \( \lambda_1(\Omega \setminus \gamma) > 0 \) and let prove that Eq. (1) is positively stabilizable in \( L^2(\Omega) \). Due to Lemma 6 there exists \( \mu > 0 \) large enough such that \( \lambda_1^\mu > 0 \), where \( \lambda_1^\mu \) is the principal eigenvalue of (5). Let \( y_0 \in L^2(\Omega) \) be nonnegative.

Using a fixed point method (for example, as in [8, Proposition 2.7]) we can prove that the following equation:

\[
\begin{align*}
\frac{\partial \zeta}{\partial t} - \Delta \zeta + a(x)\zeta &= -\mu \delta \zeta \quad \text{in } Q_{\infty}, \\
\zeta &= 0 \quad \text{on } \Sigma_{\infty}, \\
\zeta(0) &= y_0 \quad \text{in } \Omega,
\end{align*}
\]

has a unique weak solution in \( L^2_{\text{loc}} ([0, \infty); H^1_0(\Omega)) \). Since \( \lambda_1^\mu > 0 \) we may infer that

\[
\lim_{t \to \infty} \left\| \xi(t) \right\|_{L^2(\Omega)} \leq \lim_{t \to \infty} Ke^{-\lambda_1^\mu t} \left\| y_0 \right\|_{L^2(\Omega)} = 0,
\]

where \( K > 0 \) is a constant. We can actually prove that \( \xi \) belongs to \( L^2([0, \infty); H^1_0(\Omega)) \).

Indeed, the operator \( A \xi \) is a selfadjoint operator on the Hilbert space \( L^2(\Omega) \), and it is the generator of an analytic semigroup of contractions on \( L^2(\Omega) \). Since the coefficients of the operator \( A \) are not regular the domain \( D(A) \) is not equal to \( H^2(\Omega) \cap H^1_0(\Omega) \). However, there exists \( \alpha \in (1/2, 1) \) such that \( D((-A)\alpha) \subset H^1_0(\Omega) \). With the estimate

\[
\left\| \xi(t) \right\|_{H^1_0(\Omega)} \leq C_1 \left\| (-A)\alpha \xi(t) \right\|_{L^2(\Omega)} \leq C_2 e^{-\lambda_1^\mu t} t^{-\alpha} \left\| y_0 \right\|_{L^2(\Omega)},
\]

we prove that \( \xi \in L^2(0, \infty; H^1_0(\Omega)) \).

We deduce that the feedback control

\[
u := -\mu y
\]

stabilizes (1) and that \( \nu \) belongs to \( L^2(0, \infty; H^{1/2}(\gamma)) \). The proof is complete.

Remark. Theorem 5 is obviously more general than Theorem 1. From Theorem 5 it follows that Eq. (1) is positively stabilizable in \( L^2(\Omega) \) if and only if, for all nonnegative function \( y_0 \in L^2(\Omega) \), the solution \( z_{y_0} \) to Eq. (4) (with \( T := +\infty \)) obeys

\[
\left\| z_{y_0}(t) \right\|_{L^2(\Omega)} \to 0 \quad \text{exponentially as } t \to \infty.
\]

Moreover, due to Lemma 4, for all \( y_0 \in L^2(\Omega) \) and \( u \in L^1_{\text{loc}} ([0, \infty); L^1(\gamma)) \), if the solution \( y_{y_0}^u \) to Eq. (1) is nonnegative in \( Q_{\infty} \), then

\[
0 \leq z_{y_0} \leq y_{y_0}^u.
\]

This means that when \( y_0 \in H^1_0(\Omega \setminus \gamma) \), Theorem 1 provides the control \( u \) ensuring the best positive stabilization we can expect.
4. Lower bound for $\lambda_1(\Omega \setminus \gamma)$

In this section, we give a lower bound for $\lambda_1(\Omega \setminus \gamma)$ which can be helpful to choose $\gamma$ ensuring the condition $\lambda_1(\Omega \setminus \gamma) > 0$. Let $S$ be the boundary of $\Omega \setminus \gamma$, and denote by $S^2$ the unit circle in $\mathbb{R}^2$. For every unit vector $v \in S^2$ we denote by $v^\perp \in S^2$ the vector defined by

$$\hat{(v, v^\perp)} = \frac{\pi}{2}.$$ 

We denote by $R_v$ (respectively, $R_v^\perp$) the vector space generated by $v$ (respectively, $v^\perp$). Let $v$ be in $S^2$ and set

$$d(v) = \sup\left\{\inf\left\{|t|; x + tv \in S\}; x \in \Omega \setminus \gamma\right\},$$

and

$$\ell = \inf\{d(v) | v \in S^2\}.$$

**Theorem 7.**

$$\lambda_1(\Omega \setminus \gamma) \geq \frac{\pi^2}{4\ell^2} + \text{ess inf}(a).$$

**Proof.** For any $\phi \in H^1_0(\Omega \setminus \gamma)$, we denote by $\tilde{\phi}$ the extension of $\phi$ by zero on $\mathbb{R}^2 \setminus (\Omega \setminus \gamma)$. From the Wirtinger inequality, it follows that:

$$\int_{\mathbb{R}^2} \tilde{\phi}(x+\tau v)^2 \, d\tau \leq \frac{4d(v)^2}{\pi^2} \int_{\mathbb{R}^2} |\nabla \tilde{\phi}(x+\tau v)v|^2 \, d\tau \leq \frac{4d(v)^2}{\pi^2} \int_{R_v} |\nabla \phi(x+s)|^2 \, ds,$$

for almost every $x \in \Omega \setminus \gamma$, and for every $v \in S^2$.

Therefore, we have

$$\int \int_{R_v} \int_{R_v^\perp} \tilde{\phi}(x)^2 \, dx \leq \frac{4d(v)^2}{\pi^2} \int_{R_v} |\nabla \phi(x)|^2 \, dx.$$ 

Taking the infimum with respect to $v$ we obtain

$$\int_{\Omega} \tilde{\phi}(x)^2 \, dx \leq \frac{4\ell^2}{\pi^2} \int_{\Omega} |\nabla \phi(x)|^2 \, dx,$$

and consequently

$$\left(1 + \frac{4\ell^2}{\pi^2} \text{ess inf}(a)\right) \int_{\Omega} \phi(x)^2 \, dx \leq \frac{4\ell^2}{\pi^2} \left\{ \int_{\Omega} |\nabla \phi(x)|^2 \, dx + \int_{\Omega} a(x)\phi(x)^2 \, dx \right\}.$$ 

By Rayleigh’s principle we conclude that

$$\lambda_1(\Omega \setminus \gamma) = \inf_{\phi \in H^1_0(\Omega \setminus \gamma), \|\phi\|_{L^2(\Omega)} = 1} \left\{ \int_{\Omega} [|\nabla \phi(x)|^2 + a(x)\phi(x)^2] \, dx \right\}$$

$$\geq \frac{\pi^2}{4\ell^2} + \text{ess inf}(a). \quad \Box$$
5. Further remarks

The results in this paper can be extended to the problem with homogeneous Neumann boundary conditions. The proofs of the main results are similar.

On the other hand, the results in Sections 3 and 4 can be extended to the case $N \geq 3$, where $\gamma$ is a $(N - 1)$-dimensional Lipschitz manifold.

References