On uniquely 3-colorable graphs

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Abstract


We show the following. (1) For each integer $n \geq 12$, there exists a uniquely 3-colorable graph with $n$ vertices and without any triangles. (2) There exist infinitely many uniquely 3-colorable regular graphs without any triangles. It follows that there exist infinitely many uniquely $k$-colorable regular graphs having no subgraph isomorphic to the complete graph $K_k$ with $k$ vertices for any integer $k \geq 3$.

1. Introduction

Let $G$ be a graph, $V(G)$ its vertex-set and $E(G)$ its edge-set. An assignment of colors to the vertices of $G$ in such a way that adjacent vertices are assigned different colors is called a proper coloring of $G$. Here, we shall use a coloring to mean a proper coloring. A coloring of $G$ in which $\lambda$ colors are used is called a $\lambda$-coloring of $G$. Given $\lambda$ colors, we let $P(G, \lambda)$ denote the number of ways of $\lambda$-coloring of $G$. $P(G, \lambda)$ is called the chromatic polynomial of $G$. The minimum number of colors used to color $G$ is called the chromatic number of $G$ and is denoted by $\chi(G)$. Thus, a $\lambda$-coloring of $G$ is a partition of $V(G)$ into $\lambda$ color classes such that the vertices in the same color class are not adjacent. If every $\chi(G)$-coloring of $G$ gives the same partition of $V(G)$, then $G$ is said to be a uniquely $\chi(G)$-coloring graph, or a uniquely $\chi(G)$-colorable graph.

On p. 139 in [2] and on p. 269 in [1], the two uniquely 3-colorable graphs without any triangles do not seem to be correct. However, motivated by these graphs, we shall prove the following theorems.

Theorem 1. For each integer $n \geq 12$, there exists a uniquely 3-colorable graph with $n$ vertices and without any triangles.

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Theorem 2. There exist infinitely many uniquely 3-colorable regular graphs without any triangles.

Corollary. There exist infinitely many uniquely k-colorable regular graphs having no subgraph isomorphic to the complete graph $K_k$ with $k$ vertices for any integer $k \geq 3$.

2. A proof of Theorem 1

In order to prove Theorem 1, we need the following lemmas.

Lemma 1. Let $\chi(G) = k$. Then $P(G, k) = k! \cdot t$ for some positive integer $t$, and $t$ is the number of ways of coloring $G$ in exactly $k$ colors with color indifference. Furthermore, $t = 1$ if and only if $G$ is a uniquely $k$-colorable graph.

Proof. Since $\chi(G) = k$, $P(G, x) = 0$ for $x = 0, 1, \ldots, k - 1$, and $P(G, k) = k(k - 1) \cdots (k - k + 1)q(k)$ for some polynomial $q(k)$. Let $q(k) = t$. Then $P(G, k) = k! \cdot t$. By Theorem 15 in [3], we know that $t$ is the number of ways of coloring $G$ in exactly $k$ colors with color indifference. It follows that $t = 1$, if and only if, $G$ is a uniquely $k$-colorable graph. □

Lemma 2. The graph $G_{12}$ depicted in Fig. 1 with $V(G_{12}) = \{1, 2, \ldots, 12\}$ is a uniquely 3-colorable graph with 12 vertices and without any triangles.

Proof. Let $a, b$ and $c$ denote 3 colors. We shall use the following notations: $i(a)$ means the vertex $i$ is colored with the color $a$, and $\rightarrow i(a)$ means the vertex $i$ is forced to be colored with the color $a$.

We first consider a 5-cycle subgraph $C_5$, see Fig. 2.

We know that $P(C_5, k) = (k - 1)^5 - (k - 1)$ and $\chi(C_5) = 3$. Then $P(C_5, 3) = (3 - 1)^5 - (3 - 1) = 3! \cdot 5$, and, by Lemma 1, there are 5 ways to color $C_5$ in exactly 3 colors with color indifference, i.e., there are 5 ways to partition $V(C_5)$. Namely,

I $\{1, 4\}, \{2, 5\}, \{3\}$
II $\{1, 4\}, \{3, 5\}, \{2\}$
III $\{1, 3\}, \{2, 5\}, \{4\}$
IV $\{1, 3\}, \{2, 4\}, \{5\}$
V $\{2, 4\}, \{3, 5\}, \{1\}$

For the partition I, we have $1(a), 4(a), 2(b), 5(b)$ and $3(c)$. Then $\rightarrow 10(b) \rightarrow 11(c) \rightarrow 6(a) \rightarrow 7(b) \rightarrow 8(c) \rightarrow 9(a) \rightarrow 12(c)$. Thus, we have a 3-coloring of $G_{12}$.

\{1, 4, 6, 9\}, \{2, 5, 10, 7\}, \{3, 11, 8, 12\}.

(1)

For the partition II, we have $1(a), 4(a), 3(b), 5(b)$, and $2(c)$. Then $\rightarrow 6(a) \rightarrow 7(c) \rightarrow 8(b) \rightarrow 9(a)$. Since the neighborhood of vertex 12 denoted by $N(12)$ is $\{9(a), 7(c), 5(b)\}$, this case is impossible.
For the partition III, we have $1(a), 3(a), 2(b), 5(b)$ and $4(c)$. Then $\rightarrow 8(b) \rightarrow 7(c) \rightarrow 6(a) \rightarrow 11(b) \rightarrow 10(c) \rightarrow 9(a)$. Since $N(12) = \{5(b), 7(c), 9(a)\}$, this case is impossible.

For the partition IV, we have $1(a), 3(a), 2(b), 4(b)$ and $5(c)$. Then $\rightarrow 8(c) \rightarrow 7(b) \rightarrow 9(a)$. Since $N(12) = \{5(c), 7(b), 9(a)\}$, this case is impossible.

For the partition V, we have $2(u), 4(u), 3(b), 5(b)$ and $1(c)$. Then $\rightarrow 6(c) \rightarrow 7(a) \rightarrow 8(b) \rightarrow 9(c)$. Since $N(12) = \{5(b), 7(a), 9(c)\}$, this case is impossible.

Thus, $G_{12}$ is a uniquely 3-colorable graph with 12 vertices and without any triangles. The unique partition of $V(G_{12})$ is (1).

The proof of Theorem 1 goes as follows. We use the mathematical induction on the number of vertices $n$. For $n = 12$, the graph $G_{12}$ in Lemma 2 is a uniquely 3-colorable graph with 12 vertices and without any triangles. Assume that Theorem 1 holds for $n > 12$, i.e., we assume that there exists a uniquely 3-colorable graph $G_n$, $n > 12$, with $n$ vertices and without any triangles which contains $G_{12}$ as a subgraph.

We shall construct a graph $G_{n+1}$ from $G_n$. Let $V(G_{n+1}) = V(G_n) \cup \{w\}$ and $E(G_{n+1}) = E(G_n) \cup \{\{w, u\}, \{w, v\}\}$ where $u, v \in V(G_n)$, $u$ and $v$ are not adjacent, and $u$ and $v$ are colored with different colors. Thus, $G_{n+1}$ contains $G_n$ and $G_{12}$ as subgraphs. We color $w$ by the third color which is different from the colors of $u$ and $v$. Clearly, $G_{n+1}$ is 3-colorable. Since $G_n$ is uniquely 3-colorable and the color of vertex $w$ is uniquely determined by the colors of vertices $u$ and $v$, $G_{n+1}$ is a uniquely 3-colorable graph, and there exists a uniquely 3-colorable graph with $m$ vertices and without any triangles for any integer $m \geq 12$.

**Remark.** A theorem in [1] states that for any integer $k \geq 3$, there is a uniquely $k$-colorable graph which contains no subgraph isomorphic to the complete graph $K_k$. 

![Fig. 1.](image1) 

![Fig. 2.](image2)
with \( k \) vertices. We may use Lemma 2 and the mathematical induction to prove it. Following the proof of Theorem 2, we shall give a similar result, i.e., the Corollary of Theorem 2.

3. A proof of Theorem 2

In order to prove Theorem 2, we need the following lemmas.

**Lemma 3.** There exists a uniquely 3-colorable regular graph of degree 5 with 24 vertices and without any triangles.

**Proof.** Let \( M \) and \( N \) be graphs such that each of them is isomorphic to \( G_{12} \) in Lemma 2, \( V(M) = \{1, 2, \ldots, 12\} \) and \( V(N) = \{1', 2', \ldots, 12'\} \). Then each of \( M \) and \( N \) is uniquely 3-colorable, and each can be colored the same as \( G_{12} \) with the same colors \( a, b, \) and \( c \). Since \( M \) is uniquely 3-colorable, \( V(M) \) is uniquely partitioned into 3 subsets:

\[
A_1 = \{1, 4, 6, 9\}, \quad B_1 = \{2, 5, 7, 10\}, \quad C_1 = \{3, 8, 11, 12\}.
\]

Similarly, \( V(N) \) is uniquely partitioned into 3 subsets:

\[
A_1' = \{1', 4', 6', 9'\}, \quad B_1' = \{2', 5', 7', 10'\}, \quad C_1' = \{3', 8', 11', 12'\}.
\]

Let \( G_{24} \) be a graph with \( V(G_{24}) = V(M) \cup V(N) \), and \( E(G_{24}) = E(M) \cup E(N) \cup \{(1, 3'), (2, 8'), (3, 1'), (4, 12'), (5, 11'), (6, 12'), (7, 11'), (8, 2'), (9, 10'), (10, 9'), (11, 5'), (11, 7'), (12, 4'), (12, 6')\} \). Then \( G_{24} \) is a regular graph of degree 5 with 24 vertices. The edges \((5, 11'), (7, 11'), (4, 12'), \) and \((6, 12')\) belonging to \( E(G_{24}) \) imply that the vertices in \( C_1 \) have to be colored differently from the vertices in \( A_1 \) and \( B_1 \). Since \((10, 9')\) belongs to \( E(G_{24}) \), the vertices in \( A_1 \) can only be colored the same as in \( A_1 \). Since \((9, 10')\) belongs to \( E(G_{24}) \), the vertices of \( B_1 \) can only be colored the same as \( B_1 \). Thus, \( G_{24} \) is a uniquely 3-colorable graph with the unique partition of \( V(G_{24}) \) as

\[
A_1 \cup A_1' = \{1, 4, 6, 9, 1', 4', 6', 9'\},
\]

\[
B_1 \cup B_1' = \{2, 5, 7, 10, 2', 5', 7', 10'\},
\]

\[
C_1 \cup C_1' = \{3, 8, 11, 12, 3', 8', 11', 12'\}.
\] (2)

Hence, we have constructed a uniquely 3-colorable regular graph of degree 5 with 24 vertices and without any triangles. \( \Box \)

**Lemma 4.** There exists a uniquely 3-colorable regular graph of degree 6 with 48 vertices and without any triangles.

**Proof.** Let \( P_1 \) and \( Q_1 \) be graphs such that each of them is isomorphic to \( G_{24} \) in the proof of Lemma 3, \( V(P_1) = \{1, 2, \ldots, 12, 1', 2', \ldots, 12'\} \) and \( V(Q_1) = \{1, 2, \ldots, 12, 1', 2', \ldots, 12'\} \). Then, by Lemma 3, each of \( P_1 \) and \( Q_1 \) is uniquely 3-colorable regular
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Let $G_{2.24}$ be a graph with $V(G_{2.24}) = V(P_1) \cup V(Q_1)$ and $E(G_{2.24}) = E(P_1) \cup E(Q_1) \cup E_{P_1, Q_1}$, where $E_{P_1, Q_1}$ is the set

$$E_{P_1, Q_1} = \{[1, 2], [2, 3], [3, 4], [4, 5], [5, 6], [6, 7], [7, 8], [8, 1],$$

$$[9, 12'], [10, 11'], [11, 10'], [12, 9'],$$

$$[1', 2'], [2, 3'], [3, 4'], [4, 5'], [5, 6'], [6, 7'], [7, 8'], [8, 1']$$

(3)

where $[i, j]$ is an edge in $G_{2.24}$ with $i \in V(P_1)$ and $j \in V(Q_1)$. (We use $[i, j]$ to emphasize $i \in V(P_1)$ and $j \in V(Q_1)$.) Then $G_{2.24}$ is a regular graph of degree 6 with 48 vertices.

Since $[9, 12']$ and $[10, 11']$ belong to $E(G_{2.24})$, the vertices in $C_1 \cup C_1'$ of $Q_1$ in (2) (i.e., the vertices $3, 8, 11, 12, 3', 8', 11', 12'$ in $Q_1$) have to be colored differently from $A_1 \cup A_1'$ of $P_1$ in (2) (i.e., the vertices $1, 4, 6, 9, 1', 4', 6', 9'$ of $P_1$) and $B_1 \cup B_1'$ of $P_1$ in (2) (i.e., the vertices $2, 5, 7, 10, 2', 5', 7', 10'$ in $P_1$). Since $[1, 2]$ belongs to $E(G_{2.24})$, the vertices in $B_1 \cup B_1'$ of $Q_1$ in (2) have to be colored differently from the vertices in $A_1 \cup A_1$ of $P_1$ in (2). Thus, the $V(G_{2.24})$ is uniquely partitioned into 3 subsets

$$\{A_1 \cup A_1' of P_1 in (2) \} \cup \{A_1 \cup A_1' of Q_1 in (2)\},$$

$$\{B_1 \cup B_1' of P_1 in (2) \} \cup \{B_1 \cup B_1' of Q_1 in (2)\},$$

$$\{C_1 \cup C_1' of P_1 in (2) \} \cup \{C_1 \cup C_1' of Q_1 in (2)\}.$$  (4)

Hence, $G_{2.24}$ is a uniquely 3-colorable regular graph of degree 6 with 48 vertices and without any triangles. \(\square\)

The proof of Theorem 2 goes as follows. We construct a family of graphs $G_{2n.24}$ for $k = 1, 2, \ldots$.

For $k = 1$, $G_{2.24}$ is constructed as the one in the proof of Lemma 4, i.e., $V(G_{2.24}) = V(P_1) \cup V(P_2)$ where $P_1 \simeq P_2 \simeq G_{24}$, $E(G_{2.24}) = E(P_1) \cup E(P_2) \cup E_{P_1, P_2}$ where $E_{P_1, P_2}$ is defined in (3).

For $k = 2$, let $H$ and $H'$ be graphs such that $H \simeq H' \simeq G_{2.24}$, $V(H) = \bigcup_{i=1}^{2} V(P_i)$ and $V(H') = \bigcup_{i=1}^{2} V(Q_i)$ where $Q_i \simeq P_i \simeq G_{24}$ for $i = 1, 2$. $G_{22.24}$ is the graph whose $V(G_{22.24}) = V(H) \cup V(H')$, and $E(G_{22.24}) = E(H) \cup E(H') \cup E_{P_1, Q_1} \cup E_{P_2, Q_2}$. (Thus, $E(G_{22.24}) = (E(P_1) \cup E(P_2) \cup E_{P_1, P_2}) \cup (E(Q_1) \cup E(Q_2) \cup E_{Q_1, Q_2}) \cup E_{P_1, Q_1} \cup E_{P_2, Q_2}$.)

We define $G_{2n+1.24}$ inductively. Let $K$ and $K'$ be graphs such that $K \simeq K' \simeq G_{2n.24}$,

$$V(K) = \bigcup_{i=1}^{2n} V(P_i), \quad V(K') = \bigcup_{i=1}^{2n} V(Q_i).$$
where \( P_i \cong Q_i \cong G_{24} \) for \( i = 1, 2, \ldots, 2^n \). Let \( G_{2n+1,24} \) be the graph whose \( V(G_{2n+1,24}) = V(K) \cup V(K') \), and

\[
E(G_{2n+1,24}) = E(K) \cup E(K') \cup \left( \bigcup_{i=1}^{2^n} E_{P_i,Q_i} \right).
\]

We shall use the mathematical induction to show that each of the graph \( G_{2n,24} \), for \( k = 1, 2, \ldots \), is a uniquely 3-colorable regular graph of degree \( n + 5 \) with \( 2^k \cdot 24 \) vertices and without any triangles.

For \( k = 1 \), we know that, by Lemma 4, \( G_{2,24} \) is such a graph.

Assume that \( G_{2n,24} \) is such a graph. Consider \( G_{2n+1,24} \). Since \( K \cong K' \cong G_{2n,24} \), each of \( K \) and \( K' \) is a uniquely 3-colorable regular graph of degree \( n + 5 \) with \( 2^n \cdot 24 \) vertices and without triangles. From the construction, we know that \( G_{2n+1,24} \) have \( 2^{n+1} \cdot 24 \) vertices, \( G_{2n+1,24} \) is a regular graph of degree \( n + 6 \), and \( G_{2n+1,24} \) does not contain any triangles.

Since each of \( K \) and \( K' \) is uniquely 3-colorable, each of \( K \) and \( K' \) can be colored the same as \( G_{2n,24} \) with the same colors \( a, b \) and \( c \), i.e., \( V(K) \) is uniquely partitioned into three subsets:

\[
V_1 = \bigcup_{i=1}^{2^n} \{1, 4, 6, 9, 1', 4', 6', 9' \text{ of } P_i\},
\]

\[
V_2 = \bigcup_{i=1}^{2^n} \{2, 5, 7, 10, 2', 5', 7', 10' \text{ of } P_i\}, \quad \text{and}
\]

\[
V_3 = \bigcup_{i=1}^{2^n} \{3, 8, 11, 12, 3', 8', 11', 12' \text{ of } P_i\},
\]

and \( V(K') \) is also uniquely partitioned into three subsets:

\[
U_1 = \bigcup_{i=1}^{2^n} \{1, 4, 6, 9, 1', 4', 6', 9' \text{ of } Q_i\},
\]

\[
U_2 = \bigcup_{i=1}^{2^n} \{2, 5, 7, 10, 2', 5', 7', 10' \text{ of } Q_i\}, \quad \text{and}
\]

\[
U_3 = \bigcup_{i=1}^{2^n} \{3, 8, 11, 12, 3', 8', 11', 12' \text{ of } Q_i\}.
\]

Since \( [9, 12'] \) and \( [10, 11'] \) belong to \( E_{P_i,Q_i} \subseteq E(G_{2n+1,24}) \), the vertices in \( U_1 \) have to be colored differently from \( V_1 \) and \( V_2 \). Since \( [1, 2] \) belongs to \( E_{P_i,Q_i} \subseteq E(G_{2n+1,24}) \), the vertices in \( U_2 \) have to be colored differently from \( V_1 \). Consequently, \( V(G_{2n+1,24}) \) can be uniquely partitioned into 3 subsets: \( V_1 \cup U_1 \), \( V_2 \cup U_2 \) and \( V_3 \cup U_3 \).

Hence, there exist infinitely many uniquely 3-colorable regular graphs without any triangles, and the proof is completed.

The proof of the Corollary goes as follows. We shall use the mathematical induction on \( k \). For \( k = 3 \), it follows from Theorem 2.
Assume that it holds for \( k \geq 3 \), and consider for the case of \( k+1 \). That is, we assume that there exist infinitely many uniquely \( k \)-colorable regular graphs having no subgraph isomorphic to \( K_3 \). Let \( G_k \) be anyone of these graphs with \( V(G_k) = n \) and degree \( r \). We construct a graph \( G_{k+1} \) as follows. Let \( N_{n-r} \) be the null graph with \( n-r \) vertices, \( V(G_{k+1}) = V(G_k) \cup V(N_{n-r}) \), and \( E(G_{k+1}) = E(G_k) \cup \{(u, v) : u \in V(G_k) \text{ and } v \in V(N_{n-r})\} \), i.e., \( E(G_{k+1}) \) consists of all of the edges in \( G_k \) and all possible edges with one vertex in \( G_k \) and the other vertex in \( N_{n-r} \). Thus, \( |V(G_{k+1})| = 2n-r \) and \( G_{k+1} \) is a regular graph of degree \( n \). Since \( G_k \) is uniquely \( k \)-colorable, \( G_{k+1} \) is uniquely \((k+1)\)-colorable. Since \( G_k \) has no subgraph isomorphic to \( K_3 \), \( G_{k+1} \) has no subgraph isomorphic to \( K_{k+1} \). Since we have constructed \( G_{k+1} \) for each \( G_k \) and non-isomorphic \( G_k \) graphs produce non-isomorphic \( G_{k+1} \) graphs, we have infinitely many uniquely \((k+1)\)-colorable regular graphs having no subgraph isomorphic to \( K_{k+1} \).

References