Lattice Isomorphisms of Alternative Algebras*

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The lattice of subalgebras of an alternative algebra can determine the algebraic structure of the algebra. Here it is shown that alternative algebras with a lattice of subalgebras isomorphic to nondivision semisimple alternative algebras are closely related to them. © 1990 Academic Press, Inc

INTRODUCTION

Let $A$ be an alternative algebra over the field $F$. It is known that the set of subalgebras of $A$ has a lattice structure. We denote this lattice by $\mathcal{L}(A)$. Let $B$ be another algebra over the field $F$. By an $\mathcal{L}$-isomorphism, or lattice isomorphism of the algebra $A$ onto an algebra $B$, we mean a one-to-one map $\Psi: \mathcal{L}(A) \to \mathcal{L}(B)$ such that $\Psi(A_1 \vee A_2) = \Psi(A_1) \vee \Psi(A_2)$ and $\Psi(A_1 \cap A_2) = \Psi(A_1) \cap \Psi(A_2)$, for all $A_1$ and $A_2$ subalgebras of $A$, where we denote by $A_1 \vee A_2$ the least subalgebra of $A$ containing $A_1$ and $A_2$.

We are interested in the study of the lattice isomorphisms of alternative algebras. We want to inquire into the algebraic relationships between a semisimple alternative algebra and an alternative algebra $\mathcal{L}$-isomorphic to it.

Here, we solve the problem when $A$ is a simple nondivision finite dimensional algebra, that is, when $A$ is a matrix algebra, $M_n(D)$, with $n \geq 2$, and $D$ a division associative algebra or when $A$ is a split Cayley–Dickson algebra. Then, if $n \geq 3$, it is shown that $B$ is isomorphic or semisomorphic to $A$ (if $n = 2$, $B = M_2(D)$, with $D$ a division associative algebra). When $A$ is a division central nonassociative algebra, that is, a division Cayley–Dickson algebra over $F$, it is shown that $B$ must be a division Cayley–Dickson algebra.

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algebra or a purely inseparable $p^3$-dimensional extension field of $F$, with $p$
characteristic of $F$. If $A$ is a finite dimensional semisimple algebra, we extend Barnes' results for associative algebras [5], and we show that $B$ is
also semisimple and the images under $\Psi$ of simple direct summands of $A$, with dimension greater than one, are simple direct summands of $B$.

In the following we denote by

\begin{align*}
[a_1, a_2, a_3] & : (a_1, a_2) a_3 - a_1 (a_2 a_3), \text{ with } a_1, a_2, a_3 \in A \\
(a_1, a_2, ..., a_n) & : \text{the subspace of } A \text{ spanned by } a_1, a_2, ..., a_n \in A \\
\langle a_1, a_2, ..., a_n \rangle & : \text{the subalgebra of } A \text{ spanned by } a_1, a_2, ..., a_n \in A \\
R(A) & : \text{the nilpotent radical of } A \\
\dim_F(A) & : \text{the dimension of vectorial space } A \text{ over the ground field } F \\
l(A) & : \text{the length of the longest chain in } \mathcal{L}(A) \\
A_1 \lor A_2 & : \text{the least subalgebra of } A \text{ containing } A_1 \text{ and } A_2.
\end{align*}

We always consider algebras with finite length. It is clear that $\dim_F(A) \geq l(A)$.

We recall [6, 7, 13] that simple alternative algebras are associative or
Cayley-Dickson algebras (C-D algebras). If char $F \neq 2$ we can find a basis $e_0 = 1, e_1, ..., e_7$ over the center of the algebra such that it has the multi-
plication table [8] shown in Table I with $\alpha, \beta, \gamma \in F$. C-D algebras can be
division algebras or have zero divisors (split). In the latter case it is possible to find a basis, $\{x_i\} \cup \{y_i\}, i = 0, 1, 2, 3$, over the center such that

\begin{align*}
x_i y_0 &= x_i, \quad x_0 x_i = x_i, \quad y_i x_0 = y_i, \quad y_0 y_i = y_i, \\
y_i x_i &= -y_0, \quad x_i y_i = -x_0, \quad x_i x_{i+1} = y_{i+2}, \quad y_i y_{i+1} = x_{i+2}, \\
x_0 x_0 &= x_0, \quad y_0 y_0 = y_0, \quad 0 < i \leq 3,
\end{align*}

where the indices are taken modulo 3 [15, Chap. II, Lemma 11].

| \begin{tabular}{c|cccccccc}
| \hline
| & $e_1$ & $e_2$ & $e_3$ & $e_4$ & $e_5$ & $e_6$ & $e_7$ \\
| \hline
| $e_1$ & $\alpha e_0$ & $e_3$ & $\alpha e_2$ & $e_5$ & $\alpha e_4$ & $-e_7$ & $-\alpha e_6$ \\
| $e_2$ & $-e_3$ & $\beta e_0$ & $-\beta e_1$ & $e_6$ & $e_7$ & $\beta e_4$ & $\beta e_5$ \\
| $e_3$ & $\alpha e_2$ & $\beta e_1$ & $-\alpha e_0$ & $e_7$ & $\alpha e_6$ & $-\beta e_5$ & $\alpha \beta e_4$ \\
| $e_4$ & $-e_5$ & $-e_6$ & $-e_7$ & $\gamma e_0$ & $-\gamma e_1$ & $-\gamma e_2$ & $-\gamma e_3$ \\
| $e_5$ & $-\alpha e_4$ & $-e_7$ & $-\alpha e_6$ & $\gamma e_1$ & $-\gamma e_0$ & $\gamma e_3$ & $\gamma e_2$ \\
| $e_6$ & $e_7$ & $-\beta e_4$ & $\beta e_5$ & $\gamma e_2$ & $-\gamma e_3$ & $-\beta e_0$ & $-\beta e_1$ \\
| $e_7$ & $\alpha e_6$ & $-\beta e_5$ & $\alpha \beta e_4$ & $\gamma e_3$ & $-\gamma e_2$ & $\beta e_1$ & $\alpha \beta e_0$
\end{tabular}
1. ALTERNATIVE ALGEBRAS WITH SMALL LENGTH

We begin our study with the following proposition that extends the associative case [5, Sect. 2].

**Proposition 1.1.** Let \( A \) be an alternative algebra with \( l(A) \) finite; then \( \dim_F A \) is finite.

**Proof:** Let \( R \) be the nilpotent radical of \( A \). First we show that \( \dim_F(R) = l(R) \). Since \( R \) is nilpotent, there exists \( n \in \mathbb{Z}^+ \) such that \( R^{n+1} = 0 \). So, we have the chain \( 0 = R^{n+1} \subseteq \cdots \subseteq R^2 \subseteq R \). But \( R^i/R^{i+1} \) is null (that is, \( R^2 = 0 \)). Then each subspace of \( R^i/R^{i+1} \) is a subalgebra and it follows that \( l(R^i/R^{i+1}) = \dim_F(R^i/R^{i+1}) \). But \( \dim_F(R) = \sum \dim_F(R^i/R^{i+1}) = \sum l(R^i/R^{i+1}) \leq l(R) \). Therefore \( l(R) - \dim_F(R) \) is finite.

Now it suffices to see that \( \dim_F(A/R) \) is finite, or also, \( \dim_F(A) \) is finite if \( R = 0 \). In this case \( A = A_1 \oplus \cdots \oplus A_n \) [14], where \( A_i \)'s are \( C-D \) algebras over its center or simple associative algebras with finite length. Barnes showed that simple associative algebras with finite length are finite dimensional [4, Sect. 2]. And if \( A_i \) is a \( C-D \) algebra over its center and \( K_i \) is an extension field of \( F \), since \( A_i \) is an eight-dimensional over \( K_i \), it will suffice to show that \( K_i \) is a finite extension field of \( F \). Let \( t \in K_i \) and \( F[t] \) be the algebra of polynomials in \( t \). If \( t \) is transcendental over \( F \), then \( l(F[t]) \) is infinite. Therefore \( K \) is algebraic over \( F \). Since \( l(K_i) \) is finite, \( K_i \) is finitely generated over \( F \). Therefore \( K_i \) is finite dimensional over \( F \). Thus \( \dim_F(A_i) \) is finite and also \( \dim_F(A) \) is finite.

**Lemma 1.2.** Let \( A \) be an alternative algebra over \( F \) with \( l(A) = 1 \). Then \( \dim_F(A) = 1 \). Thus if \( A \) is an alternative algebra with \( l(A) \) finite and only one minimal subalgebra, then \( A \) is nilpotent or a division algebra.

**Proof:** Let \( A \) be nilpotent. From the proof of Proposition 1.1, \( \dim_F(A) = l(A) = 1 \). If \( A \) is not nilpotent, it is known, from the fact that \( l(A) \) is finite and from Proposition 1.1, that \( A \) contains some idempotent \( c \) [12, Proposition 3.3]. Thus \( (c) = A \).

Therefore the minimal subalgebras of an algebra \( A \) are spanned by idempotents or they are null.

Now, let \( A \) be an algebra with only one minimal subalgebra and \( l(A) \) finite. If \( A \) is not nilpotent, it contains some idempotent which spans the unique minimal subalgebra of \( A \), and thus \( R(A) = 0 \). Then \( A \) is a direct sum of simple algebras. But each of these summands contains a minimal subalgebra. Therefore \( A \) is simple. If \( A \cong M_n(D) \), the algebra of all \( n \times n \) matrices over the division algebra \( D \), it is clear that \( n = 1 \). If it is a \( C-D \) algebra, it could be a division algebra or have zero divisors. In the second
case \( A \) will have more than one minimal subalgebra. Therefore \( A \) is a division algebra. It is clear in this case that \( A \) has only one minimal subalgebra.

**Lemma 1.3.** (a) \( l(M_2(F)) = 4 \)

(b) Let \( C \) be the Cayley–Dickson central algebra over \( F \). Then:

(i) If \( C \) is a division algebra, \( l(C) = 4 \)

(ii) If \( C \) is split, \( l(C) = 7 \).

**Proof.** (a) We denote by \( \{e_{11}, e_{12}, e_{21}, e_{22}\} \) the usual basis of \( M_2(F) \). We can build the chain of subalgebras

\[
0 \leq (e_{11}) \leq (e_{11}, e_{22}) \leq (e_{11}, e_{12}, e_{22}) \leq (e_{11}, e_{12}, e_{21}, e_{22})
\]

(b)(i) Racine [11] showed in this case that the maximal subalgebras of \( C \) are division quaternion algebras. In a quaternion algebra the unique proper nonminimal subalgebras are quadratic field extensions of \( F \), and there are no subalgebras between these subalgebras and the division quaternion algebra. Therefore \( l(C) = 4 \).

(ii) From the multiplication table of \( C \) and [11, Theorem 5], we can check that the following chain of subalgebras is the longest:

\[
0 \leq (y_0) \leq (y_0, y_1) \leq (y_0, y_1, x_2) \leq (y_0, y_1, x_2, x_0, x_3) \leq C
\]

**Lemma 1.4.** Let \( A \) be an alternative algebra with \( l(A) = 2 \). Then \( A \) is associative.

**Proof.** Consider \( R(A) \). If \( l(R(A)) = 0 \), then \( A \) is semisimple, that is, a direct sum of simple algebras. From Lemma 1.3 the summands are associative. If \( l(R(A)) = 1 \), then from Lemma 1.2 \( \dim_F(R(A)) = 1 \) and

<table>
<thead>
<tr>
<th>Type</th>
<th>Defining relations</th>
<th>No. of minimal subalgebras</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Extension field ( K ) of ( F ) with ( F ) as a maximal subfield</td>
<td>1</td>
</tr>
<tr>
<td>II</td>
<td>((a, a^2)) with ( a^2 = 0 )</td>
<td>1</td>
</tr>
<tr>
<td>III(a)</td>
<td>((e, r), e^2 = e, r^2 = 0, er = re = 0 )</td>
<td>2</td>
</tr>
<tr>
<td>III(b)</td>
<td>((e, r), e^2 = e, r^2 = 0, er = re = r )</td>
<td>2</td>
</tr>
<tr>
<td>IV</td>
<td>( F \otimes F )</td>
<td>3</td>
</tr>
<tr>
<td>V</td>
<td>((a_i, a_j), a_i a_j = 0 ) for ( i, j )</td>
<td>( k + 1 )</td>
</tr>
<tr>
<td>VI(a)</td>
<td>((e, r), e^2 = e, r^2 = 0, er = r, re = 0 )</td>
<td>( k + 1 )</td>
</tr>
<tr>
<td>VI(b)</td>
<td>((e, r), e^2 = e, r^2 = 0, er = 0, re = r )</td>
<td>( k + 1 )</td>
</tr>
</tbody>
</table>
\[ \dim_F(A/R(A)) = l(A/R(A)) = 1. \] Therefore \( \dim_F A = 2 \) and from Artin's theorem, \( A \) is associative. If \( l(R(A)) = 2 \), then \( A \) is nilpotent and thus \( \dim_F (A) = 2 \). From Artin's theorem again \( A \) is associative.

From Lemma 1.4 and [5] we can now classify the alternative algebras with length two.

**Lemma 1.5.** Let \( k \) be the cardinal of \( F \). Suppose \( l(A) = 2 \). Then \( A \) is isomorphic to one of the algebras listed in Table II.

### 2. The Central Division Quaternion Algebra

In this section we suppose \( \text{char } F \neq 2 \).

Now we study the algebras \( \mathcal{L} \)-isomorphic to a division quaternion algebra. This is the first step before we consider the algebras \( \mathcal{L} \)-isomorphic to a division central C-D algebra.

Let \( Q \) be a central division quaternion algebra. It is known that \( Q \) is a noncommutative associative composition algebra with \( \dim_F Q = 4 \) and \( l(Q) = 3 \). If \( \{e_0, e_1, e_2, e_3\} \) is the usual basis of \( Q \) over \( F \), since \( \text{char } F \neq 2 \), the subalgebras with length two are quadratic extensions of \( F \), \( (e_0, \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3) \) with \( \lambda_1, \lambda_2, \lambda_3 \in F \). Two different subalgebras with length two span all \( Q \) and their intersection is \( F \).

Let \( A \) be an algebra \( \mathcal{L} \)-isomorphic to \( Q \). Then \( A \) has only one minimal subalgebra and from Lemma 1.2, \( A \) is nilpotent or a division algebra.

**Lemma 2.1.** \( A \) is nilpotent and \( \mathcal{L} \)-isomorphic to \( Q \) if and only if \( l(A) = \dim_F A = 3 \) and \( A \) has a basis \( \{a, b, c\} \) over \( F \) for which the multiplication table is Table III, with \( \varphi = 0 \) and then \( \gamma \in F - \{0\} \) and \( F \) is a field where the equation \( X^2 = -1 \) has no solution, or with \( \varphi = 1 \) and then \( \gamma \in F - \{0\} \) and such that the equation \( X^2 + X + \gamma - 0 \) has no solution in \( F \).

If \( A \) is nilpotent and \( \mathcal{L} \)-isomorphic to \( Q \), \( l(A) = \dim_F A = 3 \). Badalov showed that nilpotent alternative algebras with dimension smaller than 5 are associative [2]. Kruse and Price [9] described all nilpotent associative

| \(\begin{array}{c|ccc}
 a & b & c \\
 a & a & \varphi c & 0 \\
b & 0 & \gamma c & 0 \\
c & 0 & 0 & 0 \\
\end{array}\) |
algebras $A$ with $\dim_F A = 3$. Thus $A = (a, b, c)$ with $c \in \text{Ann}(A)$ and satisfies one of the following conditions:

1. $a^2 = b^2 = ab = ba = 0$
2. $a^2 = b^2 = 0$, $ab = -ba = c$
3. $a^2 = c$, $ab = ba = b^2 = 0$
4. $a^2 = c$, $ab = ba = 0$, $b^2 = \gamma c$ where $\gamma$ is a predetermined representative of its coset of $(F^*)^2$ in $F^*$
5. $a^2 = ab = c$, $ba = 0$, $b^2 = \gamma c$, some $\gamma \in F$
6. $a^2 = b$, $a^3 = c$, $a^4 = 0$ (power algebra).

Only types (4) and (5) give algebras $A$ $L'$-isomorphic to $Q$, because $A$ has only one minimal subalgebra. But in both cases it is necessary that the field $F$ have restrictions, because elements $a + \mu b$ cannot have square zero. These restrictions are in (4) that $X^2 = -1$ has no solution over $F$, and in (5) that $\gamma \neq 0$ such that $X^2 + X + \gamma = 0$ has no solution over $F$.

Conversely, we show that $A$ is $L'$-isomorphic to $Q$. $A$ has only one minimal subalgebra (c). The subalgebras with length 2 are $(c, a + \lambda b)$ and $(c, b)$ with $\lambda \in F$. We remark that two different subalgebras with length 2 span the algebra $A$ and their intersection is (c). The cardinal of $\mathcal{D} = \{A_1 \leq A : l(A_1) = 2\}$ is $|\mathcal{D}| = |F| + 1$.

From the Wedderburn theorem and since $Q$ is not a field, $|F|$ is infinite. Thus $|\mathcal{D}| = |F|$. But $\mathcal{C} = \{S \leq Q : l(S) = 2\} = \{(e_0, \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3) : \lambda_1, \lambda_2, \lambda_3 \in F\}$ and $|F| \leq |\mathcal{C}| \leq |F \times F \times F| = |F|$. Therefore there exists a one-to-one map from $\mathcal{C}$ onto $\mathcal{D}$, $\varphi$.

Now we can build the $L'$-isomorphism $\Psi : L'(Q) \to L'(A)$ such that $\Psi((e_0)) = (c)$, $\Psi(S) = \varphi(S)$ if $S \in \mathcal{C}$, $\Psi(Q) = A$, and $\Psi(0) = 0$.

**Lemma 2.2.** Let $A$ be a division algebra $L'$-isomorphic to $Q$. Then one of the following conditions holds:

(a) $A$ is an extension field of $F$, purely inseparable and $p^2$-dimensional with $p = \text{char } F$.

(b) $A$ is an $m^2$-dimensional central division associative algebra, where $m$ is the dimension of the subalgebras of $A$ with length 2. All these subalgebras will be extension fields of $F$ and at least one of them will be separable.

**Proof.** If $A$ is a division algebra, the center of $A$ can be $F$ or all $A$, because subalgebras with length two are maximal. Thus (i) $A$ is a field or (ii) $A$ is a central division algebra.

From Lemma 1.5 subalgebras of $A$ with length 2 are extension fields of $F$ without intermediate fields. Thus they are separable extensions or purely
inseparable extensions (p.i. extensions) of $F$. We denote by $\mathcal{P}$ the $\mathcal{L}$-isomorphism of $Q$ onto $A$. Let $\mathcal{P}((e_0, e_1))$, $\mathcal{P}((e_0, e_2))$, $\mathcal{P}((e_0, e_3))$ be subalgebras. These subalgebras indicate that two of the extensions are separable of p.i.

(i) We suppose that $A$ is a field. If the two field extensions mentioned above are separable, $A$ is finite dimensional separable extension of $F$. Moreover from the fact that $F$ is an infinite field, $A$ has an infinite number of subfields with length two: $\{\mathcal{P}((e_0, e_1 + \lambda_2 e_2 + \lambda_3 e_3)) | \lambda_2, \lambda_3 \in F\}$. Let $L$ be a normal closure of $A$. $L$ is a Galois extension of $F$. Its Galois group is a finite group which has a finite number of subgroups. From the Fundamental Theorem of Galois theory, $L$ will have a finite number of subfields. Contradiction. Thus all subfields of $A$ are p.i. extensions. Since $\mathcal{P}(Y((e_0, e_1))) = \mathcal{P}(Y((e_0, e_2)))$ then $\mathcal{P}(Y((e_0, e_1))) = \mathcal{P}(Y((e_0, e_2))) = p$ with $p = \text{char } F$ and thus $A$ is a $p^e$-dimensional over $F$.

(ii) We suppose that $A$ is a central division algebra. If will be associative because its length is 3. From [10, Sect. 13.1] each maximal subfield of $A$ will be $m$-dimensional with $d(A) = m^2$. But not all those maximal subfields can be purely inseparable because, from [10, Sect. 13.5], $A$ would be equal to $F$.

Now we give an example which shows that there exist extensions of $F$, p.i., with a lattice of subalgebras $\mathcal{L}$-isomorphic to $Q$.

**Example.** Consider $\mathbb{Z}_5(X, Y)(\sqrt[5]{X}, \sqrt[5]{Y})$. It is an algebra over $\mathbb{Z}_5(X, Y)$ (field of rational functions in $X$, $Y$ over $\mathbb{Z}_5$) with dimension 25. Let $Q$ be a division quaternion algebra over $\mathbb{Z}_5(X, Y)$. Then $\mathbb{Z}_5(X, Y)(\sqrt[5]{X}, \sqrt[5]{Y})$ and $Q$ are $\mathcal{L}$-isomorphic.

3. **Alternative Algebras $\mathcal{L}$-Isomorphic to Some Central Division Cayley–Dickson Algebra**

We suppose in all this section that $\text{char } F \neq 2$.

Let $A$ be an alternative algebra over $F$ and $\mathcal{P}: \mathcal{L}(C) \to \mathcal{L}(A)$ an $\mathcal{L}$-isomorphism. $A$ has only one minimal subalgebra and from Lemma 1.2, $A$ is a division algebra or a nilpotent algebra.

**Lemma 3.1.** Let $A$ be a division algebra that is not a purely inseparable extension. Then $A$ is a central division Cayley–Dickson algebra.

**Proof.** If $A$ is a division algebra, its center, $Z(A)$, could be $F$, an algebra with length two, or all $A$, because $l(A) = 4$. 
First we suppose that \( A \) is a field. Let \( Q \) be a maximal subalgebra of \( C \). We know that \( Q \) is a quaternion algebra and \( \mathcal{P}(Q) \) is a field. From Lemma 2.2, \( \mathcal{P}(Q) \) is also a \( p^2 \)-dimensional purely inseparable extension field of \( F \), with \( p = \text{char } F \). Therefore all subalgebras with length two of \( A \) will be purely inseparable extensions of \( F \) with dimension \( p \). Thus as is a purely inseparable \( p^2 \)-dimensional extension of \( F \), which is false.

If \( l(Z(A)) = 2 \), we can suppose that \( Z(A) = \mathcal{P}((e_0, e_1)) \) (from the Cayley–Dickson process). Therefore for all \( Q \leq C \) such that \( l(Q) = 3 \) and \( (e_0, e_1) \leq Q \), \( \mathcal{P}(Q) \) is a field. From Lemma 2.2, \( \mathcal{P}(Q) \) is a \( p^2 \)-dimensional purely inseparable (p.i.) extension field of \( F \), with \( p = \text{char } F \). Now we show that each proper subalgebra of \( A \) is also a p.i. extension field of \( F \). Let \( S \leq A \) be such that \( l(S) = 2 \). Thus \( S = \mathcal{P}(e_0, \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 + \mu_4 e_4 + \mu_5 e_5 + \mu_6 e_6 + \mu_7 e_7) \) with \( \mu_i \in F \) and for some \( i > 1 \) \( \mu_i \neq 0 \). There exist \( Q_1 \leq C \) such that \( l(Q_1) = 3 \) and \( \mathcal{P}(Q_1) = S \vee \mathcal{P}(e_0, e_1) \). We have seen that \( \mathcal{P}(Q_1) \) is a p.i. extension field of \( F \) and therefore \( S \) is a p.i. extension field of \( F \). Thus all subalgebras of \( A \) with length 2 are p.i. extension fields and therefore from Lemma 2.2 all proper subalgebras of \( A \) are p.i. extension fields. Think of \( A \) as an algebra over its center \( \mathcal{P}(e_0, e_1) \). It is a division associative algebra with length 3 whose proper subalgebras are p.i. extension fields. From [10, Sect. 13.5], \( A = \mathcal{P}(e_0, e_1) \), but this is false. Therefore \( A \) is a central algebra.

Now if we show that \( A \) is not associative, since \( A \) is a central division algebra, \( A \) will be a C-D division algebra. We suppose that \( A \) is associative. From Lemma 2.2 maximal subalgebras of \( A \) are either

(a) p.i. \( p^2 \)-dimensional extension fields of \( F \) or
(b) \( m^2 \)-dimensional division central associative algebras.

Thus one of the following three cases holds:

1. All the maximal subalgebras of \( A \) are of the type (a). Then from [10, Sect. 13.5] we have \( A = F \). False.

2. All the maximal subfields of these maximal subalgebras will be \( m \)-dimensional. Since \( A \) is a division central associative algebra, \( A \) has dimension \( m^2 \). False.

3. There exist maximal subalgebras of types (a) and (b). Let \( \mathcal{P}(Q) \) be of type (b). From Lemma 2.2 there exists \( \mathcal{P}(K) \leq \mathcal{P}(Q) \) such that \( \mathcal{P}(K) \) is a separable extension of \( F \). If \( \mathcal{P}(K) \) is maximal subfield of \( A \), since \( A \) has dimension \( n^2 \) with \( n \) the dimension of every maximal subfield of \( A \), \( A = \mathcal{P}(Q) \), but this is false. Therefore there exists \( \mathcal{P}(Q_1) \leq A \) of type (a) such that \( \mathcal{P}(K) \leq \mathcal{P}(Q_1) \). Contradiction, because \( \mathcal{P}(K) \) is a separable extension over \( F \). Therefore \( A \) is nonassociative alternative and thus is a C-D division algebra.
Lemma 3.2. Let \( C \) be a division central \( C-D \) algebra over \( F \), \( F \) a field with \( \text{char} \, F \neq 2 \). Then there exist maximal subalgebras of \( C \), \( Q_1 \) and \( Q_2 \), such that \( Q_1 \cap Q_2 = F \).

Proof. We take \( Q_1 = (e_0, e_1, e_2, e_3) \), \( Q_2 = (e_0, e_4, e_1 + e_6, e_5 + \gamma e_2) \). It is easy to check \( Q_1 \cap Q_2 = F \).

Lemma 3.3. There exists no nilpotent algebra \( \mathcal{L} \)-isomorphic to \( C \).

Proof. Suppose \( A \) is a nilpotent algebra such that there exists an \( \mathcal{L} \)-isomorphism \( \Psi: \mathcal{L}(C) \rightarrow \mathcal{L}(A) \). Let \( A_1 = \Psi(e_0, e_1, e_2, e_3) \). \( A_1 \) is nilpotent and from Lemma 2.1 it has a basis \( \{a, b, c\} \) with a multiplication table as in Table III.

Let \( A_2 \) be such that \( A_2 \leq A \), \( A_2 \) is not a subalgebra of \( A_1 \), and \( l(A_2) = 2 \). From the fact that \( A \) has a unique minimal subalgebra, \( A_2 = (c, d_1) \) with \( d_1 \in A \) and \( d_1^2 = \lambda c \), \( 0 \neq \lambda \in F \). If now we consider \( (c, a) \lor (c, d_1) \) we have a maximal subalgebra of \( A \). From Lemma 2.1 and changing \( d_1 \) by a suitable \( d \in A \), we can find a basis \( \{a, b, d, c\} \) with multiplication as in Table II.

Thus \( A \) has a basis \( \{a, b, d, c\} \) with multiplication as in Table IV.

From Lemma 3.2, \( (e_0, e_1, e_2, e_3) \cap (e_0, e_4, e_1 + e_6, e_5 + \gamma e_2) = F \). Now suppose \( \Psi(e_0, e_4, e_1 + e_6, e_5 + \gamma e_2) = (\lambda_1 a + \lambda_2 b + \lambda_3 d), \mu_1 a + \mu_2 b + \mu_3 d \) \( = A_3 \) with \( \lambda_i, \mu_i \in F \) for \( i = 1, 2, 3 \) and some \( \lambda_i \neq 0 \) and some \( \mu_i \neq 0 \) and with \( (\lambda_1, \lambda_2, \lambda_3) \) not proportional to \( (\mu_1, \mu_2, \mu_3) \). Then \( l(A_1 \cap A_2) = 2 \). Contradiction.

We can now present the following theorem:

Theorem 3.4. Let \( A \) be an alternative algebra over \( F \), \( F \) a field with \( \text{char} \, F \neq 2 \), \( \mathcal{L} \)-isomorphic to a central division Cayley–Dickson algebra. Then \( A \) is a division central Cayley–Dickson algebra or, if \( \text{char} \, F = p > 0 \), a purely inseparable \( p^3 \)-dimensional extension field of \( F \).

Corollary 3.5. Let \( F \) be a perfect field and \( \Psi: \mathcal{L}(C) \rightarrow \mathcal{L}(A) \) an \( \mathcal{L} \)-isomorphism of alternative algebras over \( F \). If \( C \) is a division central

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Note. \( \theta, \varepsilon \in F \).
Cayley–Dickson algebra, then $A$ is a division central Cayley–Dickson algebra.

The following example shows that in an extension field of $F$, $K = F(a, b, c)$, with $[K : F] = p^3$ and $a^p$, $b^p$, $c^p \in F$, there are maximal subalgebras whose intersection is $F$ and others whose intersection is a subalgebra with length two.

**Example.** $K = \mathbb{Z}_3(X, Y, Z)(\sqrt[3]{X}, \sqrt[3]{Y}, \sqrt[3]{Z})$ with $F = \mathbb{Z}_3(X, Y, Z)$. Take $A_1 = F(\sqrt[3]{X} + \sqrt[3]{Y})$ and $A_2 = F(\sqrt[3]{X} - \sqrt[3]{Y})$. Then $A_1 \cap A_2 = F$. Take $A_1 = F(\sqrt[3]{X})$ and $A_2 = F(\sqrt[3]{Y})$. Then $A_1 \cap A_2 = F(\sqrt[3]{X} + \sqrt[3]{Y})$.

4. **ALTERNATIVE ALGEBRAS $\mathcal{L}$-ISOMORPHIC TO THE CENTRAL SPLIT CAYLEY–DICKSON ALGEBRA**

Let $M = M_2(F)$ be the $2 \times 2$ matrix algebra over the ground field $F$. We denote by $e_i$ the usual $F$-basis of $M$. From Lemma 1.3 we can show, as in the associative case [5], the following.

**Lemma 4.1.** Let $\Psi : \mathcal{L}(M) \rightarrow \mathcal{L}(A)$ be an $\mathcal{L}$-isomorphism of alternative algebras over the same field $F$. Then $A$ is isomorphic to $M$ and we can check that $a_i \in A$ such that $\Psi((e_i)) = (a_i)$ and the $a_i$'s have a multiplication table the same as or opposite that of the $e_i$'s (that is, $a_i a_{ki} = \delta_{ik} a_i$ or $a_i a_{ki} = \delta_{ik} a_i$ for all $i, j, k, l$)

Now, we can use this to show

**Theorem 4.2.** Let $C$ be a central split Cayley–Dickson algebra and let $A$ be an algebra $\mathcal{L}$-isomorphic to it. Then $A$ is also a central split Cayley–Dickson algebra.

**Proof.** Consider the $F$-basis of $C$ mentioned above. We remark that each of the subalgebras $(x_0, -x_1, y_0, y_1)$, $(x_0, -x_2, y_0, y_2)$, and $(x_0, -x_3, y_0, y_3)$ has a basis with a multiplication table like that of the usual $F$-basis of $M_2(F)$. Let $\Psi$ be the $\mathcal{L}$-isomorphism from $C$ onto $A$.

From Lemma 4.1, $\Psi((x_0, -x_1, y_0, y_1)) \cong M_2(F)$ and there exist $z_0, z_1, w_0, w_1 \in A$ such that $\Psi((x_0)) = (z_0)$, $\Psi((x_1)) = (z_1)$, $\Psi((y_0)) = (w_0)$, $\Psi((y_1)) = (w_1)$, with $\{z_0, -z_1, w_0, w_1\}$ a basis of $\Psi((x_0, -x_1, y_0, y_1))$ with a multiplication table the same as or opposite that of $\{x_0, -x_1, y_0, y_1\}$. Similarly there exist $z_2, w_2 \in A$ such that $\Psi((x_2)) = (z_2)$, $\Psi((y_2)) = (w_2)$, and $\{z_0, -z_2, w_0, w_2\}$ is a basis of $\Psi((x_0, -x_2, y_0, y_2))$ with a multiplication table the same as or opposite that of
\{x_0, -x_2, y_0, y_2\}. Also, there exist \(z_3, w_3 \in A\) such that \(\Psi((x_3)) = (z_3)\), \(\Psi((y_3)) = (w_3)\), and \(\{z_0, -z_3, w_0, w_3\}\) is a basis of \(\Psi((x_0, -x_3, y_0, y_3))\) with a multiplication table the same as or opposite that of \(\{x_0, -x_3, y_0, y_3\}\).

1) Consider \(R = R(B)\). If \(l(R) = 0\), then \(B\) is semisimple. But this is not possible because \(z_1, z_2, w_3 \in B\) and they are nilpotent. Nor is \(l(R) = 1, 2\) because \(B/R\) does not have nilpotent elements and \(z_1 + R, z_2 + R, w_3 + R\) are nilpotent. Therefore \(B = R\) and \(d(B) = l(B) = 3\). From \(B = \Psi((x_1) \vee (y_1) \vee (x_2)) = (z_1) \vee (z_2) \vee (w_3)\), it follows that \(z_1, z_2, w_3\) are a basis of \(B\).

2) Suppose \(\{z_0, z_1, w_0, w_1\}\) has a multiplication table like that of \(\{x_0, -x_1, y_0, y_1\}\). We show that \(\{z_0, -z_2, w_0, w_2\}\) and \(\{z_0, -z_3, w_0, w_3\}\) have the same multiplication tables as those of \(\{x_0 - x_2, y_0, y_2\}\) and \(\{x_0 - x_3, y_0, y_3\}\), respectively.

If we suppose \(\{z_0, -z_2, w_0, w_2\}\) has multiplication opposite that of \(\{x_0 - x_2, y_0, y_2\}\), then \(z_0z_2 = 0 = w_0w_2, z_2z_0 = z_2,\) and \(w_2w_0 = w_2\). We have shown above in (1) that \((z_1, z_2, w_3)\) is a subalgebra with basis \(\{z_1, z_2, w_3\}\). Thus there exist \(\lambda_1, \lambda_2, \lambda_3 \in F\) such that \(z_1z_2 = \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 w_3\). From the middle Moufang identity

\[z_1z_2 = (z_0z_1)(z_2z_0) = z_0(z_1z_2) = z_0(\lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 w_3)z_0 = 0.\]

From \(l((x_1) \vee (x_2)) > 2\), it follows that \(0 \neq z_2z_1 = \mu_1 z_1 + \mu_2 z_2 + \mu_3 w_3\) with \(\mu_1, \mu_2, \mu_3 \in F\) and \(\mu_3 \neq 0\). From the left Moufang identity

\[z_0(z_2(z_0z_1)) = (z_0z_2z_0)z_1 = 0.\]

But

\[z_0(z_2(z_0z_1)) = \mu_1 z_1 + \mu_3 z_0 w_3,\]

Therefore \(\mu_1 = 0\) and from \(\mu_3 \neq 0\) we will have \(z_0w_3 = 0\). Thus \(w_3z_0 = w_3\).

But from the right Moufang identity

\[0 = (z_2)(z_0z_1z_0) = ((z_2z_0)z_1)z_0 = \mu_2 z_2 + \mu_3 w_3 z_0 = \mu_2 z_2 + \mu_3 w_3.\]

Contradiction. Thus \(z_0z_2 = z_2, z_2z_0 = 0 = w_2w_0\). In the same way \(\{z_0, -z_3, w_0, w_3\}\) has the same multiplication table as that of \(\{x_0, -x_3, y_0, y_3\}\), but not the opposite.

If we had supposed that \(\{z_0, -z_2, w_0, w_2\}\) has a multiplication table opposite that of \(\{x_0, -x_1, y_0, y_1\}\), then we would have shown that \(\{z_0, -z_2, w_0, w_2\}\) and \(\{z_0, -z_3, w_0, w_3\}\) have the same multiplication table as that of \(\{x_0, -x_2, y_0, y_2\}\) and \(\{x_0, -x_3, y_0, y_3\}\).
(3) We want to obtain the multiplication table of \( \{z_0, z_1, z_2, z_3, w_0, w_1, w_2, w_3 \} \) in order to show that they form a basis of \( A \) over \( F \). We have seen above in (1) that \( \{z_1, z_2, w_3\} \) is a subalgebra of \( A \). Thus \( z_1 z_2 = v_1 z_1 + v_2 z_2 + v_3 w_3 \) for \( v_1, v_2, v_3 \in F \). From the left Moufang identity

\[
0 = (z_0 z_1 z_0) z_2 = z_0 (z_1 (z_0 z_2)) = v_1 z_0 z_1 + v_2 z_0 z_2 + v_3 z_0 w_3 \]

Therefore \( v_1 = v_2 = 0 \) and \( z_1 z_2 = v_3 w_3 \). In the same way \( z_2 z_1 = \sigma_3 w_3 \) for some \( \sigma_3 \in F \). Now take the subalgebras \( \{x_2, x_3, y_1\}, \{x_1, x_3, y_2\}, \{y_1, y_2, x_3\}, \{y_2, y_3, x_1\}, \{y_3, y_1, x_2\} \) and do the same thing. Changing the notation we have

\[
\begin{align*}
z_1 z_2 &= \alpha_3 w_3 \quad w_1 w_2 = \gamma_3 z_3 \\
z_2 z_1 &= \beta_3 w_3 \quad w_2 w_1 = \theta_3 z_3 \\
z_3 z_2 &= \alpha_1 w_1 \quad w_3 w_2 = \gamma_1 z_1 \\
z_3 z_1 &= \beta_1 w_1 \quad w_3 w_2 = \theta_1 z_1 \\
z_1 z_3 &= \alpha_2 w_2 \quad w_1 w_3 = \theta_2 z_2 \\
z_2 z_3 &= \alpha_2 w_2 \quad w_1 w_3 = \theta_2 z_2
\end{align*}
\]

with \( \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3, \theta_1, \theta_2, \theta_3 \in F \). Thus

\[
[z_1, z_2, z_3] = (z_1 z_2) z_3 - z_1 (z_2 z_3) = \alpha_3 w_0 - \alpha_1 z_0 \\
= -[z_1, z_3, z_2] = -(z_1 z_3) z_2 + z_1 (z_2 z_3) = -\beta_2 w_0 + \beta_1 z_0 \\
= -[z_2, z_1, z_3] = -(z_2 z_1) z_3 + z_2 (z_1 z_3) = -\beta_3 w_0 + \beta_2 z_0 \\
= -[z_3, z_2, z_1] = -(z_3 z_2) z_1 + z_3 (z_2 z_1) = -\beta_1 w_0 + \beta_3 z_0 \\
= [z_2, z_3, z_1] = (z_2 z_3) z_1 - z_2 (z_3 z_1) = \alpha_1 w_0 - \alpha_2 z_0,
\]

and then \( \alpha_1 = \alpha_2 = \alpha_3 = -\beta_1 = -\beta_2 = -\beta_3 \).

In the same way with the \( w_i \)'s and the associator \( [w_1, w_2, w_3] \) we obtain \( \gamma_1 = \gamma_2 = \gamma_3 = -\theta_1 = -\theta_2 = -\theta_3 \). Next we look for a relation between \( \alpha_3 \) and \( \gamma_2 \). We take the subalgebras

\[
(z_1) \lor (w_3), (z_1) \lor (w_2), (z_2) \lor (w_3), (z_2) \lor (w_1), (z_3) \lor (w_1), (z_3) \lor (w_2).
\]

From Lemma 1.5, \( z_i^2 - w_i^2 = 0 \). Thus

\[
\begin{align*}
[z_1, z_2, w_1] &= (z_1 z_2) w_1 - z_1 (z_2 w_1) = \alpha_3 w_3 w_1 = \alpha_3 \gamma_2 z_2 \\
[z_1, w_1, z_2] &= -(z_1 w_1) z_2 + z_1 (w_1 z_2) = -z_0 z_2 = -z_2.
\end{align*}
\]
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Therefore $\alpha_3 = -\gamma_2^{-1}$. Now changing the notation

$$
\begin{align*}
 z_1 z_2 &= w_3 = -z_2 z_1 \\
 z_1 z_3 &= -w_2 = -z_3 z_1 \\
 z_2 z_3 &= w_1 = -z_3 z_2 \\
 w_1 w_2 &= z_3 = -w_2 w_1 \\
 w_1 w_3 &= -z_2 = -w_3 w_1 \\
 w_2 w_3 &= z_1 = -w_3 w_2
\end{align*}
$$

This completes the multiplication table of $\{z_0, z_1, z_2, z_3, w_0, w_1, w_2, w_3\}$, and then it is easy to check that these elements are an $F$-basis of $A$.

5. ALTERNATIVE ALGEBRAS $\mathcal{L}$-ISOMORPHIC TO SOME ARBITRARY MATRIX ALGEBRA

**Lemma 5.1.** Let $D$ be a finite dimensional division algebra, let $M = M_n(D)$ with $n \geq 2$, and let $N$ be a nilpotent subalgebra of $M$. If $\Psi$ is an $\mathcal{L}$-isomorphism of alternative algebras, then $\Psi(N)$ is nilpotent and $d(\Psi(N)) = d(N)$.

**Proof.** If $d(N) = 1$, then there exist $U \leq M$ with $N \leq U$ and $\varphi : U \to M_2(F)$ an isomorphism such that $\varphi(N) = (e_{12})$ [1, Chap. III, Lemma 18]. Thus, from Lemma 4.1, $N$ is nilpotent. For general $N$, every one-dimensional subalgebra of $N$ is nilpotent. Hence every minimal subalgebra of $\varphi(N)$ is nilpotent and therefore $\varphi(N)$ is nilpotent. Then we have $d(\varphi(N)) = I(\varphi(N)) = I(N) = d(N)$.

**Lemma 5.2.** Let $M = M_n(F)$ with $n \geq 2$. If $A$ is $\mathcal{L}$-isomorphic to $M$, then $A \cong M$.

**Proof.** Let $\{e_{ij} : i = 1, \ldots, n\}$ be the usual basis of $M$. Each subalgebra $(e_{ii}, e_{ij}, e_{ji}, e_{jj})$ with $i \neq j$ is isomorphic to $M_2(F)$. We denote $A_{ij} = \psi((e_{ij}))$ if $\Psi$ is the $\mathcal{L}$-isomorphism from $M$ onto $A$. From Lemma 4.1, $A_{ii}^2 = A_{ii}$ for every $i$. Let $a_{ii}$ be the unique idempotent not zero in $A_{ii}$. Then $a_{ii}a_{jj} = 0$ for every $i \neq j$.

From Lemma 4.1 we know there exists $a_{ij} \in A_{ij}$ such that $(a_{ij}) \in A_{ij}$ and (i) $a_{ii}a_{ij} = a_{ij} = a_{ij}a_{ij}$, $a_{ij}a_{jj} = 0 = a_{ij}a_{jj}$ or (ii) $a_{ii}a_{jj} = 0 = a_{ii}a_{jj}$, $a_{ii}a_{ij} = a_{ij}a_{ij}$.

(1) We consider $(a_{ij}, a_{kl})$ with $i, j, k, l$ distinct. From Lemma 4.1, $(a_{ij})^2 = 0 = (a_{kl})^2$ and $(a_{ij}, a_{kl})$ has $k + 1$ minimal subalgebras with $k = |F|$; therefore from Lemma 1.5 it is a zero algebra and thus $a_{ij}a_{kl} = 0$.

(2) In the same way for $i, j, k$ distinct we obtain, considering $(a_{ij}, a_{ik})$, $a_{ij}a_{ik} = 0$. Also, with $(a_{ij}, a_{kj})$, $a_{ij}a_{kj} = 0$.

We remark that (1), (2) are true for every $0 \neq a_{ij} \in A_{ij}$.

(3) We now consider $(a_{ii}, a_{jk})$ with $i \neq j, k$. Then $(a_{ii}, a_{jk})$ has two
minimal subalgebras; therefore from Lemma 1.7 for every \( 0 \neq a_{jk} \in A_{jk} \) it holds that
\[
a_{ii}a_{jk} = 0 = a_{jk}a_{ii} \quad \text{or} \quad a_{ii}a_{jk} = a_{jk} = a_{jk}a_{ii}.\]
We show that the second case is not possible. If \( a_{ii}a_{jk} = a_{jk} = a_{jk}a_{ii} \), since
\[
a_{jk}a_{ij} = a_{jk} \quad \text{or} \quad a_{jj}a_{jk} = a_{jk},
\]
we have in the first case
\[
0 = a_{jk}(a_{ji}a_{ii}a_{jj}) = ((a_{ik}a_{jj})a_{ii})a_{ii} = a_{jk},
\]
a contradiction, and in the second case
\[
0 = (a_{ji}a_{ii}a_{jj})a_{jk} = a_{jj}(a_{ii}(a_{jj}a_{jk})) = a_{jj}a_{jk} = a_{jk},
\]
also a contradiction. Thus \( a_{ii}a_{jk} = 0 = a_{jk}a_{ii} \) if \( i \neq j, k \).

(4) Now suppose \( a_{ii}a_{ij} = a_{ij} \) with \( 0 \neq a_{ij} \in A_{ij} \). We have that \( A_{ij} \vee A_{jk} \)
is a three-dimensional nilpotent subalgebra for \( i, j, k \) distinct, with \( F \)-basis
\[
\{a_{ij}, a_{jk}, a_{ik}\}
\]
(we can show this as in (1) of the Proof of Theorem 5.1). Thus \( a_{jk}a_{ij} = \lambda_1 a_{ik} + \lambda_2 a_{ij} + \lambda_3 a_{ik} \) with \( \lambda_1, \lambda_2, \lambda_3 \in F \). Since \( a_{jk}a_{ij} = a_{jk}(a_{ii}a_{ij}) \), from (3)
\[
0 = (a_{ii}a_{jk}a_{ii})a_{ij} = a_{ii}(a_{jk}(a_{ii}a_{ij})) = \lambda_2 a_{ij} + \lambda_3 a_{ii}a_{ik}
\]
and \( a_{ii}a_{jk} \in A_{ik} \).

Now we separate the following cases:

(i) \( a_{ii}a_{jk} \neq 0 \). Then \( \lambda_2 = \lambda_3 = 0 \) and therefore \( a_{jk}a_{ij} = \lambda_1 a_{jk} \). Since
\[
a_{jk}a_{rr} = a_{jk} \quad \text{for} \ r = j \quad \text{or} \ r = k,\]
it holds that
\[
0 = a_{jk}(a_{rr}a_{ij}a_{rr}) = ((a_{jk}a_{rr})a_{ij})a_{rr} = \lambda_1 a_{jk}a_{rr} = \lambda_1 a_{jk}.
\]
Therefore \( \lambda_1 = 0 \). But \( d(A_{ij} \vee A_{jk}) = 3 \), hence \( a_{ij}a_{jk} \neq 0 \).

(ii) \( a_{ii}a_{jk} = 0 \). Then \( a_{kk}a_{ik} \neq 0 \), and from \( 0 = \lambda_2 a_{ij} + \lambda_1 a_{ii}a_{ik} \), we obtain \( \lambda_2 = 0 \). Therefore \( a_{jk}a_{ij} = \lambda_1 a_{jk} + \lambda_3 a_{ik} \). From (3), \( a_{ik}a_{kk} = 0 \) and
\[
a_{jk}a_{rr} = a_{jk} \quad \text{for} \ r = j \quad \text{or} \ k,\]
it holds that
\[
0 = a_{jk}(a_{rr}a_{ij}a_{rr}) = ((a_{jk}a_{rr})a_{ij})a_{rr} = \lambda_1 a_{jk}.
\]
Thus \( \lambda_1 = 0 \) and \( a_{jk}a_{ij} = \lambda_3 a_{ik} \). But \( 0 \neq (a_{kk}a_{ik})a_{ii} \in A_{ik} \), therefore
\[
(a_{kk}(a_{jk}a_{ij}))a_{ii} = \lambda_3 (a_{kk}a_{ik})a_{ii} = \lambda_3 a_{ik}' \quad \text{with} \ a_{ik}' \in A_{ik}.
\]
The linearization of the middle Moufang identity gives us
\[
(a_{kk}(a_{jk}a_{ij}))a_{ii} = -(a_{ii}(a_{jk}a_{ij}))a_{kk} + (a_{kk}a_{jk})(a_{ij}a_{ii})
\]
\[
+ (a_{ii}a_{jk})(a_{ij}a_{kk}).
\]
Since $a_{ij}a_{ii} = 0$ and $a_{ij}a_{kk} = 0$, it holds that

$$\lambda_3 a_{ik} = (a_{kk}(a_{jk}a_{ij})) a_{ii} = -(a_{ii}(a_{jk}a_{ij})) a_{kk}$$

$$= -(a_{ii}(a_{jk}(a_{ij}a_{jk}))) a_{kk}$$

$$= -((a_{ii}a_{jk}a_{ii}) a_{ij}) a_{kk} = 0.$$

Thus $\lambda_3 = 0$. Hence $d(A_{ij} \vee A_{jk}) = 3$; we obtain also in this case $a_{ij}a_{jk} \neq 0$.

(5) Again suppose that $a_{ij}a_{ij} = a_{ij}$. We will show that $a_{ij}a_{jk} \neq 0$ with $j \neq k$. We proceed supposing $a_{ij}a_{jk} = 0$. Then $a_{jk}a_{jj} = a_{jk}$, $a_{kk}a_{jk} = a_{jk}$, and $a_{jk}a_{kk} = 0$. Since $a_{ij}a_{jk} = \mu_1 a_{ij} + \mu_2 a_{jk} + \mu_3 a_{ik}$ for $\mu_1, \mu_2, \mu_3 \in F$, one of which is not zero, from (3) we have

$$0 = a_{ij}(a_{ij}a_{jk}a_{ij}) = ((a_{ij}a_{ij}) a_{jk}) a_{ij} = \mu_1 a_{ij} + \mu_2 a_{jk}.$$

Thus $\mu_1 = \mu_2 = 0$ and $a_{ij}a_{jk} = \mu_3 a_{ik}$. We distinguish two cases:

(i) $0 \neq (a_{ij}a_{jk}) a_{kk} = a_{jk} \in A_{ik}$.

Then from the linearization of the middle Moufang identity

$$\mu_3 a_{ik} = (a_{ii}(a_{ij}a_{jk})) a_{kk} = (a_{ii}a_{ij})(a_{jk}a_{kk}) + (a_{kk}a_{ij})(a_{jk}a_{ii})$$

$$= -(a_{kk}(a_{ij}a_{jk})) a_{ii}.$$

But $a_{jk}a_{kk} = 0 = a_{kk}a_{ij}$, therefore

$$\mu_3 a_{ik} = -(a_{kk}(a_{ij}a_{jk})) a_{ii} = -(a_{kk}(a_{ij}(a_{kk}a_{jk}))) a_{ii}$$

$$= -((a_{kk}a_{ij}a_{kk}) a_{jk}) a_{ij} = 0.$$

Thus $\mu_3 = 0$ and we have a contradiction with $a_{ij}a_{jk} \neq 0$. Therefore $a_{ij}a_{jk} = 0$.

(ii) $a_{kk}a_{jk} = a_{jk}$. Then

$$\mu_3 a_{jk} = a_{kk}(a_{ij}(a_{kk}a_{jk})) = (a_{kk}a_{ij}a_{kk}) a_{jk} = 0.$$

Therefore $\mu_3 = 0$. Again we have a contradiction with $a_{ij}a_{jk} \neq 0$. Thus $a_{ij}a_{jk} = a_{jk}$ also in this case.

(6) From the hypothesis $a_{ii}a_{ij} = a_{ij}$ we had $a_{ij}a_{jk} = \mu_1 a_{ij} + \mu_2 a_{jk} + \mu_3 a_{ik}$ with $\mu_1, \mu_2, \mu_3 \in F$, one of which is not zero (4). We want to find $\mu_1, \mu_2, \mu_3$. Hence we start from the left and right Moufang identities

$$0 = a_{ij}(a_{ij}(a_{jk}a_{jk})) = \mu_2 a_{jk} \quad \text{and thus } \mu_2 = 0,$$

$$0 = ((a_{ij}a_{ij}) a_{jk}) a_{jj} = \mu_1 a_{ij} \quad \text{and thus } \mu_1 = 0.$$
Therefore $a_j a_{jk} = \mu_0 a_k$. It shows that we can choose $\mu_3$ for every $k$ so that $\mu_3 = 1$. Suppose $i = 1$ and $j = 2$, then $a_{11} a_{12} = a_{12}$. We pick $0 \neq a_{12} \in A_{12}, \ldots, 0 \neq a_{1n} \in A_{1n}$ arbitrarily. From Lemma 4.1, we can find $a_{ij} \in A_{ij}$ such that $a_{ij} a_{ij} = a_{ij}$ for $i = 2, \ldots, n$. The $a_{ij}$'s are univocally determined by this condition and they satisfy $a_{ij} a_{ij} = a_{ij}$. For $i \neq j$ and $i, j \neq 1$, we check $a_{ij} \in A_{ij}$ such that $a_{ij} a_{ij} = a_{ij}$. Thus the $a_{ij}$'s are univocally determined. Then

$$a_{1k} = a_{1j} a_{jk} = (a_{1i}, a_{ij}) a_{jk} = [a_{1i}, a_{ij}, a_{jk}] + a_{1k}(a_{ij} a_{jk})$$

$$= -[a_{1i}, a_{jk}; a_{ij}] + a_{1k}(a_{ij} a_{jk}) = -(a_{1i} a_{jk}) a_{ij} + a_{1i}(a_{jk} a_{ij})$$

$$+ a_{1k}(a_{ij} a_{jk}) = a_{1k}(a_{ij} a_{jk}).$$

Thus $a_{ij} a_{jk} = a_{jk}$. Therefore \{a_{ij}\} have the same multiplication as \{e_{ij}\}, if $a_{11} a_{12} \neq 0$. In the same way if $a_{11} a_{12} = 0$ we should find \{a_{ij}\} with a multiplication table opposite that of \{e_{ij}\}.

**Lemma 5.3.** Let $\eta$ be an idempotent not zero in $M = M_n(D)$, where $D$ is a division associative algebra and $n \geq 2$. Then if $\Psi$ is an $L$-isomorphism from $M$ onto $A$, $\Psi((\eta))$ is not nilpotent and $\Psi((\eta)) = (e)$ for some idempotent $e \in A$.

**Proof.** Similar to Lemma 13 in [5].

Barnes [4] showed that in associative rings with the minimum condition for left ideals, the radical is the intersection of the maximal nilpotent subrings. This result can be extended to alternative rings, following the Barnes process and using the next proposition,

**Proposition 5.4.** Let $C$ be a Cayley–Dickson algebra over the field $F$. Then there exist maximal nilpotent subrings $U$, $V$, $L$, of $A$ such that $U \cap V \cap L = 0$.

**Proof.** It is clear if $C$ is a division C–D algebra. If $C$ is a split C–D algebra, consider the usual basis of $C$. The subalgebras $(x_1, y_2, x_3)$, $(x_1, x_2, y_3)$, $(x_2, x_3, y_1)$ are nilpotent. Moreover, they are maximal nilpotent subalgebras. We consider, for example, $(x_1, y_2, x_3)$, and we suppose there exists $N \subseteq C$ such that it is nilpotent and $(x_1, y_2, x_3) \subseteq N$. Take $a \in N - (x_1, y_2, x_3)$. We can suppose

$$a = \mu_0 x_0 + \mu_2 x_2 + \mu_4 y_0 + \mu_5 y_1 + \mu_7 y_3$$

with $\mu_0, \mu_2, \mu_4, \mu_5, \mu_7 \in F$.

We have $x_1 a = \mu_2 y_3 + \mu_4 x_1 - \mu_5 x_0 \in N$ and thus $x_1 a$ is nilpotent. But $(x_1 a)^2 = -\mu_2 \mu_4 x_0 - \mu_4 \mu_5 y_3 + \mu_2^2 x_0$. Since $x_0$ is idempotent $\mu_5 = 0$. Using the same $y_2 a = \mu_0 y_0 + \mu_7 x_1 \in N$ is nilpotent. Since $(y_2 a)^2 = -\mu_2 \mu_0 y_0 + \mu_2^2 y_0 - \mu_2 \mu_7 x_1$ and $y_0$ is idempotent, it is necessary that $\mu_2 = 0$. 
Then we have \( a = \mu_0 x_0 + \mu_4 y_0 + \mu_7 y_3 \) is nilpotent. Thus \( \mu_0 = 0 \) and \( \mu_4 = 0 \), because \( x_0, y_0 \) are idempotent. Finally, \( a = \mu_7 y_3 \) and \( x_3 a = -\mu_7 x_0 \in N \) is nilpotent. Then \( \mu_7 = 0 \). Therefore \((x_1, y_2, x_3)\) is a maximal nilpotent subalgebra. Now we let \( U = (x_1, y_2, x_3) \), \( V = (x_1, x_2, y_3) \), and \( L = (x_2, y_3, y_1) \) and we have shown the lemma.

**Theorem 5.5** Let \( S = M_n(\Delta) \), where \( n \geq 2 \) and \( \Delta \) is a finite dimensional division associative algebra. Let \( A \) be an algebra \( L \)-isomorphic to \( S \) by the \( L \)-isomorphism \( \Psi \). Then \( A \cong M_n(D) \), where \( D \) is a division associative algebra \( L \)-isomorphic to \( \Delta \) such that \( d(D) = d(\Delta) \).

**Proof.** From Lemmas 5.1, and 5.3, for \( U \subseteq S \), \( \Psi(U) \) is nilpotent if and only if \( U \) is nilpotent. Thus the maximal nilpotent subalgebras of \( A \) are the images under \( \Psi \) of maximal nilpotent subalgebras of \( S \). From the commentary before Proposition 5.4, \( R(S) = 0 \) implies \( R(A) = 0 \).

Let \( N = M_n(F) \subseteq S \) and \( 1_S \) be the identity of \( S \). We can identify \( \Delta \) with the subalgebra \( 1_S A \). Then \( S = N \lor A, N \cap A = (1_S) \). Let \( B \) be some simple direct summand of \( A \). Then \( B \) contains an idempotent \( e \). Let \( U = \Psi^{-1}((e)) \). If \( U \) is nilpotent, from Lemma 5.1, \( \Psi(U) = (e) \) is nilpotent and thus \( e \) will not be idempotent. Therefore \( U \) is not nilpotent and it contains an idempotent, \( e' \). Clearly \( U = (e') \) and \( (e) = \Psi((e')) \). But \( e' \in \alpha(N) \) for any automorphism \( \alpha \) of \( S \). Hence \( \Psi(\alpha(N)) \cong M_n(F) \) and \( B \cap \Psi(\alpha(N)) \geq (e) \neq 0 \), we have \( \Psi(\alpha(N)) \leq B \), and therefore \( \Psi(\alpha(1)) \leq B \). Now \( \Psi(\alpha(1)) \) is the unique minimal subalgebra of \( \Psi(\alpha(A)) \) and is not nilpotent; therefore \( \Psi(\alpha(A)) \) is a division algebra, and since \( \Psi(\alpha(A)) \cap B \geq \Psi(\alpha(1)) \neq 0 \) and \( B \) is an ideal, then \( \Psi(\alpha(A)) \leq B \). Thus \( B \geq \Psi(\alpha(N)) \vee \Psi(\alpha(A)) = \Psi(x(V \lor A)) = \Psi(x(S)) = A \). Therefore \( A \) is simple. Because \( A \) has many minimal subalgebras, \( A \) cannot be a division \( C-D \) algebra. Since there exists a split central \( C-D \) algebra \( C \leq A \), from Theorem 4.2, \( \Psi^{-1}(C) \leq A \) is also a split central \( C-D \) algebra. Thus \( A \) cannot be a split \( C-D \) algebra over \( K \), an extension field of \( F \). Therefore \( A \cong M_m(D) \), and from the associative case \( m = n \) and \( D \) is a division associative algebra \( L \)-isomorphic to \( \Delta \) such that \( (D) = d(\Delta) \) [5].

6. **Alternative Algebras \( L \)-Isomorphic to Some Alternative Simple Algebra**

We suppose that \( A \) and \( B \) are alternative algebras over \( F \). A one-to-one map \( \sigma \) from \( A \) onto \( B \) is called a semiisomorphism if

(i) \( \sigma \) is semilinear, that is, for some \( \alpha \) an automorphism of \( F \),

\[
\sigma(\lambda_1 a_1 + \lambda_2 a_2) = \alpha(\lambda_1) \sigma(a_1) + \alpha(\lambda_2) \sigma(a_2) \quad \forall a_1, a_2 \in A \quad \text{and} \quad \forall \lambda_1, \lambda_2 \in F;
\]
(ii) \( \sigma \) is multiplicative or antimultiplicative, that is, either

\[
\sigma(xy) - \sigma(x) \sigma(y) \quad \text{or} \quad \sigma(xy) - \sigma(y) \sigma(x) \quad \forall x, y \in A.
\]

**Theorem 6.1.** Let \( S = M_n(A) \) with a finite-dimensional division algebra and \( n \geq 3 \). Let \( A \) be an alternative algebra \( \mathcal{L} \)-isomorphism to \( S \). Then \( S \) is semi-isomorphism to \( A \).

**Proof.** It follows from Theorem 3 in \([5]\).

**Theorem 6.2.** Let \( C \) be a split Cayley–Dickson algebra over \( K \), extension field of \( F \). Let \( A \) be an alternative algebra \( \mathcal{L} \)-isomorphic to \( C \) by \( \Psi \). Then \( A \cong C \) and \( Z(A) = \Psi(K) \), with \( d(\Psi(K)) = d(K) \).

**Proof:** We let \( M_1 = K \in \{x_0, x_1, y_0, y_1\} \), \( M_2 = K \in \{x_0, -x_1, y_0, y_2\} \), \( M_3 = K \in \{x_0, -x_2, y_0, y_3\} \). We have \( M_i \cong M_2(D_i) \) for \( i = 1, 2, 3 \). From Theorem 5.5, \( \Psi(M_i) \cong M_2(D_i) \) for \( i = 1, 2, 3 \), where \( D_i \) are finite dimensional division algebras \( \mathcal{L} \)-isomorphic to \( K \) and with \( d(D_i) = d(K) \). We let \( (z_i) = \Psi((x_i)) \) and \( (w_i) = \Psi((y_i)) \) for \( i = 0, 1, 2, 3 \). Remark that \( C' = \langle x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3 \rangle \) is a split \( C \)-\( D \) algebra over \( F \).

From Theorem 4.2, \( \Psi(C') \) is a split \( C \)-\( D \) algebra over \( F \) with a basis \( \{z_0, z_1, z_2, z_3, w_0, w_1, w_2, w_3\} \) whose multiplication table is the same as or the opposite of the multiplication table of \( \{x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3\} \). We have also \( C' \cong K = C \) and thus \( \Psi(C') \cong \Psi(K) = A \).

We will show that \( D_1 = D_2 = D_3 \). Consider \( x_0 K \). It is the unique maximal subalgebra in \( M_i \) containing \( x_0 \). But \( z_0 D_i \) is the unique maximal subalgebra in \( \Psi(M_i) \) containing \( z_0 \), for \( i = 1, 2, 3 \). Thus \( \Psi(x_0 K) = z_0 D_1 = z_0 D_2 = z_0 D_3 \).

We have also that \( y_0 K \) is the unique division subalgebra in \( M_i \) containing \( y_0 \), and since \( w_0 D_i \) is the unique maximal division subalgebra in \( \Psi(M_i) \) containing \( w_0 \), then \( \Psi(y_0 K) = w_0 D_1 = w_0 D_2 = w_0 D_3 \).

Hence by \( 1_A = z_0 + w_0 \) it holds that

\[
D_1 = (z_0 + w_0) D_1 \leq z_0 D_1 + w_0 D_1
\]

\[
D_2 = (z_0 + w_0) D_2 \leq z_0 D_2 + w_0 D_2.
\]

But the unique maximal division subalgebras in \( (z_0 D_1) \vee (w_0 D_1) \) and in \( (z_0 D_2) \vee (w_0 D_2) \) containing \( 1_A = z_0 + w_0 \) are \( (z_0 + w_0) D_1 \) and \( (z_0 + w_0) D_2 \), respectively. Thus \( D_1 = D_2 \). In the same way \( D_1 = D_3 = D \).

Therefore \( A = \Psi(C') \vee D \), and \( A \) is a \( D \)-vector space with basis \( \{z_0, z_1, z_2, z_3, w_0, w_1, w_2, w_3\} \) over \( D \).

Now we check that \( A \) is a simple algebra. Suppose the \( D \)-basis above has a multiplication table like that of \( \{x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3\} \) but not the opposite (this is not a loss of generality). Let \( 0 \neq I \) be an ideal of \( A \). For
0 \neq x \in I \text{ we have } x = \sum \delta_i \bar{z}_i + \sum \mu_i z_i, \text{ where the sum is with } i \in \{0, 1, 2, 3\} \text{ and } \delta_i, \mu_i \in D \text{ and some } \delta_i \text{ or } \mu_i \neq 0. \text{ Suppose } \delta_i \neq 0 \text{ for } i \in \{1, 2, 3\} \text{ (it is the same if } \mu_i \neq 0 \text{ for some } i \in \{1, 2, 3\}. \text{ Then } (z_0 x) w_i = \delta_i w_0 \in I \text{ and } w_i (z_0 x) = \delta_i z_0 \in I. \text{ Therefore } 1 \in I \text{ and thus } I = A. \text{ If } \delta_0 \neq 0 \text{ (it is the same if } \mu_0 \neq 0), \text{ we obtain also } I = A \text{ from } x z_1 \in I \text{ and because its coefficient in } z_1 \text{ is not zero. Thus } A \text{ is a simple algebra and contains a nonassociative subalgebra, } \Psi(M), \text{ with divisors of zero. Therefore } A \text{ is a split } C-D \text{ algebra.}

Let } \Psi(K_1) \text{ be the center of } A, \text{ with } K_1 \subseteq C. \text{ We had } d(A) = 8d(D) \text{ and now } d(A) = 8d(\Psi(K_1)). \text{ Thus } d(\Psi(K_1)) = d(D). \text{ Moreover, } z_0 \Psi(K_1) \text{ is a maximal division subalgebra containing } z_0. \text{ Thus } z_0 \Psi(K_1) = z_0 D. \text{ Similarly } w_0 \Psi(K_1) = w_0 D.

Now } \Psi(K_1) = (z_0 + w_0) \Psi(K_1) \leq z_0 D + w_0 D \text{ and } D \text{ is the unique maximal division subalgebra of } z_0 D + w_0 D \text{ containing } z_0 + w_0; \text{ therefore } \Psi(K_1) - D \text{ and } \Psi((x_0 + y_0)) = (z_0 + w_0). \text{ Thus } K_1 \text{ is a division algebra and since } K \text{ is the unique maximal division subalgebra of } x_0 K + y_0 K \text{ containing } x_0 + y_0, \text{ then } K_1 = K = \Psi^{-1}(Z(A)).

**Theorem 6.3.** Let } C \text{ be a split Cayley–Dickson algebra over } K, K \text{ an extension field of } F, \text{ and let } A \text{ be an alternative algebra } \mathcal{L} \text{-isomorphic to } C. \text{ Then } A \text{ is isomorphic to } C.

**Proof.** We denote by } \Psi \text{ the } \mathcal{L} \text{-isomorphism from } C \text{ onto } A. \text{ From Theorem 6.2 and its proof we know the following: } A \text{ is a split } C-D \text{ algebra. If } \{x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3\} \text{ is the usual } K\text{-basis of } C, \text{ then } \Psi(K) = Z(A) \text{ and there exists } \{z_0, z_1, z_2, z_3, w_0, w_1, w_2, w_3\} \text{ such that } \Psi((x_i)) = (z_i) \text{ and } \Psi((y_i)) = (w_i) \text{ with } i = 0, 1, 2, 3. \text{ Also } \{z_0, z_1, z_2, z_3, w_0, w_1, w_2, w_3\} \text{ is a } \Psi(K)\text{-basis of } A \text{ and has a multiplication table the same as or opposite that of the } K\text{-basis of } C \text{ given above. If the multiplication table is the same, we will show that there exists } \sigma: K \rightarrow \Psi(K), \text{ a semilinear map such that } \sigma(\delta_1 \delta_2) = \sigma(\delta_1) \sigma(\delta_2) \text{ for all } \delta_1, \delta_2 \in K. \text{ Similarly if the multiplication table is the opposite we will show the same.}

If } K = F \text{ the proof of the theorem is finished. Suppose thus } K \neq F.

Let } V = K x_1 + K y_2. \text{ It is a zero algebra and therefore each subspace is a subalgebra. Also, } V \text{ is a } K\text{-vector space. We will show that the } K\text{-subspaces of } V \text{ are the subalgebras } P \leq V \text{ such that } P = V \cap (K \vee P). \text{ Suppose } P \text{ is a } K\text{-subspace of } V. \text{ Then } K \vee P = K + P. \text{ Let } x = \delta + p \in V \text{ have } \delta \in K \text{ and } p \in P. \text{ Then } \delta \in V \text{ and thus } \delta = 0. \text{ Therefore } x \in P. \text{ Thus } V \cap (K \vee P) \leq P \text{ and it is clear that } P = V \cap (K \vee P). \text{ Conversely, suppose } P = V \cap (K \vee P). \text{ Since } V \text{ is an ideal of } K \vee V, P = V \cap (K \vee P) \text{ is an ideal in } (K \vee P). \text{ Therefore } K \cdot P \leq P \text{ and } P \text{ is a } K\text{-subspace of } V.

Let } W = \Psi(K) \cdot z_1 + \Psi(K) \cdot w_2. \text{ We remark that } \Psi(K x_1) = \Psi(K) \cdot z_1 \text{ because } K x_1 \text{ is the maximal nilpotent subalgebra of } K \vee (x_1) \text{ containing } (x_1), \text{ and } \Psi(K) \cdot z_1 \text{ is the maximal nilpotent subalgebra of } \Psi(K) \vee (z_1) \text{ con-}
taining \((z_1)\) and \(\mathcal{P}(K \vee (x_1)) = \mathcal{P}(K) \vee (z_1)\). Similarly \(\mathcal{P}(K y_2) = \mathcal{P}(K) \cdot w_2\).

Thus \(\mathcal{P}(V) = \mathcal{W}\), and therefore the \(\mathcal{P}(K)\)-subspaces of \(\mathcal{W}\) are the subalgebras \(Q \leq \mathcal{W}\) such that \(Q = \mathcal{W} \cap (\mathcal{P}(K) \vee Q)\). Thus \(P\) is a \(K\)-subspace of \(V\) if and only if \(\mathcal{P}(P)\) is a \(\mathcal{P}(K)\)-subspace of \(\mathcal{W}\). Therefore we have an isomorphism, \(\mathcal{P}\), from the lattice of \(F\)-subspaces of \(V\) onto the lattice of \(F\)-subspaces of \(\mathcal{W}\), which applies \(K\)-subspaces of \(V\) in \(\mathcal{P}(K)\)-subspaces of \(\mathcal{W}\).

From \(K \neq F\), \(d(V) = 2d(K) > 3\); and thus, applying the First Fundamental Theorem of Projective Geometry [3, Chap. III], we see that there exists a semilinear map \(\sigma : V \rightarrow \mathcal{W}\) which induces \(\mathcal{P}_V\).

The elements \(z_1\) and \(w_2\) in \(\mathcal{P}(x_1)\) and \(\mathcal{P}(y_2)\), respectively, can be checked arbitrarily according to the proof of Theorem 4.2 (then to complete the basis of the split C-D algebra, as in part (2) of the proof, it is necessary to choose \(z_3\) and \(w_3\)). Thus we can suppose \(\sigma(x_1) = z_1\) and \(\sigma(y_2) = w_2\). For each \(\delta \in K\), \(\sigma(\delta x_1) \in \mathcal{P}(K) z_1\) because \(\sigma\) applies \(K\)-subspaces of \(V\) in \(\mathcal{P}(K)\)-subspaces of \(\mathcal{W}\). Let \(\sigma(\delta x_1) = \lambda z_1\). Then we have a one-to-one semilinear map \(\sigma' : K \rightarrow \mathcal{P}(K)\) such that \(\sigma'(\delta) = \lambda\) with \(\sigma(\delta x_1) = \lambda z_1\).

For each \(\delta \in K\), \(\sigma(\delta(x_1 + y_2)) = \sigma(\delta x_1) + \sigma(\delta y_2) = \sigma'(\delta) \cdot z_1 + \lambda^* w_2\) with \(\lambda^* \in \mathcal{P}(K)\). But \(\sigma(\delta(x_1 + y_2)) \in \mathcal{P}(K)(z_1 + w_2)\) and therefore \(\sigma'(\delta) = \lambda^*\). Thus, for \(\delta_1, \delta_2 \in K\)

\[
\sigma(\delta_1(x_1 + \delta_2 y_2)) = \sigma(\delta_1 x_1) + \sigma(\delta_1 \delta_2 y_2) = \sigma(\delta_1) z_1 + \sigma'(\delta_1 \delta_2) w_2.
\]

But \(\sigma(\delta_1(x_1 + \delta_2 y_2)) \in \mathcal{P}(K)(z_1 + \sigma'(\delta_2) w_2)\) and thus \(\sigma'(\delta_1 \delta_2) = \sigma'(\delta_1) \cdot \sigma'(\delta_2)\).

7. ALTERNATIVE ALGEBRAS \(\mathcal{L}\)-ISOMORPHIC TO A SEMISIMPLE ALGEBRA

In the following the proofs are omitted because the results can be shown as in the associative case, using the study about the simple alternative case already made.

**Theorem 7.1.** Let \(A\) be a finite dimensional semisimple algebra over the field \(F\), and let \(\mathcal{P} : \mathcal{L}(A) \rightarrow \mathcal{L}(B)\) be an \(\mathcal{L}\)-isomorphism. Let \(S_1, \ldots, S_n\) be the simple direct summands of \(A\). Suppose \(A\) is not a division algebra and, if \(F\) is not the field of two elements, that not all \(S_i\) are one-dimensional. Then \(B\) is semisimple. For each \(S_i\) with \(\dim_F(S_i) > 1\), \(\mathcal{P}(S_i)\) is a simple direct summand of \(B\). If \(S_i \cong S_j\), then \(\mathcal{P}(S_i) \cong \mathcal{P}(S_j)\).

**Corollary 7.2.** Let \(A\) be a finite dimensional semisimple algebra over an algebraically closed field \(F\). Suppose \(\dim_F(A) > 1\). Let \(\mathcal{P} : \mathcal{L}(A) \rightarrow \mathcal{L}(B)\) be an \(\mathcal{L}\)-isomorphism. Then \(A \cong B\).
LATTICE ISOMORPHISMS

REFERENCES