Threshold resummation for electroweak annihilation from DIS data

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Abstract

We show that higher-order coefficients required to perform threshold resummation for electroweak annihilation processes, such as Drell–Yan or Higgs production via gluon fusion, can be computed using perturbative results derived in deep inelastic scattering. As an example, we compute the three-loop coefficient $D^{(3)}$, generating most of the fourth tower of threshold logarithms for the Drell–Yan cross section in the $\overline{\text{MS}}$ scheme, using the recent three-loop results for splitting functions and for the quark form factor, as well as a class of exponentiating two-loop contributions to the Drell–Yan process.
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1. Introduction

Soft gluon resummations [1–3] have proven to be a valuable tool in perturbative QCD. They have provided a deep understanding of the structure of perturbation theory to all orders, which has in turn opened the door to studies on nonperturbative effects, and they have also been extensively used in phenomenology, broadening the range of QCD predictions towards the edges of phase space, where even hard processes are dominated by multiple soft gluon radiation.

Resummation is closely related to factorization [4]. For threshold resummations, the hard partonic cross section for a given QCD process can be expressed as a convolution (with respect to the energy fraction carried by hard partons, x) of different functions responsible for soft, collinear and hard radiation. The convolution turns into an ordinary product upon taking a Mellin transform. Logarithmic enhancements as $x \to 1$ turn into logarithms of the Mellin variable N, and these logarithms can be shown to exponentiate, using evolution equations for the various functions involved in the factorization.

To be precise, the resummed exponent is expressed in terms of moments of distributions singular as $x \to 1$,

$$
\begin{aligned}
\mathcal{D}_k(N) &\equiv \int_0^1 dx\, x^{N-1} \left(\frac{\log^k(1-x)}{1-x} \right)_+ \\
&= \frac{(-1)^{k+1}}{k+1} \log^{k+1} N + \mathcal{O}(\log^k N),
\end{aligned}
\tag{1.1}
$$

as well as terms independent of N, corresponding to moments of $\delta(1-x)$ [5]. The pattern of exponentiation is nontrivial: in general, a perturbative calculation will contain terms of the form $\alpha_s^k \log^{2k} N$ multiplying the Born cross section, whereas in the exponent one finds at most terms of the form $\alpha_s^k \log^{k+1} N$. Furthermore, a g-loop resummed calculation will determine completely the coefficients of the terms in the exponent proportional to $\alpha_s^k \log^{k+2-g} N$, to all orders in α_s. Such terms are usually described as N^{g-1}LL, with leading logarithms (LL) determined at one loop, next-to-leading logarithms (NLL) determined at two loops, and so forth.

Recently, the scope and expected precision of a range of QCD calculations have been extended in a remarkable series of papers by Moch, Vermaseren and Vogt (MVV), who computed first the three-loop contribution to the QCD splitting functions [6,7], and then the complete three-loop DIS coefficient func-

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tions [8], in what is arguably the most complex perturbative calculation ever carried out in quantum field theory. Their results both test and extend the range of threshold resummation for DIS, which can now be performed exactly to N^2LL accuracy. Furthermore, N^3LL terms can also be determined, up to a single unknown coefficient requiring a four-loop calculation, the fourth-order contribution to the cusp anomalous dimension of a Wilson line in the $\overline{MS}$ scheme. It can, however, be argued convincingly that the numerical effect of this coefficient is negligible [9]. Thus soft resummation for DIS can now be tested at the level of the fourth tower of logarithms, providing nontrivial checks on the convergence of the expansion as the logarithmic accuracy is increased.

Another class of benchmark cross sections for soft gluon resummation is given by electroweak annihilation processes in hadronic collisions, comprising Drell–Yan dimuon production, electroweak boson production, and Higgs production via gluon fusion. The inclusive cross sections for these processes are known to NNLO [10–12], and with the knowledge of the three-loop splitting functions the corresponding resummation can now be performed exactly at N^2LL level, both in the $\overline{MS}$ and in the DIS factorization schemes. Lacking a three-loop calculation, however, N^3LL terms are unknown, except for running coupling effects. It is the purpose of this Letter to show that, using only results extracted from the three-loop DIS calculations of MVV, as well as known two-loop perturbative results for electroweak annihilation, one can bring the accuracy of threshold resummation for these processes in line with DIS, performing N^3LL resummation up to the unknown, and very likely negligible, contribution of the four-loop cusp anomalous dimension.

In the following, we will concentrate on the Drell–Yan cross section in the $\overline{MS}$ factorization scheme, although the reasoning is readily generalized to other electroweak annihilation processes and to the DIS scheme. We will make use of a factorization derived in [5], where the complete exponentiation of N-independent terms was proven, to show that the coefficients of single-logarithmic contributions at g loops in the resummed exponent are completely determined by the knowledge of the g-loop nonsinglet splitting function, simple poles in the g-loop quark form factor, and N-independent terms at $g-1$ loops in the Drell–Yan cross section. We will explicitly compute these coefficients at the three-loop level, and provide a general ansatz for their expression to all orders. These results will be useful in refining the theoretical prediction for processes of great interest at the LHC, such as Z_0 production and Higgs production via gluon fusion, by extending our knowledge of soft-gluon effects, and our control of the theoretical uncertainty due to uncalculated higher-order perturbative as well as nonperturbative corrections.

2. Factorization and exponentiation

Our starting point is the unsubtracted partonic cross section for the Drell–Yan process. Near partonic threshold, its Mellin moments can be factorized as [1,5]

$$\omega(N,\epsilon) = \left|\Gamma(Q^2,\epsilon)\right|^2 \left(\psi_R(N,\epsilon)\right)^2 U_R(N,\epsilon) + \mathcal{O}\left(\frac{1}{N}\right). \tag{2.1}$$

Here $\psi_R(N,\epsilon)$ is the Mellin transform of a quark distribution at defined energy fraction, responsible for collinear divergences, $U_R(N,\epsilon)$ is an eikonal function describing the effects of soft gluon radiation at large angles, and $\Gamma(Q^2,\epsilon)$ is the (timelike) quark form factor. Near threshold, where all gluon radiation is soft, the quark distribution obeys a Sudakov-type evolution equation which can be solved in exponential form, as

$$\psi_R(N,\epsilon)$$
$$= \exp\left\{\int_0^1 dz \frac{z^{N-1}}{1-z} \int_z^1 \frac{dy}{1-y} \kappa_\psi\left(\bar{\alpha}\left((1-y)^2 Q^2\right),\epsilon\right)\right\}. \tag{2.2}$$

Similarly, eikonal exponentiation applies to the soft function U_R, which can be written as

$$U_R(N,\epsilon)$$
$$= \exp\left\{-\int_0^1 dz \frac{z^{N-1}}{1-z} g_U\left(\bar{\alpha}\left((1-z)^2 Q^2\right),\epsilon\right)\right\}. \tag{2.3}$$

The electromagnetic quark form factor Γ, on the other hand, is defined by

$$\Gamma_\mu(p_1,p_2;\mu^2,\epsilon) \equiv \langle 0|J_\mu(0)|p_1,p_2\rangle$$
$$= -iee_q \bar{v}(p_2)\gamma_\mu u(p_1)\Gamma(Q^2,\epsilon), \tag{2.4}$$

and it is one of the best understood amplitudes in perturbative QCD. Its logarithmic dependence on the scale Q^2 can be determined using renormalization group and gauge invariance [13–15], and the resulting evolution equation can be solved explicitly in dimensional regularization [16], yielding the exponential expression

$$\Gamma(Q^2,\epsilon) = \exp\left\{\frac{1}{2}\int_0^{-Q^2} \frac{d\xi^2}{\xi^2}\left[K(\alpha_s,\epsilon) + G\left(\bar{\alpha}(\xi^2),\epsilon\right)\right.\right.$$
$$\left.\left. + \frac{1}{2}\int_{\xi^2}^{\mu^2} \frac{d\lambda^2}{\lambda^2}\gamma_K\left(\bar{\alpha}(\lambda^2)\right)\right]\right\}, \tag{2.5}$$

where $\gamma_K(\alpha_s)$ is the cusp anomalous dimension [17,18], $G(\alpha_s,\epsilon)$ collects all other scale-dependent terms, and is finite as $\epsilon \to 0$, while $K(\alpha_s,\epsilon)$ is a pure counterterm. A key feature of Eqs. (2.2)–(2.5) is the usage of the d-dimensional running coupling $\bar{\alpha}(\xi^2)$, defined in $d = 4 - 2\epsilon$ by the equation

$$\xi\frac{\partial\bar{\alpha}}{\partial\xi} \equiv \beta(\epsilon,\bar{\alpha}) = -2\epsilon\bar{\alpha} + \hat{\beta}(\bar{\alpha}),$$
$$\hat{\beta}(\bar{\alpha}) = -\frac{\bar{\alpha}^2}{2\pi}\sum_{n=0}^\infty b_n\left(\frac{\bar{\alpha}}{\pi}\right)^n, \tag{2.6}$$

where $b_0 = (11C_A - 2n_f)/3$ and $b_1 = (17C_A^2 - 5C_A n_f - 3C_F n_f)/6$ in our normalization. Through $\bar{\alpha}$, integration over the scale of the coupling generates *all* infrared and collinear poles in Eqs. (2.2)–(2.5), so that all functions appearing in the exponents are finite as $\epsilon \to 0$, with the exception of the counterterm K in the quark form factor, whose only effect however is to cancel singularities arising from the ξ-independent limit of integration in the integral of the anomalous dimension γ_K. Further, dimensional continuation of the coupling regulates the Landau pole, which lies on the integration contour in $d = 4$, allowing for an explicit evaluation of the exponents in terms of analytic functions of α_s and ϵ [19,20].

Our next task is to perform mass factorization on Eq. (2.1). We do it here in the $\overline{\text{MS}}$ scheme, where we can make use of the expression [4]

$$
\phi_{\overline{\text{MS}}}(N, \epsilon) = \exp\Bigg[\int_0^{Q^2} \frac{d\xi^2}{\xi^2} \bigg\{ \int_0^1 dz \frac{z^{N-1} - 1}{1 - z} A(\bar{\alpha}(\xi^2)) \\
+ B_\delta(\bar{\alpha}(\xi^2)) \bigg\} \Bigg] + \mathcal{O}\bigg(\frac{1}{N}\bigg).
\tag{2.7}
$$

Here $A(\alpha_s)$ can be extracted from the singular behavior of the nonsinglet QCD splitting functions as $z \to 1$, and is known to be related to the cusp anomalous dimension by $A(\alpha_s) = \gamma_K(\alpha_s)/2$, while $B_\delta(\alpha_s)$ is the coefficient of $\delta(1 - x)$ in the same splitting function. Once again, it is easy to see that $\phi_{\overline{\text{MS}}}(N, \epsilon)$ is a pure counterterm, with all poles generated by integration over the running coupling. Clearly, Eq. (2.7) is a simple exponentiation of the splitting function in the IR limit, including running coupling effects. Since it does not have an obvious diagrammatic interpretation (see, however, Ref. [21]), there is a certain amount of arbitrariness in distinguishing real and virtual contributions in Eq. (2.7). This arbitrariness was exploited in Ref. [5] to define

$$
\phi_{\overline{\text{MS}}}(N, \epsilon) = \phi_V(\epsilon) \phi_R(N, \epsilon),
\tag{2.8}
$$

where

$$
\phi_V(\epsilon) = \exp\Bigg\{ \frac{1}{2} \int_0^{Q^2} \frac{d\xi^2}{\xi^2} \bigg[K(\alpha_s, \epsilon) + \tilde{G}(\bar{\alpha}(\xi^2)) \\
+ \frac{1}{2} \int_{\xi^2}^{\mu^2} \frac{d\lambda^2}{\lambda^2} \gamma_K(\bar{\alpha}(\lambda^2)) \bigg] \Bigg\}.
\tag{2.9}
$$

The structure of Eq. (2.9) clearly mimics that of the quark form factor, Eq. (2.5), and in fact it is designed so that $\phi_V(\epsilon)$ will precisely cancel all IR and collinear poles arising from $\Gamma(Q^2, \epsilon)$. This requirement, together with the requirement that $\phi_V(\epsilon)$ be a pure counterterm, uniquely fixes the new function $\tilde{G}(\alpha_s)$, which can be determined recursively from $G(\alpha_s, \epsilon)$, as was done explicitly in Ref. [5]. We are now ready to give our final expression for the Drell–Yan partonic cross section in the $\overline{\text{MS}}$ scheme, which is

$$
\hat{\omega}_{\overline{\text{MS}}}(N) \equiv \frac{\omega(N, \epsilon)}{(\phi_{\overline{\text{MS}}}(N, \epsilon))^2} \\
= \bigg(\frac{|\Gamma(Q^2, \epsilon)|^2}{\phi_V(\epsilon)^2} \bigg) \bigg[\frac{(\psi_R(N, \epsilon))^2 U_R(N, \epsilon)}{(\phi_R(N, \epsilon))^2} \bigg] \\
+ \mathcal{O}\bigg(\frac{1}{N}\bigg).
\tag{2.10}
$$

This expression has the important feature that virtual and real contributions are separately finite. Factoring out the virtual part $\hat{\omega}_{\overline{\text{MS}}}^{(V)}(N) \equiv |\Gamma(Q^2, \epsilon)|^2/(\phi_V(\epsilon))^2$, and mapping the real terms to the conventional expression for the resummed Drell–Yan cross section in the $\overline{\text{MS}}$ scheme, including N-independent terms as done in Ref. [5], we are lead to our basic equation

$$
\hat{\omega}_{\overline{\text{MS}}}^{(R)}(N) \equiv \lim_{\epsilon \to 0} \bigg[\frac{(\psi_R(N, \epsilon))^2 U_R(N, \epsilon)}{(\phi_R(N, \epsilon))^2} \bigg] \\
= \exp\Bigg[F_{\overline{\text{MS}}}(\alpha_s) \\
+ \int_0^1 dz \frac{z^{N-1} - 1}{1 - z} \bigg\{ 2 \int_{Q^2}^{(1-z)^2 Q^2} \frac{d\mu^2}{\mu^2} A(\alpha_s(\mu^2)) \\
+ D(\alpha_s((1-z)^2 Q^2)) \bigg\} \Bigg] + \mathcal{O}\bigg(\frac{1}{N}\bigg).
\tag{2.11}
$$

Eq. (2.11) spells out our basic strategy to determine the resummation coefficients: $\hat{\omega}_{\overline{\text{MS}}}^{(R)}(N)$ must be finite by the factorization theorem, given our construction of the virtual part $\hat{\omega}_{\overline{\text{MS}}}^{(V)}(N)$; the poles arising from the denominator, furthermore, are completely determined by the splitting functions and by the quark form factor, as seen from Eqs. (2.7) and (2.9); requiring their cancellation determines a subset of the perturbative coefficients of the numerator functions, which are sufficient to control the expansion of the functions A and D.

3. Constraints from finiteness

The scale dependence of $\hat{\omega}_{\overline{\text{MS}}}^{(R)}(N)$ can be explicitly computed order by order making use of the exponential expressions for the functions ψ_R, U_R and ϕ_R. An important point is the fact that ψ_R and U_R are renormalization group invariant [1], which determines explicitly the scale dependence of their exponents. Consider, for example, the quark distribution ψ_R. Imposing RG invariance leads to

$$
\bigg(\mu \frac{\partial}{\partial \mu} + \beta(\epsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} \bigg) \kappa_\psi \bigg(\frac{(1-y)Q}{\mu}, \alpha_s(\mu^2), \epsilon \bigg) = 0, \tag{3.1}
$$

which can be solved perturbatively using the explicit expression for the β function, Eq. (2.6), and writing

$$
\kappa_\psi(\xi, \alpha_s, \epsilon) = \sum_{n=1}^{\infty} \bigg(\frac{\alpha_s}{\pi} \bigg)^n \kappa_\psi^{(n)}(\xi, \epsilon), \tag{3.2}
$$

where from now on ξ will denote the ratio of the relevant scale (here $(1-x)Q$) to the renormalization scale, for which we take $\mu = Q$. Alternatively, one can impose

$$\kappa_\psi(\xi, \alpha_s, \epsilon) = \kappa_\psi\big(1, \bar{\alpha}(\xi), \epsilon\big) = \sum_{n=1}^{\infty} \left(\frac{\bar{\alpha}(\xi)}{\pi}\right)^n \kappa_\psi^{(n)}(1, \epsilon), \quad (3.3)$$

which also determines the scale dependence of the perturbative coefficients $\kappa_\psi^{(n)}(\xi, \epsilon)$. Using for the running coupling the solution of Eq. (2.6) expanded to three loops

$$\bar{\alpha}\big(\xi^2, \alpha_s, \epsilon\big)$$
$$= \alpha_s \xi^{-2\epsilon} + \alpha_s^2 \xi^{-4\epsilon} \frac{b_0}{4\pi\epsilon} \big(1 - \xi^{2\epsilon}\big)$$
$$+ \alpha_s^3 \xi^{-6\epsilon} \frac{1}{8\pi^2\epsilon} \left[\frac{b_0^2}{2\epsilon} \big(1 - \xi^{2\epsilon}\big)^2 + b_1\big(1 - \xi^{4\epsilon}\big)\right], \quad (3.4)$$

one finds

$$\kappa_\psi^{(1)}(\xi, \epsilon) = \kappa_\psi^{(1)}(1, \epsilon)\xi^{-2\epsilon},$$
$$\kappa_\psi^{(2)}(\xi, \epsilon) = \kappa_\psi^{(2)}(1, \epsilon)\xi^{-4\epsilon} + \frac{b_0}{4\epsilon}\kappa_\psi^{(1)}(1, \epsilon)\xi^{-2\epsilon}\big(\xi^{-2\epsilon} - 1\big),$$
$$(3.5)$$

$$\kappa_\psi^{(3)}(\xi, \epsilon) = \kappa_\psi^{(3)}(1, \epsilon)\xi^{-6\epsilon}$$
$$+ \frac{b_0}{2\epsilon}\left(\kappa_\psi^{(2)}(1, \epsilon) + \frac{b_0}{4\epsilon}\kappa_\psi^{(1)}(1, \epsilon)\right)\xi^{-4\epsilon}\big(\xi^{-2\epsilon} - 1\big)$$
$$- \frac{1}{8\epsilon}\kappa_\psi^{(1)}(1, \epsilon)\left(\frac{b_0^2}{2\epsilon} - b_1\right)\xi^{-2\epsilon}\big(\xi^{-4\epsilon} - 1\big), \quad (3.6)$$

with analogous results holding for the function $g_U(\xi, \alpha_s, \epsilon)$. The last formal step is to use the finiteness of κ_ψ and g_U as $\epsilon \to 0$ to expand the ϵ-dependent coefficients as

$$\kappa_\psi^{(p)}(1, \epsilon) = \sum_{k=0}^{\infty} \kappa_{\psi,k}^{(p)}\epsilon^k, \qquad g_U^{(p)}(1, \epsilon) = \sum_{k=0}^{\infty} g_{U,k}^{(p)}\epsilon^k, \quad (3.7)$$

as well as

$$G(\alpha_s, \epsilon) = \sum_{p=0}^{\infty} G^{(p)}(\epsilon)\left(\frac{\alpha_s}{\pi}\right)^p = \sum_{p=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^p \sum_{k=0}^{\infty} G_k^{(p)}\epsilon^k. \quad (3.8)$$

Expanding, in a similar way, the various other functions involved in Eq. (2.10) in powers of α_s/π, one can easily determine the structure of IR-collinear poles, by computing simple integrals.

It is instructive to briefly examine the information that can be extracted at the one-loop level. From Eq. (2.11) one derives

$$\lim_{\epsilon \to 0}\left\{\frac{1}{2\epsilon^2}\big(\kappa_{\psi,0}^{(1)} - \gamma_K^{(1)}\big) + \frac{1}{\epsilon}\left[\frac{g_{U,0}^{(1)} + \kappa_{\psi,1}^{(1)}}{2} + 2B_\delta^{(1)} - \tilde{G}^{(1)}\right.\right.$$
$$+ \big(2A^{(1)} - \kappa_{\psi,0}^{(1)}\big)\mathcal{D}_0(N)\bigg] + 2\kappa_{\psi,0}^{(1)}\mathcal{D}_1(N)$$
$$\left.- \big(g_{U,0}^{(1)} + \kappa_{\psi,1}^{(1)}\big)\mathcal{D}_0(N) + \frac{g_{U,1}^{(1)} + \kappa_{\psi,2}^{(1)}}{2}\right\}$$
$$= F_{\overline{\text{MS}}}^{(1)} + D^{(1)}\mathcal{D}_0(N) + 4A^{(1)}\mathcal{D}_1(N). \quad (3.9)$$

The cancellation of double poles requires, unsurprisingly, that $\kappa_{\psi,0}^{(1)} = \gamma_K^{(1)}$. Cancellation of single poles yields two equations, since the coefficient of the distribution $\mathcal{D}_0(N)$ must separately vanish. One finds that $A^{(1)} = \kappa_{\psi,0}^{(1)}/2 = \gamma_K^{(1)}/2$ (the factor of

2 being a matter of historical conventions); further, one finds that a combination of coefficients of U_R and ψ_R is determined by ϕ_R, yielding

$$g_{U,0}^{(1)} + \kappa_{\psi,1}^{(1)} = -4B_\delta^{(1)} + 2\tilde{G}^{(1)}. \quad (3.10)$$

Turning our attention to finite terms, we see first that the coefficient of the leading distribution $\mathcal{D}_1(N)$ is confirmed to be $A^{(1)} = \gamma_K^{(1)}/2$: had we not assumed the function $A(\alpha_S)$ appearing in ϕ_R to be the same as the one featuring in the resummation, this result would now have been derived at one loop. Next we see that single logarithms are given by *the same* combination of Drell–Yan coefficients that was determined by the cancellation of simple poles. This determines $D^{(1)}$ in terms of DIS data as

$$D^{(1)} = 4B_\delta^{(1)} - 2\tilde{G}^{(1)}. \quad (3.11)$$

Finally, the one-loop exponentiated constants are given by $F_{\overline{\text{MS}}}^{(1)} = (g_{U,1}^{(1)} + \kappa_{\psi,2}^{(1)})/2$.

Clearly, all the coefficients involved at one loop are known or easily computed. For example, one finds [1], in the $\overline{\text{MS}}$ scheme,

$$\kappa_\psi^{(1)}(1, \epsilon) = 2C_F e^{\gamma_E \epsilon} \frac{\Gamma(2 - \epsilon)}{\Gamma(2 - 2\epsilon)},$$
$$g_U^{(1)}(1, \epsilon) = -2C_F e^{\gamma_E \epsilon} \frac{\Gamma(1 - \epsilon)}{\Gamma(2 - 2\epsilon)}, \quad (3.12)$$

while, as derived in [5], $\tilde{G}^{(1)} = G_0^{(1)} = 3C_F/2$. It is well known that $B_\delta^{(1)} = 3C_F/4$, so one finds consistently

$$D^{(1)} = 0, \qquad F_{\overline{\text{MS}}}^{(1)} = -\frac{3}{2}\zeta(2)C_F, \quad (3.13)$$

as confirmed by a direct one-loop calculation of the Drell–Yan cross section.

At two loops, the pattern repeats itself with a few twists. The cancellation of triple and double poles brings in no new information, except the fact that the function κ_ψ begins to differ from γ_K by running coupling effects,

$$\kappa_{\psi,0}^{(2)} = \gamma_K^{(2)} + \frac{b_0}{2}\left(g_{U,0}^{(1)} + \frac{3}{2}\kappa_{\psi,1}^{(1)}\right) = \gamma_K^{(2)} + \frac{1}{2}b_0 C_F. \quad (3.14)$$

This however is just a reshuffling between ψ_R and U_R, in fact at the level of single poles the effect cancels and one finds, as expected, that requiring the cancellation of $\mathcal{D}_0(N)/\epsilon$ terms yields $A^{(2)} = \gamma_K^{(2)}/2$ [22,23]. N-independent single-pole terms, on the other hand, constrain a combination of coefficients of g_U and κ_ψ, namely

$$g_{U,0}^{(2)} + \frac{\kappa_{\psi,1}^{(2)}}{2} = -4B_\delta^{(2)} + 2\tilde{G}^{(2)} + \frac{b_0}{4}\left(g_{U,1}^{(1)} + \frac{3}{2}\kappa_{\psi,2}^{(1)}\right). \quad (3.15)$$

Turning to finite terms, one finds that once again running coupling effects involving ψ_R and U_R cancel, and single logarithms are determined by

$$D^{(2)} = 4B_\delta^{(2)} - 2\tilde{G}^{(2)} - \frac{b_0}{4}\big(g_{U,1}^{(1)} + \kappa_{\psi,2}^{(1)}\big)$$
$$= 4B_\delta^{(2)} - 2\tilde{G}^{(2)} - \frac{b_0}{2}F_{\overline{\text{MS}}}^{(1)}. \quad (3.16)$$

All required ingredients are known: $B_\delta^{(2)}$ from Refs. [24,25], while $\tilde{G}^{(2)} = G_0^{(2)} - b_0 G_1^{(1)}/4$ was given in [5].[1] One finds then

$$D^{(2)} = \left(-\frac{101}{27} + \frac{11}{3}\zeta(2) + \frac{7}{2}\zeta(3)\right)C_A C_F$$
$$+ \left(\frac{14}{27} - \frac{2}{3}\zeta(2)\right)n_f C_F, \tag{3.17}$$

which agrees with a direct comparison [4,26] with the two-loop calculation of Ref. [10], in the spirit of [27]. Exponentiated two-loop constants are also constrained by

$$F_{\overline{\mathrm{MS}}}^{(2)} = \frac{1}{4}\left(g_{U,1}^{(2)} + \frac{\kappa_{\psi,2}^{(2)}}{2}\right) - \frac{b_0}{16}\left(g_{U,2}^{(1)} + \frac{3}{2}\kappa_{\psi,3}^{(1)}\right), \tag{3.18}$$

where running coupling effects are readily evaluated using Eq. (3.12).

4. The coefficients $D^{(k)}$ at higher orders

It is straightforward to continue the analysis at three loops. As expected, the cancellation of quartic and triple poles at three loops in Eq. (2.11) is achieved automatically as a consequence of lower-order constraints. Double poles specify the relationship between κ_ψ and γ_K at the three-loop level; using Eq. (3.15) one can write

$$\kappa_{\psi,0}^{(3)} = \gamma_K^{(3)} + \frac{b_0}{4}\kappa_{\psi,1}^{(2)} - \frac{b_0^2}{16}\kappa_{\psi,2}^{(1)} + b_1\left(\kappa_{\psi,1}^{(1)} + \frac{3}{4}g_{U,0}^{(1)}\right). \tag{4.1}$$

As before, running coupling effects do not affect the known relationship between $A(\alpha_s)$ and $\gamma_K(\alpha_s)$: demanding the cancellation of $\mathcal{D}_0(N)/\epsilon$ terms at this order in fact yields $A^{(3)} = \gamma_K^{(3)}/2$. N-independent single-pole terms, on the other hand, yield the constraint

$$g_{U,0}^{(3)} + \frac{\kappa_{\psi,1}^{(3)}}{3}$$
$$= -4B_\delta^{(3)} + 2\tilde{G}^{(3)} + \frac{b_0}{4}\left(g_{U,1}^{(2)} + \frac{5}{6}\kappa_{\psi,2}^{(2)}\right)$$
$$- \frac{b_0^2}{16}\left(g_{U,2}^{(1)} + \frac{11}{6}\kappa_{\psi,3}^{(1)}\right) + \frac{b_1}{4}\left(g_{U,1}^{(1)} + \frac{4}{3}\kappa_{\psi,2}^{(1)}\right). \tag{4.2}$$

The finite coefficients of $\mathcal{D}_i(N)$ with $i = 1, 2, 3$ provide nontrivial tests of the results achieved so far. Further, concentrating on single logarithms, and using Eq. (4.2), one finds that

$$D^{(3)} = 4B_\delta^{(3)} - 2\tilde{G}^{(3)} - \frac{b_0}{4}\left(g_{U,1}^{(2)} + \frac{\kappa_{\psi,2}^{(2)}}{2}\right)$$
$$+ \frac{b_0^2}{16}\left(g_{U,2}^{(1)} + \frac{3}{2}\kappa_{\psi,3}^{(1)}\right) - \frac{b_1}{4}\left(g_{U,1}^{(1)} + \kappa_{\psi,2}^{(1)}\right)$$
$$= 4B_\delta^{(3)} - 2\tilde{G}^{(3)} - b_0 F_{\overline{\mathrm{MS}}}^{(2)} - \frac{b_1}{2}F_{\overline{\mathrm{MS}}}^{(1)}. \tag{4.3}$$

The detailed structure of the coefficients in terms of the functions g_U and κ_ψ, as before, turns out to be irrelevant, and the answer is simply expressed in terms of lower order contributions to the function $F_{\overline{\mathrm{MS}}}(\alpha_s)$. This is remarkable, but easily understood: in fact the details of the factorization given in Eq. (2.1), while conceptually crucial to prove formally the exponentiation of logarithms to all orders, cannot affect the overall structure of IR-collinear poles: one could, for example, define a modified quark density including eikonal effects, and poles would still cancel. Inspection of Eqs. (3.11), (3.16) and (4.3) leads us then to the following all-order ansatz for the function $D(\alpha_s)$, which summarizes the results of our work.

$$D(\alpha_s) = 4B_\delta(\alpha_s) - 2\tilde{G}(\alpha_s) + \hat{\beta}(\alpha_s)\frac{d}{d\alpha_s}F_{\overline{\mathrm{MS}}}(\alpha_s). \tag{4.4}$$

The function $D(\alpha_s)$, governing threshold resummation for electroweak annihilation at the single-logarithmic level, is thus completely determined at order α_s^n by the knowledge of virtual contributions to the nonsinglet splitting function, and IR-collinear poles of the quark form factor, to the same order, plus the value of exponentiated N-independent terms arising from real emission at order α_s^{n-1}.

We are now in a position to evaluate the three-loop contribution to the function $D(\alpha_s)$, thanks to the recent results of MVV. The three-loop contribution to the function $B_\delta(\alpha_s)$, in fact, is given in Ref. [6]; the three-loop coefficient of the function $\tilde{G}(\alpha_s)$ is given (in [5]) by the expression

$$\tilde{G}^{(3)} = G_0^{(3)} - \frac{b_0}{4}G_1^{(2)} - \frac{b_1}{4}G_1^{(1)} + \frac{b_0^2}{16}G_2^{(1)}, \tag{4.5}$$

and all relevant coefficients in the expansion of the function $G(\alpha_s, \epsilon)$ can be found in Ref. [28], where MVV use their results for DIS structure functions to evaluate explicitly the quark form factor at three loops; finally, the value of $F_{\overline{\mathrm{MS}}}(\alpha_s)$ at two loops can be extracted by comparing our exponentiated expression with the two-loop calculation of Ref. [10]. We find

$$F_{\overline{\mathrm{MS}}}^{(2)} = \left(\frac{607}{324} - \frac{469}{144}\zeta(2) + \frac{1}{4}\zeta^2(2) - \frac{187}{72}\zeta(3)\right)C_A C_F$$
$$+ \left(-\frac{41}{162} + \frac{35}{72}\zeta(2) + \frac{17}{36}\zeta(3)\right)n_f C_F. \tag{4.6}$$

Collecting all ingredients, or result for $D^{(3)}$ is

$$D^{(3)} = \left(-\frac{297029}{23328} + \frac{6139}{324}\zeta(2) - \frac{187}{60}\zeta^2(2) + \frac{2509}{108}\zeta(3)\right.$$
$$\left.- \frac{11}{6}\zeta(2)\zeta(3) - 6\zeta(5)\right)C_A^2 C_F$$
$$+ \left(\frac{31313}{11664} - \frac{1837}{324}\zeta(2) + \frac{23}{30}\zeta^2(2)\right.$$
$$\left.- \frac{155}{36}\zeta(3)\right)n_f C_A C_F$$
$$+ \left(\frac{1711}{864} - \frac{1}{2}\zeta(2) - \frac{1}{5}\zeta^2(2) - \frac{19}{18}\zeta(3)\right)n_f C_F^2$$
$$+ \left(-\frac{58}{729} + \frac{10}{27}\zeta(2) + \frac{5}{27}\zeta(3)\right)n_f^2 C_F. \tag{4.7}$$

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[1] Notice however a misprint in Eq. (4.6) of Ref. [5]: the coefficient of $C_A C_F$ in $G_0^{(2)}$ should read $(2545/108 + 11\zeta(2)/3 - 13\zeta(3))/4$.

The coefficient of the highest power of n_f in $D^{(3)}$ can be independently checked by comparing it with the renormalon calculations of [29] and [30]: indeed, their results agree with the last line of Eq. (4.7).[2]

5. Discussion

We have analyzed threshold resummation for the Drell–Yan process in the $\overline{\text{MS}}$ scheme, in light of the recent results obtained for deep inelastic scattering by MVV. Building upon a factorization proposed in Ref. [5], we have been able to derive a general relationship expressing the function $D(\alpha_s)$, responsible for threshold logarithms in the Drell–Yan cross section at single-logarithmic level, in terms of data requiring the knowledge of the virtual part of the nonsinglet splitting function, and the singular terms in the quark form factor, at the same perturbative order, plus a well-defined set of N-independent terms arising in the Drell–Yan cross section at lower orders. Our main result is Eq. (4.4), and, using MVV results, it has enabled us to evaluate the three-loop coefficient $D^{(3)}$, given in Eq. (4.7).

An immediate question is whether our results extend to the case in which the hard annihilating partons are gluons, which is relevant for the process of Higgs production via gluon fusion, in the effective theory with the top quark integrated out. It is, in fact, easy to show that an equation identical in form to Eq. (4.4) holds also for gluon-initiated electroweak annihilation, provided the various functions involved are appropriately redefined: in fact, threshold resummation in that case can still be cast in the form of Eq. (2.11), with $2A(\alpha_s)$ replaced by the cusp anomalous dimension for a Wilson line in the adjoint representation, $2A_g(\alpha_s)$, and two new functions $D_g(\alpha_s)$ and $F_{\overline{\text{MS}}}^g(\alpha_s)$. The $\overline{\text{MS}}$ distribution can be similarly defined for initial gluons, with $B_\delta(\alpha_s)$ replaced by the virtual part of the appropriate gluon splitting function. The gluon form factor obeys an exponentiation identical in form to Eq. (2.5).[3] All ingredients are thus in place to yield Eq. (4.4). A more delicate question is whether this implies a simple relationship between the perturbative coefficients of D and D_g. Up to two loops, one verifies by explicit calculation [12,31] that D_g can be obtained from D by simply replacing the overall factor of C_F with C_A [34], just as one does in deriving A_g from A. It is unlikely, however, that such a simple behavior will persist to all orders: in fact, while it is natural to expect that purely eikonal quantities such as A or the function U_R will be sensitive only to the representation of the gauge group in which the eikonal line is placed, not all information encoded in Eq. (4.4) arises from eikonal lines; it is known, for example [35], that subleading poles in the gluon form factor cannot be obtained from the quark form factor with such a simple prescription. Even eikonal functions would probably require a more careful treatment at high enough order, when

high-rank Casimir operators constructed out of the symmetric $SU(N)$ tensors d_{abc} come into play.

All this notwithstanding, we argue that at the three-loop level the simple prescription is still valid, and one can in fact compute $D_g^{(3)}$ by simply replacing the overall factor of C_F with C_A. To see it, one can make use of an observation of Ref. [31], already exploited in Ref. [35]. According to this observation, it is possible to isolate in the quark form factor, and specifically in the function $G(\alpha_s, \epsilon)$, a class of maximally non-Abelian contributions, dubbed $f_n^{(q,g)}$ in Ref. [35], which exhibit the same behavior as the eikonal anomalous dimension A (i.e., they obey the simple replacement rule, as verified up to three loops in [35]). We have explicitly checked up to three loops that in fact the leading terms of our equation, $4B_\delta^{(k)} - 2\tilde{G}^{(k)}$, coincide with the maximally non-Abelian factors f_k^q up to an irrelevant multiplicative factor. Since the remaining term in our Eq. (4.4) is a running coupling effect, determined at lower orders where the replacement rule is known to apply, we conclude that indeed $D_g^{(3)}$ is also given by Eq. (4.7), with the overall C_F replaced by C_A.

We conclude by noting that we expect these results to be useful for hadron collider phenomenology. In fact, along the lines of [9], the knowledge of $D^{(3)}$ allows to perform N³LL threshold resummation for Drell–Yan and Higgs production, to what is expected to be a very good approximation. This can be used not only to provide a more accurate QCD prediction for these processes, but also to check for the stability and the convergence properties of both ordinary perturbation theory and the expansion of its resummed counterpart in towers of logarithms. Finally, we note that several of the building blocks of our analysis also enter in resummations and high-order perturbative calculations for more complicated processes at hadron colliders (see for example [36]). It would be interesting to study the extent to which our techniques can be applied also in that context.

Note added

While our Letter was being written, S. Moch and A. Vogt completed their own calculation of $D^{(3)}$, for both quark- and gluon-initiated scattering [37], using a different line of argument. Their results completely agree with ours.

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[2] We thank Einan Gardi for pointing out this check to us and providing us with his results.

[3] The effects of the extra renormalization of the effective gluon–gluon–Higgs vertex [32,33] can be reabsorbed into a redefinition of the function G. This has been shown to two loops in Ref. [31] and can be shown to all orders by following the arguments in Ref. [5].

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