A new nonlinear integro-differential inequality and its application

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In this paper, a new nonlinear integro-differential inequality is established. Using the properties of M-cone and a generalization of Barbalat’s lemma, the boundedness and asymptotic behavior for the solution of the inequality are obtained. Applying this nonlinear integro-differential inequality, the invariant and attracting sets for Cohen–Grossberg neural networks with mixed delays are obtained. The results extend and improve the earlier publications. An example is given to illustrate the efficiency of the obtained results.

1. Introduction

The significance of differential and integral inequalities in the qualitative investigation of functional equations has been fully illustrated during the last 40 years [1, 2]. Various inequalities have been established such as the delay integral inequality in [3], the differential inequalities in [4], the impulsive differential inequalities in [5–8], and the Halanay inequalities in [9, 10]. By using the linear inequality technique, various advanced results on the invariant and attracting sets for differential systems have been reported [8, 11–15]. However, the linear differential inequalities are ineffective for studying the invariant and attracting sets of some nonlinear differential equations, such as Lotka–Volterra system and Cohen–Grossberg neural network model. Therefore, new nonlinear inequalities should be developed.

Motivated by the above discussions, in this paper, a new nonlinear integro-differential inequality is established. Using the properties of M-cone and a generalization of Barbalat’s lemma, the boundedness and asymptotic behavior for the solution of the inequality are obtained. Applying this nonlinear integro-differential inequality, the invariant and attracting sets for Cohen–Grossberg neural networks with mixed delays are obtained. The results extend and improve the earlier publications. An example is given to illustrate the efficiency of the obtained results.

2. Preliminaries

Let E denote the n-dimensional unit matrix, N = {1, 2, ..., n}, and \( \mathbb{R}_+ \triangleq (0, \infty) \). For \( A, B \in \mathbb{R}^{m \times n} \) or \( A, B \in \mathbb{R}^n \), \( A \succeq_B (A \succ_B) \) means that each pair of corresponding elements of A and B satisfies the inequality "\( \geq \) (\( > \))". Especially, A is called a nonnegative matrix if \( A \succeq 0 \), and z is called a positive vector if \( z > 0 \).

\( C[X, Y] \) denotes the space of continuous mappings from the topological space X to the topological space Y. In particular, \( C \triangleq C(\mathbb{R}, \mathbb{R}^n) \) denote the family of all bounded continuous \( \mathbb{R}^n \)-valued functions \( \phi \) defined on \( (-\infty, 0] \) with the norm \( \| \phi \| = \sup_{-\infty < \theta < 0} \| \phi(\theta) \| \), where \( \| \cdot \| \) is the Euclidean norm of \( \mathbb{R}^n \).

\( L^\infty(\Theta) \) denotes the space of real functions on \( \Theta \) which are measurable and essentially bounded; it is a Banach space for the norm \( \| u \|_{L^\infty(\Theta)} = \sup_{s \in \Theta} \text{ess}|u(s)| \).
For any \( x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}, \varphi \in C([-\tau, 0], \mathbb{R}^n) \) or \( \varphi \in C \), we define \( [x]^+ = ([x_1], \ldots, [x_n])^T, [A]^+ = ([a_{ij}])_{n \times n} \), and 
\[
[\varphi(t)]_r = ([u_1(t)]_r, \ldots, [u_n(t)]_r)^T, \quad [\varphi(t)]^+_r = [[[\varphi(t)]^+]_r, \quad [\varphi(t)]_r = \sup_{-\tau \leq \theta \leq 0} \varphi(t + \theta), \quad [\varphi(t)]^+_\infty = [[[\varphi(t)]^+], \quad [\varphi(t)]_\infty = \sup_{-\infty < \theta \leq 0} \varphi(t + \theta).
\]

Proof. For an \( M \)-matrix \( D [16, p114] \), we denote \( D \in \mathcal{M} \) and 
\[ \Omega_M(D) \triangleq \{ z \in \mathbb{R}^n \mid Dz > 0, \ z > 0 \}. \]
From the definition of \( M \)-matrix \([16]\), we can easily get that \( \Omega_M(D) \) is a cone without conical surface in \( \mathbb{R}^n \). We call it an "\( M \)-cone".
We also need the following generalization of Barbalat’s lemma.

**Lemma 2.1** ([17]). Let \( \psi \) be a continuous, positive definite function and \( x(t) \) be an absolutely continuous on \( \mathbb{R} \). If 
\[
\|x(\cdot)\|_{L^\infty(0, \infty)} < \infty, \quad \|\dot{x}(\cdot)\|_{L^\infty(0, \infty)} < \infty,
\]
and 
\[
\lim_{T \rightarrow \infty} \int_0^T \psi(x(t)) dt < \infty,
\]
then \( x(t) \rightarrow 0 \) as \( T \rightarrow \infty \).

### 3. Nonlinear integro-differential inequality

Assume that \( u(t) \in C([t_0, \infty), \mathbb{R}^n) \) is a solution of the following nonlinear integro-differential inequality with the initial conditions \( u(t_0 + s) = \varphi(s) \in C \),
\[
D^+[u(t)]^+ \leq R(u(t))[P[u(t)]^+] + Q[u(t)]^+ + \int_0^\infty K(s)[u(t - s)]^+ ds + I, \quad t \geq t_0,
\] \( (1) \)

where \( P = (p_{ij})_{n \times n} \) and \( p_{ij} \geq 0 \) for \( i \neq j \). \( Q = (q_{ij})_{n \times n} \geq 0, I = (I_1, \ldots, I_n)^T \geq 0, R(u) = \text{diag}(R_1(u), \ldots, R_n(u)), R_i \in C[\mathbb{R}, [0, \infty)], \) the delay kernel function \( K(t) = (k_{ij}(t))_{n \times n} \geq 0 \), \( k_{ij}(t) \) are assumed to be piecewise continuous and there is a matrix \( K \) such that \( K = (k_{ij})_{n \times n} = (\int_0^\infty k_{ij}(t) dt)_{n \times n} \).

In this section we always assume that \( D = -(P + Q + K) \in \mathcal{M} \) and \( L = D^{-1}I \).

**Theorem 3.1.** For any constant \( d \geq 1 \), the solution \( u(t) \) of \((1)\) satisfies that 
\[
[u(t)]^+ \leq dL, \quad \forall t \geq t_0,
\] \( (2) \)
provided that \( [\varphi]^+_{\infty} \leq dL \).

**Proof.** Since \( D = -(P + Q + K) \in \mathcal{M} \), from the properties of \( M \)-matrix \([16], D^{-1} \geq 0 \). Let \( \varepsilon = D^{-1}(1, \ldots, 1)^T (\varepsilon > 0 \) small enough), then \( \varepsilon > 0 \). For any given initial function \( \varphi \in C \) with \( [\varphi]^+_{\infty} \leq dL \), we will first prove that 
\[
[u(t)]^+ \leq dL + \varepsilon (\bar{x}_1, \ldots, \bar{x}_n)^T = \tilde{x}, \quad \forall t \geq t_0,
\] \( (3) \)
provided that the initial conditions satisfy \( [\varphi]^+_{\infty} \leq dL \).

If \((3)\) does not hold, then there exist \( i \in \mathcal{N} \) and \( t_1 > t_0 \) such that 
\[
[u_i(t_1)] = \bar{x}_i, \quad [u(t)]^+ \leq \bar{x}, \quad \text{for} \ t \leq t_1,
\] \( (4) \)
and 
\[
D^+[u_i(t_1)] \geq 0.
\] \( (5) \)

It follows from \((1)\) and \((4)\) that 
\[
D^+[u_i(t_1)]^+ \leq R(u(t_1))[P[u_i(t_1)]^+] + Q[u_i(t_1)]^+ + \int_0^\infty K(s)[u(t_1 - s)]^+ ds + I \]
\[
\leq R(u(t_1))[(P + Q + K)\bar{x} + I] \]
\[
= -R(u(t_1))[(d + (1, \ldots, 1)^T \varepsilon - I)] \]
\[
\leq -R(u(t_1))(1, \ldots, 1)^T \varepsilon < 0,
\]
which contradicts the inequality in \((5)\). So \((3)\) holds. This implies that the conclusion holds and the proof is complete. \( \square \)

**Theorem 3.2.** For the inequality \((1)\), suppose \( k_{ij}(t) \) satisfies 
\[
\int_0^\infty e^{s \varepsilon} k_{ij}(t) dt < \infty, \quad \text{for each} \ i, j \in \mathcal{N},
\] \( (6) \)
where $\lambda_1$ is a positive constant. Then

$$[u(t)]^+ \leq \int_0^t e^{-\lambda_1 s} \hat{R}(u(s)) \, ds + L, \quad t \geq t_0,$$

(7)

provided that the initial conditions satisfy

$$[u(\theta)]^+ \leq \int_0^\theta e^{-\lambda_1 s} \hat{R}(u(s)) \, ds + L, \quad -\infty \leq \theta \leq t_0,$$

(8)

where $\hat{R}(u) \leq \min_{|u| \leq R} R(u)$ with $\hat{R}(u) \in C[\mathbb{R}^n, \mathbb{R}_+]$, $z = (z_1, \ldots, z_n)^T \in \Omega_M(D)$ and the positive constant $\lambda$ is determined by the following inequality

$$\left[ \lambda E + P + Q e^{\lambda H} + \int_0^\infty K(s)e^{\lambda Hs} \right] z < 0.$$

(9)

where $H = \max_{|u| \leq \hat{d}} \hat{R}(u) < \infty$ and $\hat{d} \geq 1$ is a constant such that $[\phi]^+ \leq \hat{d} L$.

**Proof.** For any given initial function $\phi \in C$, there is a constant $\hat{d} \geq 1$ such that $[\phi]^+ \leq \hat{d} L$. From Theorem 3.1, we have

$$[u(t)]^+ \leq \hat{d} L, \quad \forall t \geq t_0.$$

(10)

Since $-(P + Q + K) \in \mathcal{M}$, there exists a positive vector $z \in \Omega_M(D)$ such that $(P + Q + K)z < 0$. By using continuity and (6), we know that there must exist a $\lambda > 0$ satisfying the inequality (9). That is,

$$\sum_{j=1}^n \left[ p_{ij}q_{ij} + \int_0^\infty k_{ij}(s)e^{\lambda Hs} \right] z_j < -\lambda z_i, \quad i \in \mathcal{N}.$$

(11)

Since $L = D^{-1}I$, we have $(P + Q + K)L + I = 0$. Then

$$\sum_{j=1}^n [p_{ij} + q_{ij} + k_{ij}]L_j + L_i = 0, \quad i \in \mathcal{N}.$$

(12)

Next, we shall prove that for any positive constant $\epsilon$,

$$|u_i(t)| \leq (1 + \epsilon) \left[ z_i e^{-\lambda_1 \int_0^t \hat{R}(u(s)) \, ds} + L \right] \triangleq w_i(t), \quad t \geq t_0, \quad i \in \mathcal{N}.$$

(13)

We let

$$\mathcal{V} = \{i \in \mathcal{N} \mid |u_i(t)| > w_i(t) \text{ for some } t \in [t_0, \infty)\},$$

$$\theta_i = \inf\{t \in [t_0, \infty) \mid |u_i(t)| > w_i(t), \quad i \in \mathcal{V}\}.$$

If inequality (13) is not true, then $\mathcal{V}$ is a nonempty set and there must exist some integer $m \in \mathcal{V}$ such that $\theta_m = \min_{i \in \mathcal{V}} \{\theta_i\} \in [t_0, \infty)$.

By $u_m(t) \in C([t_0, \infty), \mathbb{R})$ and the inequality (13), we can get

$$\theta_m > t_0, \quad |u_m(\theta_m)| = w_m(\theta_m), \quad D^+ \left[ u_m(\theta_m) \right] \geq w_m'(\theta_m),$$

$$|u_i(t)| \leq w_i(t), \quad t \in (-\infty, \theta_m), \quad i \in \mathcal{N}.$$

(14)

(15)

By using (1) and (13)–(15), we obtain that

$$D^+ |u_m(\theta_m)| \leq R_m(u(\theta_m)) \left[ \sum_{j=1}^n \left( p_{mj}w_j(\theta_m) + q_{mj}w_j(\theta_m) \right) \right] + \int_0^\infty k_{mj}(s)w_j(\theta_m - s) \, ds + L_m$$

$$\leq R_m(u(\theta_m)) \left[ \sum_{j=1}^n \left( p_{mj}w_j(\theta_m) + q_{mj}w_j(\theta_m) \right) \right] + \int_0^\infty k_{mj}(s)w_j(\theta_m - s) \, ds + L_m$$

$$\leq R_m(u(\theta_m)) \left[ \sum_{j=1}^n p_{mj}w_j(\theta_m) + q_{mj}w_j(\theta_m) + \sum_{j=1}^n k_{mj}(s)w_j(\theta_m - s) \, ds + L_m \right]$$

$$\leq R_m(u(\theta_m)) \left[ \sum_{j=1}^n p_{mj}(1 + \epsilon) \left( z_j e^{-\lambda_1 \int_0^t \hat{R}(u(s)) \, ds} + L \right) + \sum_{j=1}^n q_{mj}(1 + \epsilon) \left( z_j e^{-\lambda_1 \int_0^t \hat{R}(u(s)) \, ds} + L \right) \right]$$

$$+ \sum_{j=1}^n \int_0^\infty k_{mj}(s)(1 + \epsilon) \left( z_j e^{-\lambda_1 \int_0^t \hat{R}(u(s)) \, ds} + L \right) \, ds + L_m.$$

Furthermore, combining with (12) and using (11), we get
\[
D^+ |u_m(\theta_m)| \leq R_m(u(\theta_m)) \left[ \sum_{j=1}^{n} (1 + \epsilon) z_j e^{-\lambda_j \hat{r}_{H_0} \hat{R}(u(s)) ds} \right. \\
+ \left. \int_{0}^{\tau \leq R_m(u(\theta_m)) \left[ \sum_{j=1}^{n} p_{mj} + q_{mj} e^{2H_0 \hat{r}_{H_0} \hat{R}(u(s)) ds} \right]} \right] \\
\leq R_m(u(\theta_m)) \left[ \sum_{j=1}^{n} p_{mj} + q_{mj} e^{2H_0 \hat{r}_{H_0} \hat{R}(u(s)) ds} \right] \left( 1 + \epsilon \right) z_j e^{-\lambda_j \hat{r}_{H_0} \hat{R}(u(s)) ds} \\
< -\lambda \hat{R}(u(\theta_m)) z_m \left( 1 + \epsilon \right) e^{-\lambda_j \hat{r}_{H_0} \hat{R}(u(s)) ds} = u_m' (\theta_m),
\]
which contradicts the second inequality in (14). Thus the inequality (13) holds. Therefore, letting \( \epsilon \to 0 \), we have (7). The proof is completed. \( \square \)

**Remark 3.1.** When the initial conditions \( \phi \in PC \) defined in [5], Theorems 3.1 and 3.2 above still hold, therefore, some known results are easily obtained. For example, Theorem 3.1 in [5], Theorem 1 in [6], Lemma 2.1 in [7] and Lemma 1 in [8] can be derived by our Theorems 3.1 and 3.2 if we choose \( \hat{R}(u) = \hat{E}, K(t) = 0, I = 0; \hat{R}(u) = \hat{E}, Q = 0, I = 0; \hat{R}(u) \geq \hat{g}\{s_1, \ldots, s_n\} > 0, \) \( K(t) = 0, I = 0; \) and \( \hat{R}(u) = \hat{E}, K(t) = 0 \) in (1), respectively.

**Theorem 3.3.** Under the conditions of Theorem 3.2, if \( \hat{R}(u) \) is positive definite, then
\[
\lim_{t \to +\infty} |u(t)|^+ \leq L. \tag{16}
\]

**Proof.** We only need to consider the following two possible cases:
(i) If \( \int_{0}^{\infty} \hat{R}(u(s)) ds = +\infty \), then from (7), we have \( \lim_{t \to -\infty} |u(t)|^+ \leq L \).
(ii) If \( \int_{0}^{\infty} \hat{R}(u(s)) ds < +\infty \). Since \( \hat{R}(\cdot) \) is continuous, we can get \( \int_{0}^{\infty} \hat{R}(u(s)) ds < +\infty \). On the other hand, from (10), for any given initial function \( \phi \in C \), we have \( u(t) \) is bounded. Furthermore, \( \hat{u}(t) \) is bounded by (1). Thus \( u(t) \) is absolutely continuous, \( u(t) \) and \( \hat{u}(t) \) are \( L^\infty (0, +\infty) \). By Lemma 2.1, we have \( \lim_{t \to -\infty} u(t) = 0 \leq L \). Therefore the conclusion holds and the proof is completed. \( \square \)

4. Applications

In this section, we will apply the nonlinear integro-differential inequality established in Section 3 to obtain the invariant and attracting sets for the following Cohen–Grossberg neural network

\[
\begin{align*}
\dot{x}_i(t) &= -\alpha_i(x_i(t)) \left( \beta_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} \hat{y}_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} f(x_j(t - \tau_j(t))) \right) \\
&\quad - \sum_{j=1}^{n} m_{ij} \int_{-\infty}^{\tau_j(t)} h_{ij}(t - s) \nu_j(x_j(s)) ds + l_i, \quad t \geq 0, \\
x_i(s) &= \phi_i(s), \quad -\infty < s \leq 0, i \in \mathcal{N},
\end{align*}
\]
where \( 0 \leq \tau_j(t) \leq \tau \) (\( \tau \) is a constant); \( h_{ij} : [0, +\infty) \to [0, +\infty) \) are piecewise continuous functions and satisfy \( \int_{0}^{\infty} e^{\alpha \tau} h_{ij}(t) dt = \hat{h}_{ij}(\lambda \alpha) < \infty \), where \( \hat{h}_{ij}(\lambda \alpha) \) are continuous functions in \([0, \rho], \rho > 0 \).

In the following, we firstly introduce the following assumptions.

(A1) The amplification functions \( \alpha_i(\cdot) \) are positive and continuous. Furthermore, there is a continuous and positive definite function \( \hat{\alpha}(\cdot) \) such that \( \min_{1 \leq i \leq n} \alpha_i(\cdot) \geq \hat{\alpha}(\cdot) \).

(A2) The behaved functions \( \beta_i(\cdot) \) are monotone increasing, i.e., there exists a positive diagonal matrix \( \beta = \text{diag}(\beta_1, \ldots, \beta_n) \) such that
\[
\frac{\beta_i(s_1) - \beta_i(s_2)}{s_1 - s_2} \geq \beta_i > 0, \quad \text{for all } i \in \mathcal{N}, s_1 \neq s_2, s_1, s_2 \in \mathbb{R}.
\]

(A3) For any \( x \in \mathbb{R}^n \), there exist nonnegative diagonal matrices \( G = \text{diag}(G_1, \ldots, G_n), F = \text{diag}(F_1, \ldots, F_n) \) and \( V = \text{diag}(V_1, \ldots, V_n) \) such that
\[
[g(x)]^+ \leq G|x|^+, \quad [f(x)]^+ \leq F|x|^+, \quad [u(x)]^+ \leq V|x|^+.
\]

(A4) Let \( \hat{D} = -\hat{P} + \hat{Q} + \hat{K} \in \mathcal{M} \), where \( \hat{P} = \hat{P}_{ij} N \times N, \hat{P}_{ij} = -\beta_i + |a_{ij}| G_i, \hat{P}_{ij} = |a_{ij}| G_j, \) for \( i \neq j; \hat{Q} = \hat{Q}_{ij} N \times N, \hat{Q}_{ij} = |b_{ij}| F_j, \hat{K} = (k_{ij})_{N \times N}, k_{ij} = \hat{h}_{ij}(0) |m_{ij}| V_j, i, j \in \mathcal{N} \).
Theorem 4.1. If (A₁)–(A₄) hold, then \( S = \{ \phi \in C ||\phi||^\infty \leq \hat{D}^{-1}I \} \) is a positive invariant and global attracting set of (17), where \( \hat{I} = [\beta(0) + I]^+ \).

Proof. Calculating the upper right derivative \( D^+ |x(t)| \) along system (17), from conditions (A₁)–(A₄), we obtain

\[
D^+ [x(t)]^+ \leq \alpha(x(t)) \left[ \hat{P}[x(t)]^+ + \hat{Q}[x(t)]^+ + \int_0^\infty \hat{k}(s)[x(t - s)]^+ ds + \hat{I} \right], \quad t \geq 0
\]

where \( \alpha(x(t)) = \text{diag}\{\alpha_1(x_1(t)), \ldots, \alpha_n(x_n(t))\} \).

From Theorem 3.1, we get

\[
[x(t)]^+ \leq \hat{L}, \quad \forall t \geq 0,
\]

provided \( [\phi]^+ \leq \hat{L} \), where \( \hat{L} = \hat{D}^{-1}I \).

Since \( \hat{D} \in \mathcal{M} \), there exists a positive vector \( z = (z_1, \ldots, z_n)^T \) such that

\[
\hat{D}z > 0, \quad \text{or} \quad \left[ \hat{P} + \hat{Q} + \hat{K} \right] z < 0.
\]

By using continuity, we know that there must exist a positive scalar \( \lambda \) such that

\[
\left[ \lambda E + \hat{P} + \hat{Q} e^{i\lambda t} + \int_0^\infty \hat{k}(s)e^{i\lambda s} ds \right] z < 0,
\]

where \( \hat{H} = \max_{|u| \leq \hat{d}} \hat{\alpha}(u) < \infty \) and \( \hat{d} \geq 1 \) is a constant such that \( [\phi]^+ \leq \hat{d} \).

So all the conditions of Theorem 3.3 are satisfied by (18) and (20) and the condition (A₄). Then we obtain that

\[
\lim_{t \to +\infty} [x(t)]^+ \leq \hat{L}.
\]

By the definition of invariant and global attracting sets (see [11–14]), the proof is complete. \( \square \)

For the case \( \hat{I} = 0 \), we easily observe that \( x(t) = 0 \) is a solution of (17) from (A₃). We can get the attractivity of the zero solution and the proof is similar to that of Theorem 4.1.

Theorem 4.2. In addition to (A₁)–(A₄), further assume that \( \hat{\alpha}(s) = \delta > 0 \) and \( \hat{I} = 0 \). (17) has a zero solution and the zero solution is global exponential stability and the exponential convergent rate \( \lambda \) is determined by (20) with \( H = \delta \).

When \( \alpha_i(x_i) \equiv 1 \) and \( \beta_i(x_i) = \beta_i x_i \) (\( \beta_i \) is a positive constant) for each \( i \in \mathcal{N} \), system (17) becomes the following system which has been studied in [15,11]

\[
\begin{cases}
\dot{x}_i(t) = -\beta_i x_i(t) + \sum_{j=1}^n a_{ij}g_j(x_j(t)) + \sum_{j=1}^n b_{ij}f_j(x_j(t - \tau_j(t))) \\
\quad + \sum_{j=1}^n m_{ij} \int_{-\infty}^t h_j(t - s)v_j(x_j(s)) ds - l_i, \\
x_i(s) = \phi_i(s), \quad -\infty < s \leq 0, \quad i \in \mathcal{N}.
\end{cases}
\]

We can choose \( \hat{\alpha}(x) = 1 \) in (A₁). Then, we have the following corollaries.

**Corollary 4.1.** If (A₃)–(A₄) hold, then the set \( S = \{ \phi \in C ||\phi||^\infty \leq \hat{D}^{-1}[I]^+ \} \) is positive invariant and global attracting set of (22).

**Corollary 4.2.** In addition to (A₃)–(A₄), further assume that \( I = 0 \). Then the zero solution of (22) is global exponential stability and the exponential convergent rate \( \lambda \) is determined by (20) with \( H = 1 \).

**Remark 4.1.** Suppose that \( b_{ij} \equiv 0 \) and \( \int_0^{+\infty} h_i(s) ds = 1 \) (\( i, j \in \mathcal{N} \)) in (22), then we can get Theorem 3.1 and Theorem 3.2 in [11].

**Remark 4.2.** The positive invariant and global attracting set of (22) were investigated in [15], in which the differentiability of the time-varying delays \( \tau_i(t) \) (\( i, j \in \mathcal{N} \)) and \( \tau_i(t) \leq \varrho < 1 \) were required. However, Corollary 4.1 does not require these conditions.
5. An illustrative example

The following illustrative example will demonstrate the effectiveness of our results.

**Example 1.** Consider the following model:

\[
\begin{cases}
\dot{x}_1(t) &= -(2 + \sin x_1(t)) \left[ 7x_1(t) + g_1(x_1(t)) - 0.5g_2(x_2(t)) + f_1(x_1(t - \tau_1(t))) \right] + f_2(x_2(t - \tau_2(t))) - \int_{-\infty}^{t} e^{-(t-s)} |x_1(s)| \, ds + \int_{-\infty}^{t} e^{-2(t-s)} |x_2(s)| \, ds + l_1, \\
\dot{x}_2(t) &= (\cos x_2(t) - 2) \left[ 8x_2(t) - 0.5g_1(x_1(t)) + g_2(x_2(t)) - 1.5f_1(x_1(t - \tau_2(t))) \right] + 0.5f_2(x_2(t - \tau_2(t))) + 2 \int_{-\infty}^{t} e^{-2(t-s)} |x_1(s)| \, ds - 1.5 \int_{-\infty}^{t} e^{-(t-s)} |x_2(s)| \, ds + l_2,
\end{cases}
\]

(23)

where \( g_i(u) = \frac{1}{2} (|u + 1| - |u - 1|) \), \( f_i(t) = \tanh 2t \), \( \tau_i(t) = \frac{\sin(i + j)t}{1 + \tau} \), for \( i, j = 1, 2 \).

Taking \( \lambda = 0.3 \) and \( z = (1, 1)^T \), we easily verify the conditions (A1)–(A4) with \( \hat{\alpha}(s) = 1 \) and

\[
\hat{D} = \begin{pmatrix} 4 & -2 \\ 3 & 5 \end{pmatrix} \in \mathcal{M}, \quad \begin{bmatrix} \lambda E + \hat{\rho} + \hat{Q}e^{\lambda\tau} + \int_{0}^{\infty} \hat{K}(s)e^{\lambda s} \, ds \end{bmatrix} z = \begin{pmatrix} -0.325 \\ -022 \end{pmatrix} < 0.
\]

Now, we discuss the asymptotic behavior of the system (24) as follows:

(i) If \( I = (2, 2)^T \), then by Theorem 4.1, \( S = \{ x(t) \in \mathbb{R}^2 | |x|^T \leq (1, 1)^T \} \) is a positive invariant and global attracting set of (23).

(ii) If \( I = (0, 0)^T \), then \( x(t) = (0, 0)^T \) is a solution of (23). It follows from Theorem 4.2 that the zero solution of (23) is globally exponentially stable and the exponential convergent rate equals 0.3.

References